# Invariance in Property Testing

**Madhu Sudan**
**Microsoft Research**

Based on: works with/of Eli Ben-Sasson, Elena Grigorescu, Tali Kaufman, Shachar Lovett, Ghid Maatouk, Amir Shpilka.

# Property Testing

- Sublinear time algorithms:
  - Algorithms running in time o(input), o(output).
    - Probabilistic.
    - Correct on (approximation) to input.
    - Input given by oracle, output implicit.
  - Crucial to modern context
    - (Massive data, no time).
- Property testing:
  - Restriction of sublinear time algorithms to decision problems (output = YES/NO).
- Amazing fact: Many non-trivial algorithms exist!

# Example 1: Polling

- Is the majority of the population Red/Blue
  - Can find out by random sampling.
  - Sample size $\propto$ margin of error
    - Independent of size of population

- Other similar examples: (can estimate other moments …)

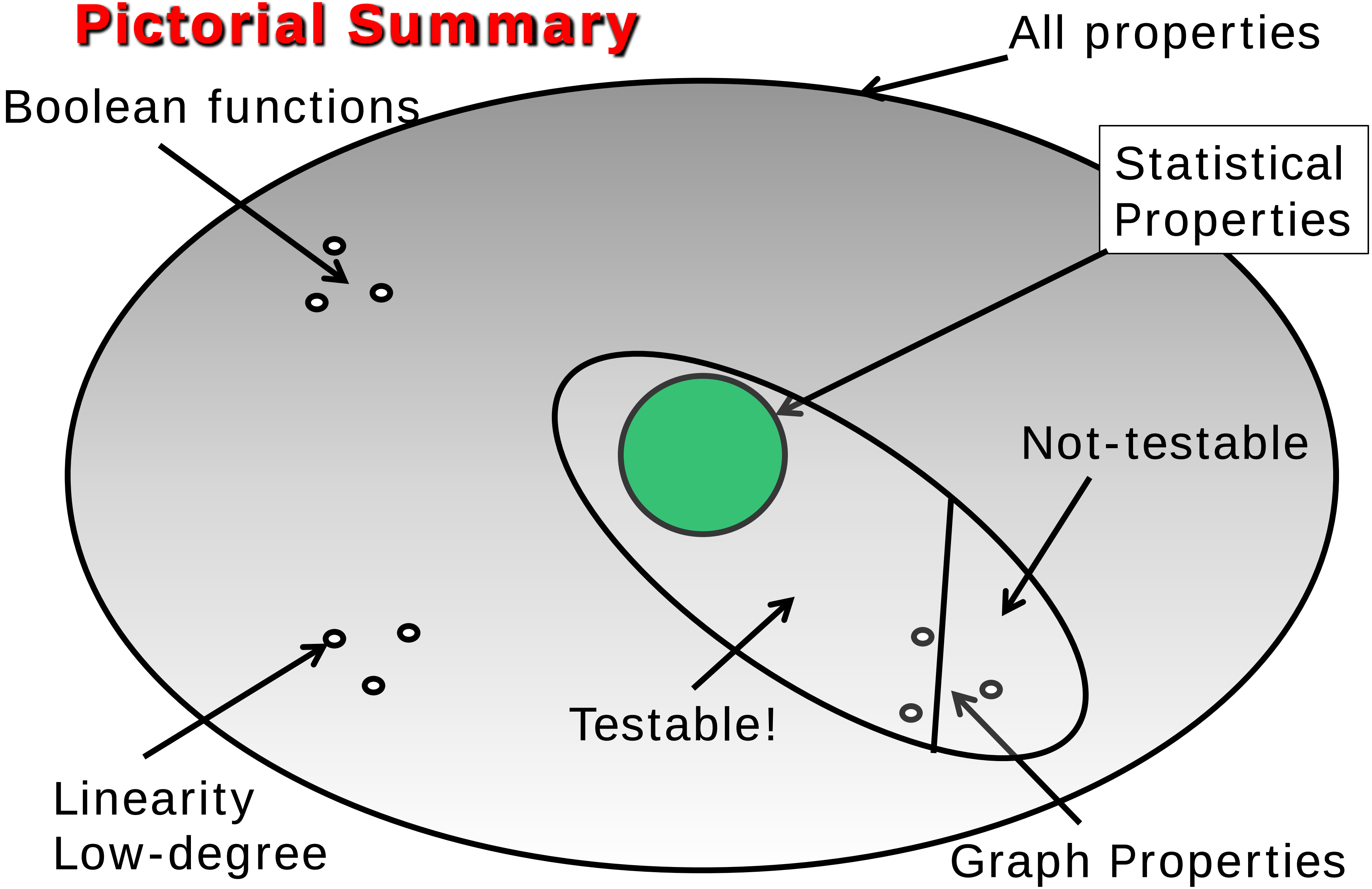
# Example 2: Linearity

- Can test for homomorphisms:
  - Given: $f: G \rightarrow H$ ($G,H$ finite groups), is $f$ essentially a homomorphism?
  - Test:
    - Pick $x,y$ in $G$ uniformly, ind. at random;
    - Verify $f(x) \cdot f(y) = f(x \cdot y)$
  - Completeness: accepts homomorphisms w.p. 1
    - (Obvious)
  - Soundness: Rejects f w.p prob. Proportional to its "distance" (margin) from homomorphisms.
    - (Not obvious, [BlumLubyRubinfeld'90])

# History (slightly abbreviated)

- [Blum,Luby,Rubinfeld – S'90]
  - Linearity + application to program testing
- [Babai,Fortnow,Lund – F'90]
  - Multilinearity + application to PCPs (MIP).
- [Rubinfeld+S.]
  - Low-degree testing
- [Goldreich,Goldwasser,Ron]
  - Graph property testing
- Since then … many developments
  - More graph properties, statistical properties, matrix properties, properties of Boolean functions …
  - More algebraic properties

# Pictorial Summary



# Some (introspective) questions

- What is qualitatively novel about linearity testing relative to classical statistics?
- Why are the mathematical underpinnings of different themes so different?
- Why is there no analog of “graph property testing” (broad class of properties, totally classified wrt testability) in algebraic world?

# Invariance?

- Property $P \subseteq \{f : D \rightarrow R\}$
- Property $P$ invariant under permutation (function) $\pi: D \rightarrow D$, if

$$f \in P \Rightarrow f \circ \pi \in P$$

- Property $P$ invariant under group $G$ if

  $\forall \pi \in G$, $P$ is invariant under $\pi$.

- Observation: Different property tests unified/separated by invariance class.

# Invariances (contd.)

- Some examples:
  - Classical statistics: Invariant under all permutations.
  - Graph properties: Invariant under vertex renaming.
  - Boolean properties: Invariant under variable renaming.
  - Matrix properties: Invariant under mult. by invertible matrix.
  - Algebraic Properties = ?
- Goals:
  - Possibly generalize specific results.
  - Get characterizations within each class?
  - In algebraic case, get new (useful) codes?

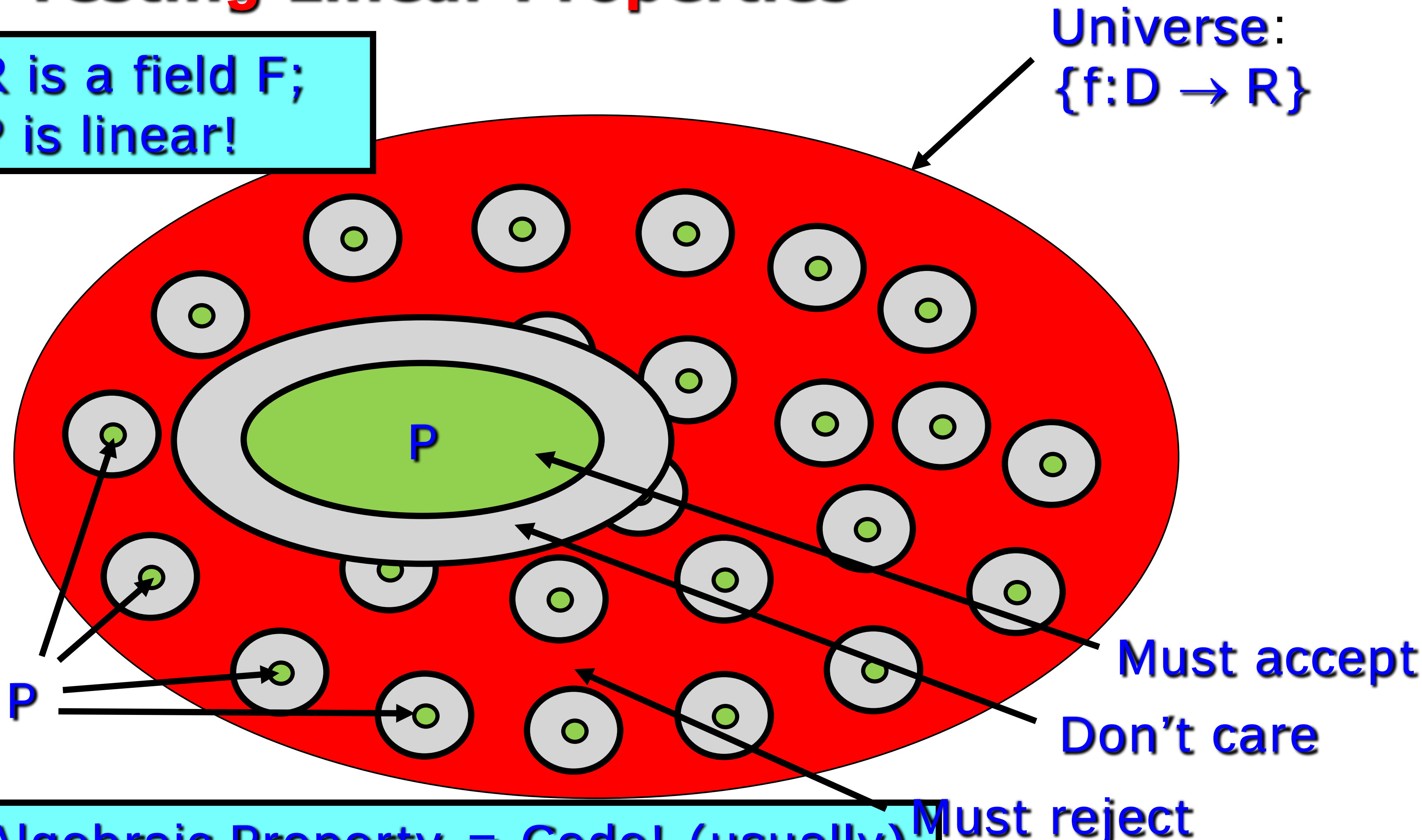
# Abstracting Linearity/Low-degree tests

- Affine Invariance:
  - Domain = Big field ($GF(2^n)$)
    or vector space over small field ($GF(2)^n$).
  - Property invariant under affine transformations of domain ($x \mapsto A.x + b$)

- Linearity:
  - Range = small field ($GF(2)$)
  - Property = vector space over range.

# Testing Linear Properties

R is a field F;
P is linear!




Algebraic Property = Code! (usually)

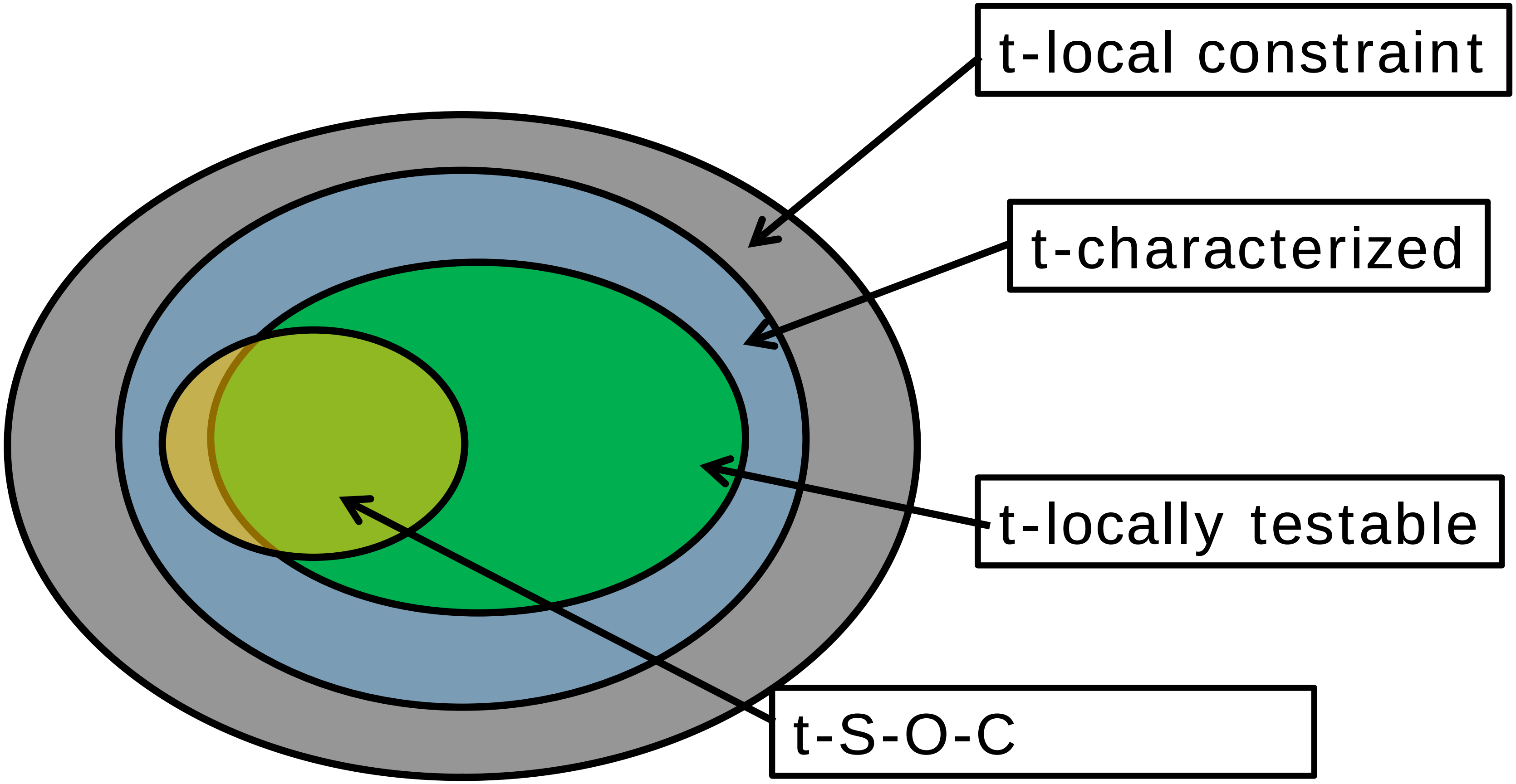
# Why study affine-invariance?

- Common abstraction of properties studied in [BLR], [RS], [ALMSS], [AKKLR], [KR], [KL], [JPRZ].
  - (Variations on low-degree polynomials)
- Hopes
  - Unify existing proofs
  - Classify/characterize testability
  - Find new testable codes (w. novel parameters)
- Rest of the talk: Brief summary of findings

# Basic terminology

- Local Constraint:
  - Example: f(1) + f(2) = f(3).
  - Necessary for testing Linear Properties [BHR]
- Local Characterization:
  - Example: $\forall$ x, y, f(x) + f(y) = f(x+y) $\Leftrightarrow$ f $\in$ P
  - Aka: LDPC code, k-CNF property etc.
  - Necessary for affine-invariant linear properties.
- Single-orbit characterization:
  - One linear constraint + implications by affine-invariance.
  - Feature in all previous algebraic properties.

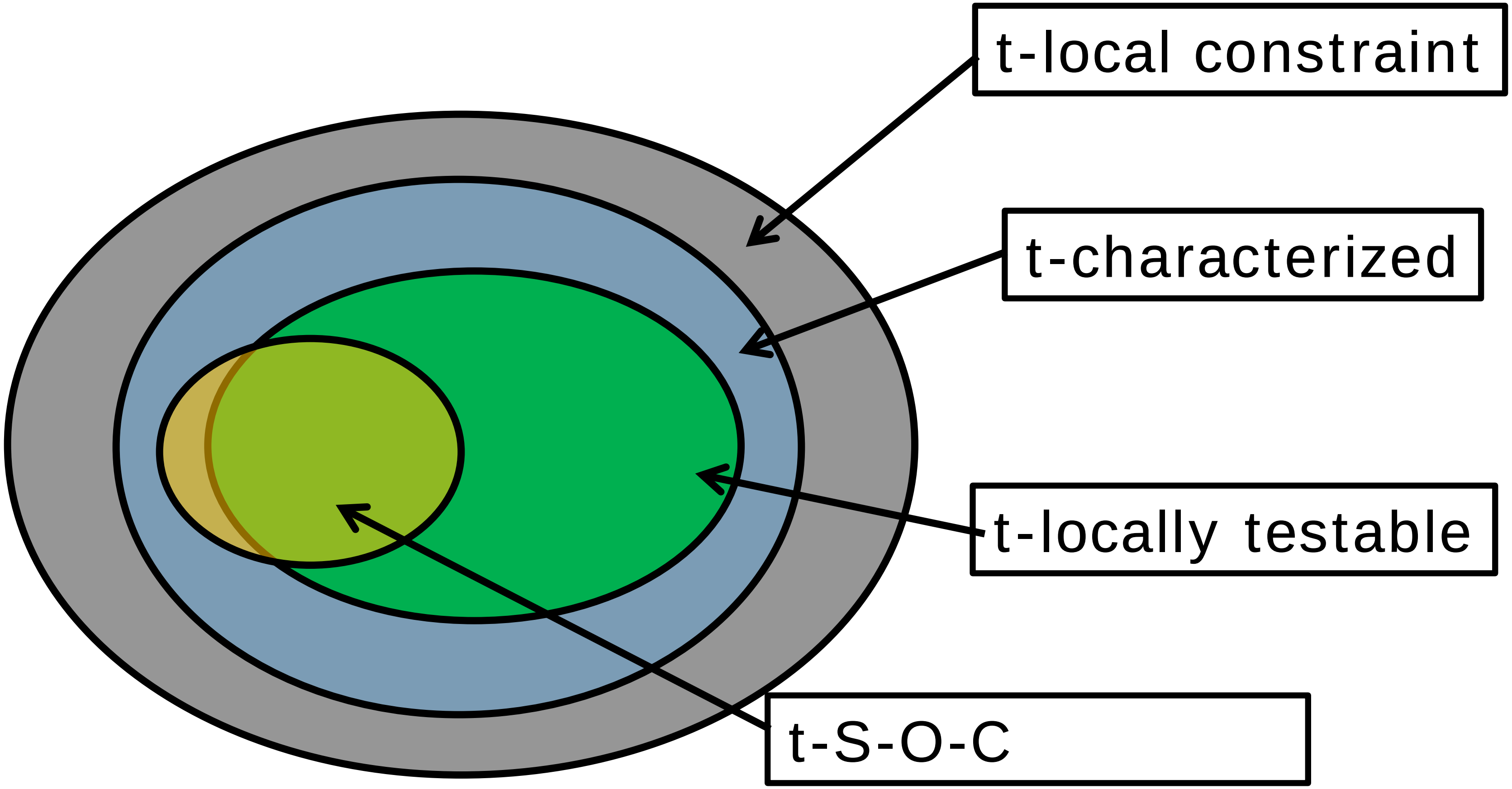
# Affine-invariance & testability



# State of the art in 2007

- [AKKLR]: k-constraint = k'-testable, for all linear affine-invariant properties?

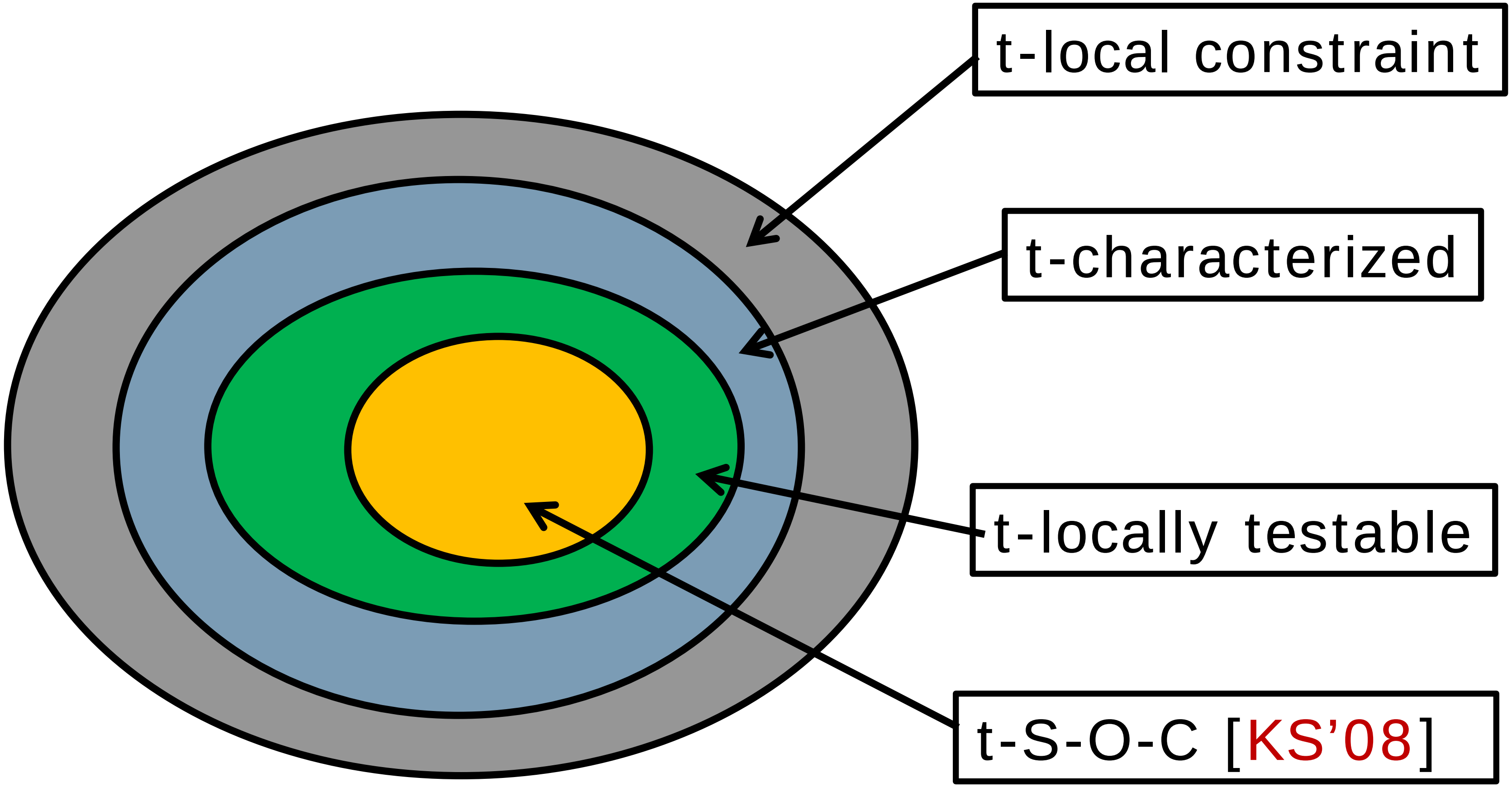
# Affine-invariance & testability



# Some results

- [Kaufman+S.'07]: Single-orbit $\Rightarrow$ Testable.

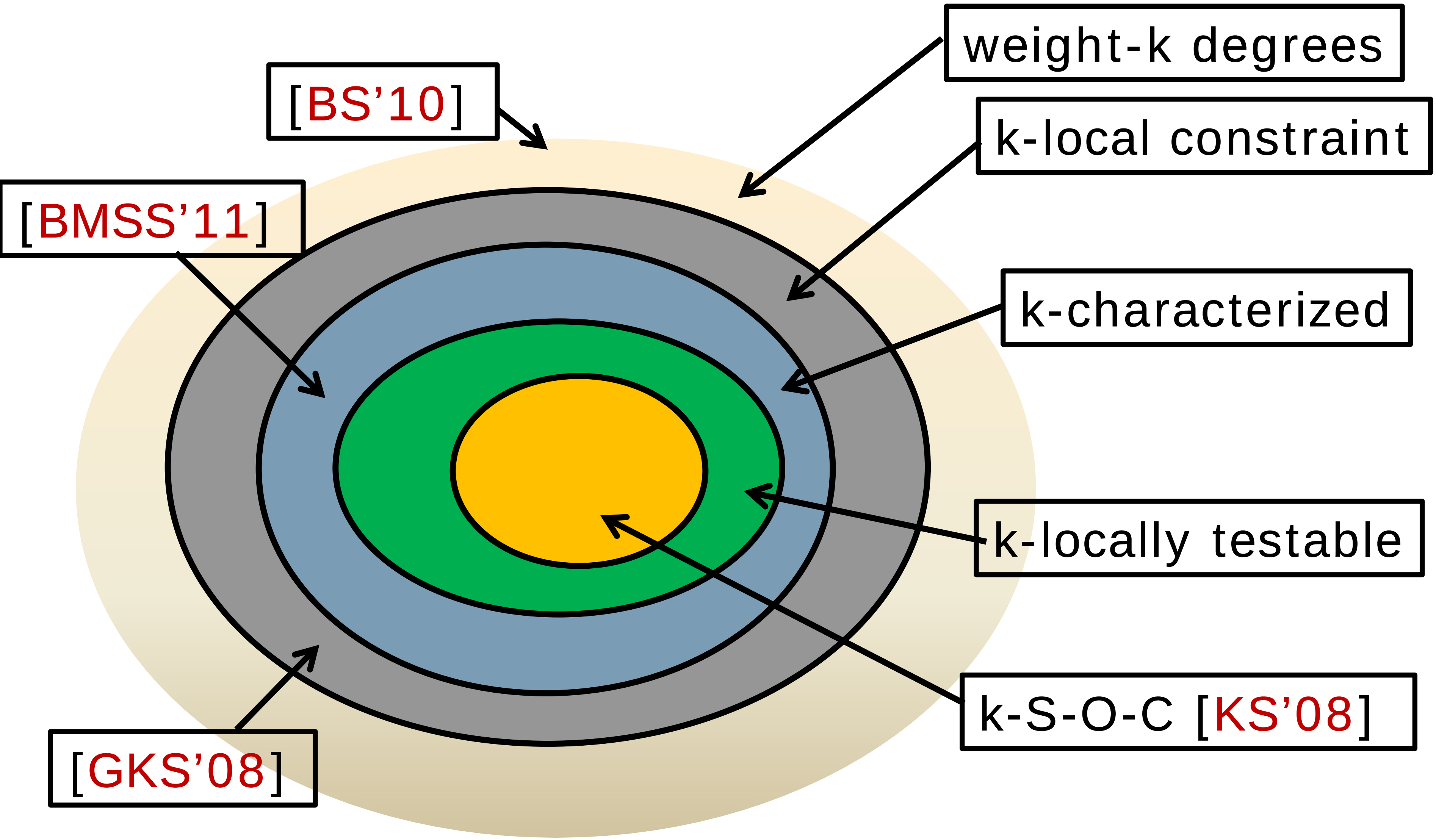
# Affine-invariance & testability



# Some results

- [Kaufman+S.'07]: Single-orbit $\Rightarrow$ Testable.
  - Unifies known algebraic testing results.
  - Converts testability to purely algebraic terms.
  - Yields "Constraints = Char. = Testability" for vector spaces over small fields.
  - Left open: Domain = Big field.
  - $\exists$ Many "non-polynomial" testable properties
- [GKS'08]: Over big fields, Constraint $\neq$ Char.
- [BMSS'11]: Over big fields, Char $\neq$ Testability.
- [BGMSS'11]: Many questions/conjectures outlining a possible characterization of affine-invariant properties.

# Affine-invariance & testability



# Hopes

- Get a complete characterization of locally testable affine-invariant properties.
- Use codes of (polynomially large?) locality to build better LTCs/PCPs?
  - In particular move from “domain = vector space” to “domain = field”.
- More broadly: Apply lens of invariance more broadly to property testing.

# Thank You!