Physical Limits of Communication

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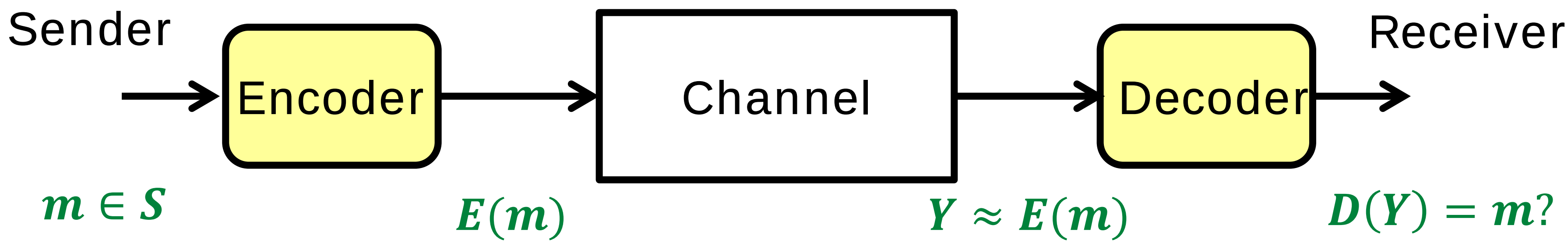
Reliable Communication



- Communication is Reliable $\Leftrightarrow$ $D(Y) = m$.
- Can be communication across space (e.g. cellphones) or time (DVD).
- Implicit axiom:
  - If Sender/Receiver are physically separated, then only finite # bits can be communicated in finite amount of time. ($|S|$ is finite.)
- This talk: Why?

Why should computer science care?

- Axioms of computation: Computation is local.
  - Works with finite state
  - Operates on/based on finite number of (preselected) bits at a time.
  - Preselection changes locally from step to step.

- Rely on communication axiom implicitly:
  - Why is state finite?
  - Why finite number of bits at a time?
  - If communication were different, computation should/would be too!

Why is finiteness restrictive?

- Physical channels are not a priori discrete.
  - Input to channel/Output of channel = signal.

nctic ... e

[0,1]

- T... ty
  - ... ibili
  - ... ssibi
  - ... = ∞?
    (Even when channel is s ...))
  - What do physically realizable channels look like?

Part I: Classical Models

Continuous-valued functions [Shannon]

- Say signals are discrete-time, continuous-valued:
  $$f: \{0,1,\ldots,T\} \to [0,1]$$
- Channel = ?
  - Error $\eta: \{0,1,\ldots,T\} \to [-\epsilon,+\epsilon]$
  - Output signal $Y: \{0,1,\ldots,T\} \to \Re$
    $$Y(t) = f(t) + \eta(t)$$
- Capacity := $\log|S| =$ finite? Infinite?
  - Analysis (two cases):
    - Adversarial error: Easy.
      - $\forall t$, adversary can fix $Y(t)$ to be multiple of ϵ.
      - Capacity $\leq T \log(1/\epsilon)$

Continuous-valued functions (contd.)

- Recall
  - Input: $f:\{0,1,\ldots,T\} \to [0,1]$
  - Error $\eta:\{0,1,\ldots,T\} \to [-\epsilon,+\epsilon]$
  - Output signal $Y:\{0,1,\ldots,T\} \to \Re$, $\quad Y(t) = f(t) + \eta(t)$
- Probabilistic error: $\eta(t) \leftarrow N(0,\epsilon^2)$ ind., $\forall\, t$.
- Spirit of Shannon's analysis:
  - Capacity of channel without noise $= \infty$
  - Entropy of noise $= \infty$
  - Capacity of noisy channel

    = cap of channel w/o noise – entropy of noise

    $= \infty - \infty = O\left(\mathrm{T}\log\frac{1}{\epsilon}\right)$.

Continuous-time [SP: Nyquist et al.]

- Signals (input/output): $f:[0,T]\to[0,1]$
  - Methodology quite different:
- Well-studied in classical Signal Processing (SP):
  - Works of [Nyquist, Shannon, Landau-Pollak-Slepian]
- Many Variations:
  - Layperson version
    - Frequency spectrum of signal $\subseteq [-W,+W]$
      $\Rightarrow$ suffices to sample signal $O(T/W)$ times.
  - Correct versions:
    - More complex (theorems + models).

Continuous-time (contd.)

- Actual versions:
  - Shannon:
    - Frequency spectrum finite subset of $[-W, +W] \Rightarrow$ suffices to sample finitely many times. (V. weak).
  - Nyquist:
    - Frequency spectrum $\subseteq [-W, +W]$

      $\Rightarrow$ signal f reconstructible from $\left\{f\left(\frac{i}{2W}\right)\right\}_{\{i \in Z\}}$
      - Infinite many samples! Finite version can't work (with exact reconstruction).

(My) Problems with SP axioms

- Why do we need Fourier transforms?
  - What are these operations in time domain?
  - (Fourier analysis should remain analysis technique – not natural operation).
  - Not clean (like Shannon for discrete-time).
- Frequency vs. time:
  - Only signal bounded in time and frequency spectrum is the zero signal
    - So we need to relax even bandwidth restrictions (some variations studied).
  - Impulse-response of low-pass filter is non-causal!
    - Are variations causal?

Part II: Our Model: Delays

Noisy and Tardy Channels

- Input: $f : [0, T] \rightarrow [0,1]$
- Noise: $\eta : [0, T] \rightarrow \Re$ (typically small $\approx \pm\epsilon$)
- Delay: $\Delta : [0, T] \rightarrow \Re^{\{\geq 0\}}$ (typically ≈ 1).
- Output: $Z : [0, T] \rightarrow \Re$ where
  - $Z(t) = \int_0^t 1\{\tau + \Delta(\tau) = t\} \cdot \left(f(\tau) + \eta(\tau)\right).$

- Noise + Delay:
  - Probabilistic or Adversarial ?
  - If one is adversarial, does it know the other?

Motivations for delay

- Channels seem to do some frequency "attenuation"/"smoothing".
- Such attenuation should be expressible in time domain (impulse response).
- Impulse response should be causal.
- Under simplifying assumptions (response is non-negative) impulse response looks like pdf of delay.
  - (Making delay probabilistic necessary to introduce some uncertainty. If not, easy to invert distortion.)

Discrete Modelling of Continuous time

- To simplify our analysis, will discretize time (and signal value), but will allow encoder/decoder to choose how fine the discretization is.
- So 1 unit of time = M micro-intervals (each microinterval is of length 1/M).
- Signal value $\in \{0,1\}$; and constant within microinterval.
- Will ask: Does $\frac{capacity(M)}{T} \to \infty$ as $M \to \infty$?

Notationally:

- Let $N = M \cdot T$.
- Encoding $= X_1, X_2, \ldots, X_N \in \{0,1\}$
- Error $= \eta_1, \eta_2, \ldots, \eta_N \in \{0,1\}$;

  $\eta_i = 1 \approx$ for ϵ-fraction of i's.
- Delay $= \Delta_1, \Delta_2, \ldots, \Delta_N$; $\Delta_i \approx M$.
- Output $= Z_1, Z_2, \ldots, Z_N \in \mathbb{Z}^{\geq 0}$;

  $$Z_i = \sum_{\{j \leq i : j + \Delta_j = i\}} \left(X_j \oplus \Delta_j\right)$$
- Will be interested in: $Capacity(M) \triangleq \frac{1}{T} \cdot \log |S|$

Questions:

- Will be interested in: $Capacity(M) \triangleq \frac{1}{T} \cdot \log |S|$
  - Does $Capacity(M) \to \infty$?
- Might depend on whether Noise/Delay are adversarial/probabilistic.
  - Furthermore, if only one is adversarial, is it adaptive wrt randomness of the other?
- Probabilistic Models:
  - Noise: η_i Bernoulli r.v. 1 w.p. ϵ and 0 o.w.
  - Delay: Δ_i Geometric r.v. with mean M.

    (So unit time delay.)

Answers:

- Adv. Noise + Adv. Delay: Capacity is finite.

- Random Noise + Random Delays: Capacity unbounded.

- Final theorem (a classification):
  - $\exists\, \epsilon > 0$ s.t. $\lim_{\{M\to\infty\}} \{Capacity(M)\} = \infty$ for random delay with adversarial/random noise of rate ϵ, provided noise independent of delay.
  - Otherwise, $\lim_{\{M\to\infty\}} \{Capacity(M)\} < \infty,\ \forall \epsilon > 0.$

Part III: Some Proofs

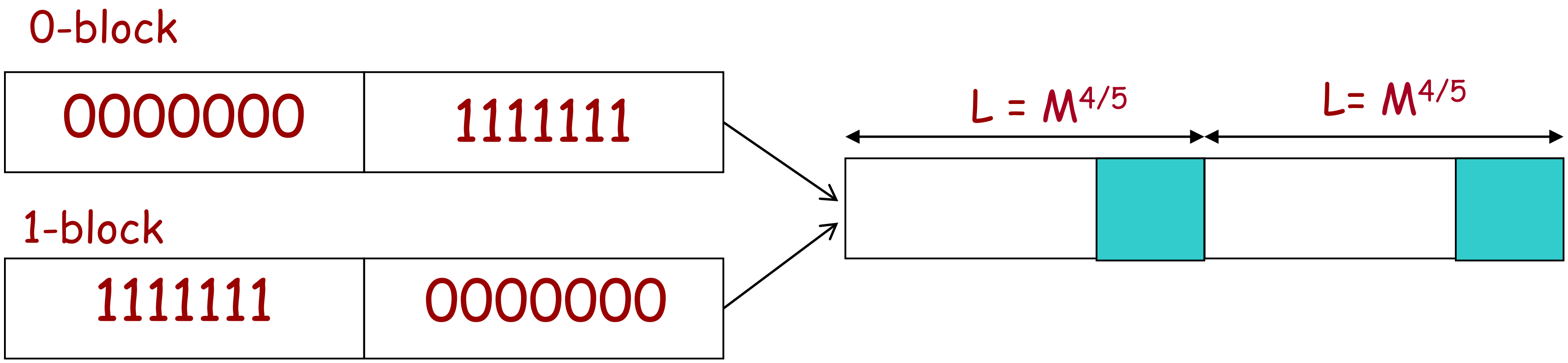
Adversarial Noise and Delay

- General view of delays:
  - Think of "delay" as a queue/buffer.
    - Incoming bits ($X_i + \eta_i$) held in this buffer and released at time $i + \Delta_i$.

- Analysis of Adversarial channel:
  - Adversary can force channel to look discrete:
    - Bits depart the queue at integral time units ($i + \Delta_i$ is a multiple of M).
    - Number of ones departing buffers are always integral mutiples of $\epsilon \cdot M$.

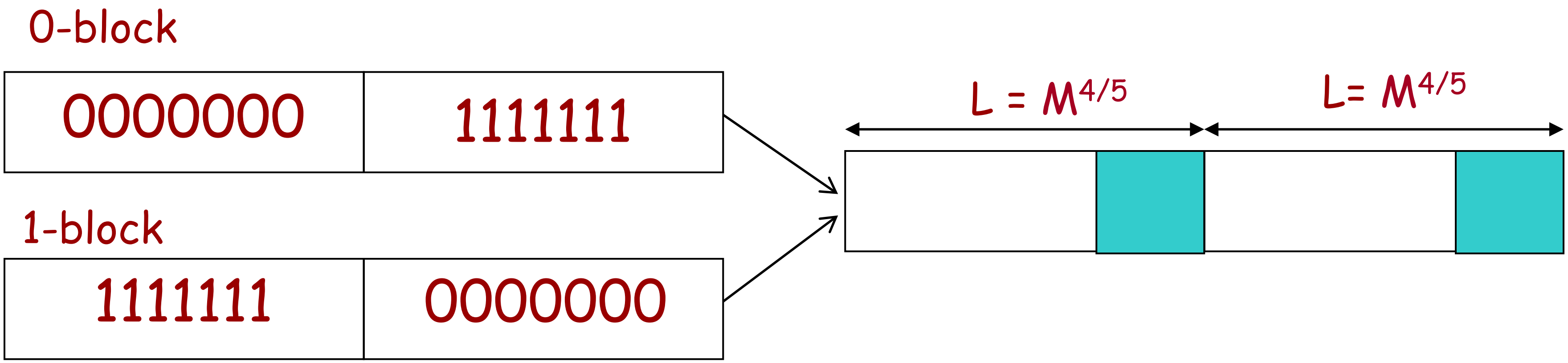
Random Noise and Delay

- **Basic idea:** Repeating bits $M^{\{1-\delta\}}$ times gives enough "signal" to overwhelm $\sqrt{M}$ deviation due to delay/noise (especially if buffer is balanced).
- **Encoder:** $0 \to 0^L 1^L; \;\; 1 \to 1^L 0^L; \;\; L \approx M^{\{\frac{4}{5}\}}$



- **Differential Decoding:**
  - Compare fraction of 1's in middle of block to end.
  - report 0 iff increase.

Random Noise and Delay (contd.)



- Differential Decoding:
  - Compare fraction of 1's in middle of block to end.
  - report 0 iff increase.

- Analysis: Chernoff bounds.
- Same works if noise is adv. but ind. of delay.

Other Finite Cases:

- Random Delay | Adversarial Noise:
  - Adv. groups signal into blocks of length $\approx \epsilon M$
  - At end of each block, round buffer contents to multiple of ϵM.
  - Also zeroes out all bits that arrive & depart within same block.
- Analysis:
  - Output process "distributionally defined" by contents of buffer at end of blocks.
  - Uses: "Geometric distribution is memoryless."

Other Finite Cases – II

- Adv. Delay | Random Noise
  - Divide input into blocks;
  - Delay enough (of the right) bits to make sure buffer contents at end of blocks are multiples of ϵM (before noise).

- Analysis:
  - Prove that output signal is "distributionally determined" by buffer contents at end of blocks.
  - Involves analysis of "signal via noise" channel.

Part IV: Conclusions

Physical Limits of Communication = ?

- Most reasonable interpretation of nature: non-adversarial.
  - In such settings capacity = infinite!
  - Counterintuitive + Contrary to SP literature.
- Did we model physics correctly?: Not sure …
- Other possible explanations:
  - Universe is finite … (was this implicit in Shannon?)
  - Precise measurements are expensive (but wasn't this taken care of?)
  - Some non-linearity?
  - No natural explanations in time domain!

Thank You!