Local Algorithms & Error-correction

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Prelude

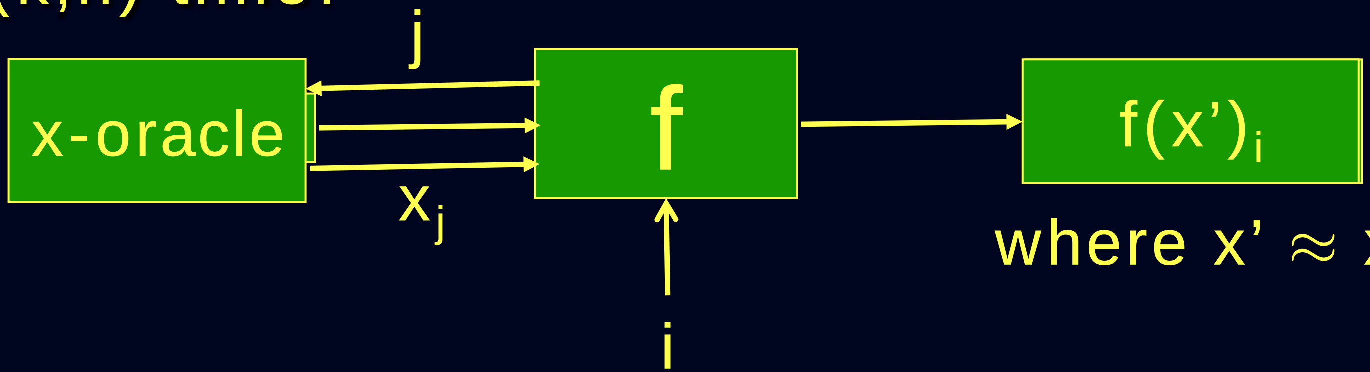
- Algorithmic Problems in Coding Theory
- New Paradigm in Algorithms
- The Marriage: Local Error-Detection & Correction

Algorithmic Problems in Coding Theory

- Code: Σ = finite alphabet (e.g., {0,1}, {A … Z})
  - $E:\Sigma^k \to \Sigma^n$; Image(E) = $C \subseteq \Sigma^n$
  - $R(C) = k/n$; $\delta(C)$ = normalized Hamming distance
- Encoding:
  - Fix code C and associated E.
  - Given $m \in \Sigma^k$, compute E(m).
- Error-detection (ϵ-Testing):
  - Given $x \in \Sigma^n$, decide if $\exists$ m s.t. x = E(m).
  - Given x, decide if $\exists$m s.t. $\delta(x,E(m)) \leq \epsilon$.
- Error-correction (Decoding):
  - Given $x \in \Sigma^n$, compute (all) m s.t.
    $\delta(x,E(m)) \leq \epsilon$ (if any exist).

Sublinear time algorithmics

- Given $f:\{0,1\}^k \to \{0,1\}^n$ can f be "computed" in o(k,n) time?



where $x' \approx x$

- Answer 2: Clearly NO, since that is the time it takes to even read the input/write the output
- Answer 1: YES, if we are willing to
  1. Present input implicitly (by an oracle).
  2. Represent output implicitly
  3. Compute function on approximation to input.

Extends to computing relations as well.

Sub-linear time algorithms

- Initiated in late eighties in context of
  - Program checking [BlumKannan,BlumLubyRubinfeld]
  - Interactive Proofs/PCPs [BabaiFortnowLund]
- Now successful in many more contexts
  - Property testing/Graph-theoretic algorithms
  - Sorting/Searching
  - Statistics/Entropy computations
  - (High-dim.) Computational geometry
- Many initial results are coding-theoretic!

Sub-linear time algorithms & Coding

- Encoding: Not reasonable to expect in sub-linear time.
- Testing? Decoding? – Can be done in sublinear time.
  - In fact many initial results do so!
- Codes that admit efficient ...
  - ... testing: Locally Testable Codes (LTCs)
  - ... decoding: Locally Decodable Codes (LDCs).

Rest of this talk

- Definitions of LDCs and LTCs
- Quick description of known results
- The first result: Hadamard codes
- Some basic constructions
- Recent constructions of LDCs.
  - [Kopparty-Saraf-Yekhanin '11]
  - [Yekhanin '07,Raghavendra '08,Efremenko '09]

Definitions

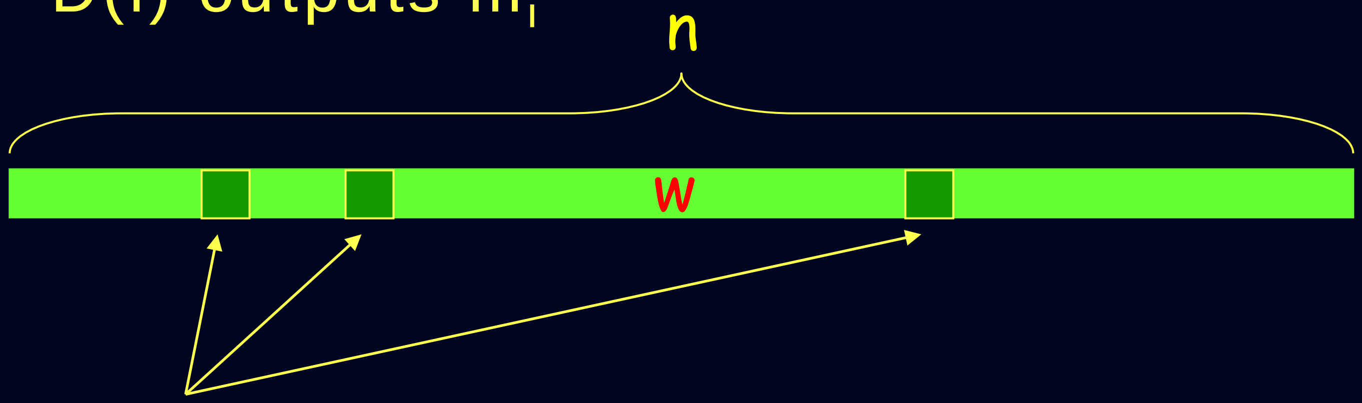
Locally Decodable Code

$C:\Sigma^k \to \Sigma^n$ is (q,ϵ)-Locally Decodable if $\exists$ decoder D

s.t. given $i \in [k]$, and oracle $w : [n] \to \Sigma$

s.t. $\exists$ m s.t. $\delta(w,C(m)) \leq \epsilon \leq \delta(C)/2$,

D(i) outputs m_i



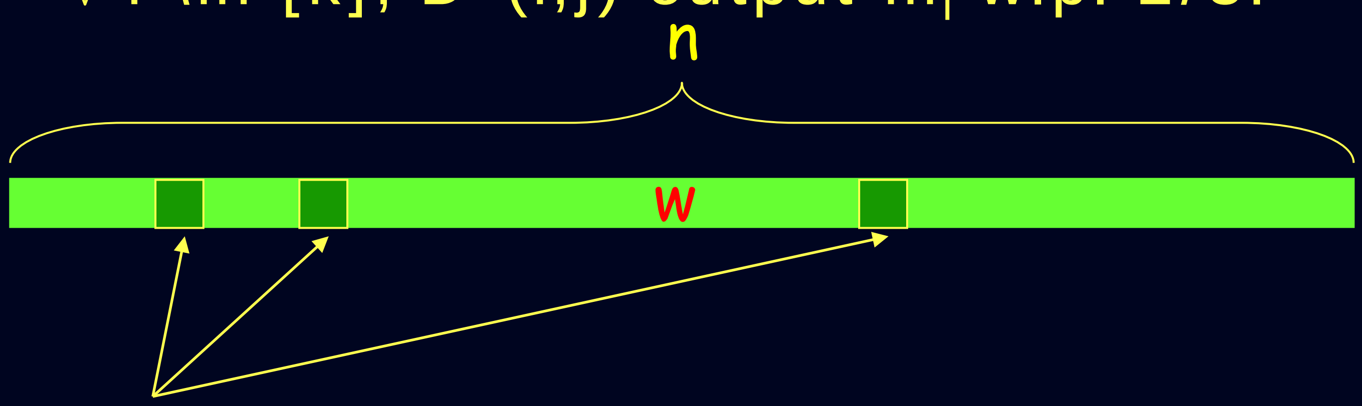
D(i) reads q(n) random positions of w and outputs m_i w.p. $\geq 2/3$.

What if $\epsilon > \delta(C)/2$? Might need to report a list of codewords.

Locally List-Decodable Code

C is (ϵ,L)-<u>list-decodable</u> if $\forall\ w \in \Sigma^n$
\# codewords $c \in C$ s.t. $\delta(w,c) \leq \epsilon$ is at most L.

C is (q,ϵ,L)-<u>locally-list-decodable</u> if $\exists$ decoder D s.t.
given oracle $w: [n] \to \Sigma$,
$\forall\ m \in \Sigma^k$, s.t. $\delta(w,C(m)) \leq \epsilon$, $\exists\ j \in [L]$ s.t.,
$\forall\ i \in [k]$, $D^w(i,j)$ output m_i w.p. 2/3.



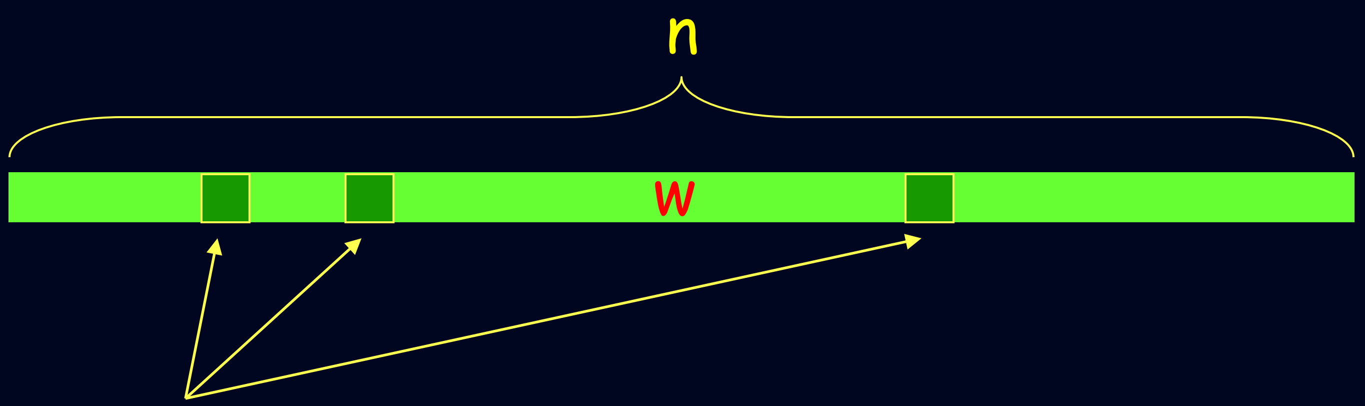
D(i,j) reads q(n) random positions of w
and outputs m_i w.p. $\geq 2/3$.

History of definitions

- Constructions predate formal definitions
  - [Goldreich-Levin '89].
  - [Beaver-Feigenbaum '90, Lipton '91].
  - [Blum-Luby-Rubinfeld '90].
- Hints at definition (in particular, interpretation in the context of error-correcting codes): [Babai-Fortnow-Levin-Szegedy '91].
- Formal definitions
  - [S.-Trevisan-Vadhan '99] (local list-decoding).
  - [Katz-Trevisan '00]

Locally Testable Codes

C is (q,ϵ)-Locally Testable if ∃ tester T s.t.



T reads q(n) positions (probabilistically):

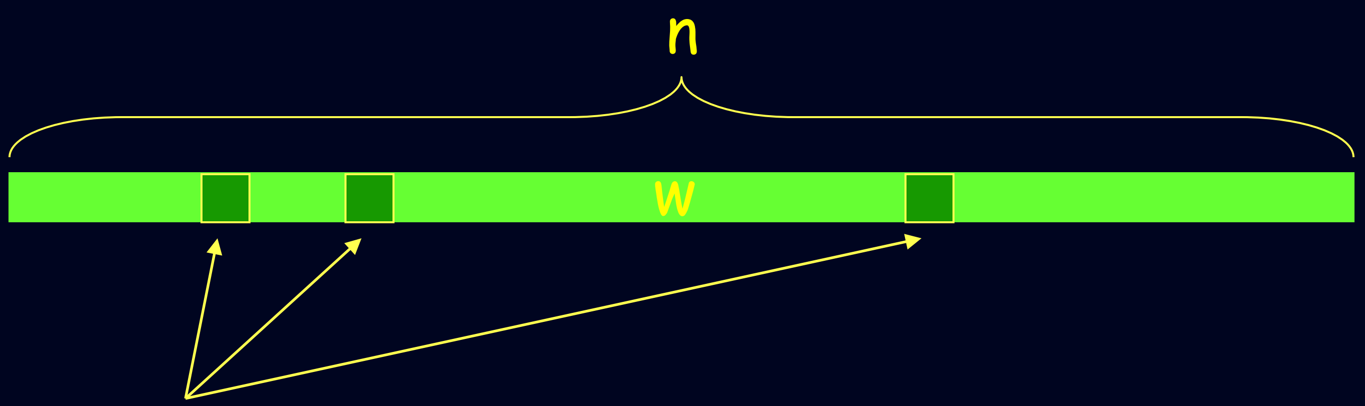
If $w \in C$, T accepts w.p. 1.

If $\delta(w,C) > \epsilon$, T rejects w.p. $\geq$ ½.

"Weak" definition: hinted at in [BFLS], explicit in [RS'96, Arora'94, Spielman'94, FS'95].

Strong Locally Testable Codes

C is (q,ϵ)-(strongly) Locally Testable if $\exists$ tester T s.t.



T reads $q(n)$ positions (probabilistically):

If $w \in C$, T accepts w.p. 1.

$\forall\, w \in \Sigma^n$, T rejects w.p. $\geq \Omega(\delta(w,C))$.

"Strong" Definition: [Goldreich-S. '02]

Motivations

Local Decoding: Worst-case vs. Average-case

- Suppose $C \subseteq \Sigma^N$ is locally-decodable for $N = 2^n$. (Furthermore assume can locally decode all bits of the codeword, and not just message bits.)
- $c \in C$ can be viewed as $c: \{0,1\}^n \rightarrow \Sigma$.
- Local decoding $\sim\Rightarrow$ can compute $c(x)$, $\forall$ x, if can compute $c(x')$ for most x'.
- Relates average case complexity to worst-case complexity. [Lipton, STV].
- Alternate interpretation:
  - Can compute $c(x)$ without revealing x.
  - Leads to Instance Hiding Schemes [BF], Private Information Retrieval [CGKS].

Motivation for Local-testing

- No generic applications known.
- However,
  - Interesting phenomenon on its own.
  - Intangible connection to Probabilistically Checkable Proofs (PCPs).
  - Potentially good approach to understanding limitations of PCPs (though all resulting work has led to improvements).

Contrast between decoding and testing

- Decoding: Property of words near codewords.
- Testing: Property of words far from code.

- Decoding:
  - Motivations happy with n = quasi-poly(k), and q = poly log n.
  - Lower bounds show q = O(1) and n = nearly-linear(k) impossible.
- Testing: Better tradeoffs possible! Likely more useful in practice.
  - Even conceivable: n = O(k) with q = O(1)?

Some LDCs and LTCs

Hadamard (1st Order RM) Codes

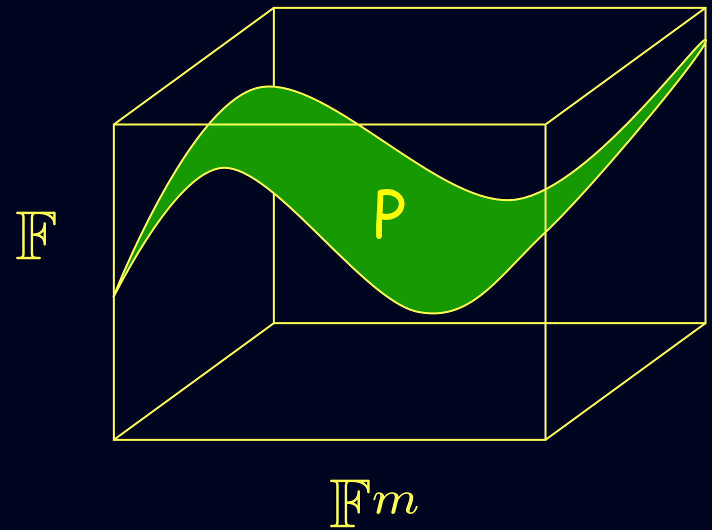
- **Messages:**
  - (Coefficients of) Linear functions $\{L : F_2^k \to F_2\}$.
- **Encoding:**
  - Evaluations of L on all of F_2^k.
- **Parameters:**
  - k bit messages $\to 2^k$ bit codewords.
- **Locality:**
  - 2-Locally Decodable [Folklore/Exercise]
  - 3-Locally Testable [BlumLubyRubinfeld]

Hadamard (1st Order RM) Codes

- Summary:
  - There exist infinite families of codes
  - With constant locality (for testing and correcting).

Codes via Multivariate Polynomials

- Message: Coefficients of degree t, m-variate polynomial over (finite field) F



((generalized) Reed-Muller Code)

- Encoding: Evaluations of P over all of F^m
- Parameters: $k \approx (t/m)^m$; $n = F^m$; $\delta(C) \approx 1 - t/F$.

Basic insight to locality

- m-variate polynomial of degree t, restricted to m' < m dim. affine subspace is poly of deg. t.
- Local Decoding:
  - Given oracle for $w \approx P$, and $x \in F^m$
  - Pick subspace A through x.
  - Query w on A and decode for $P|_A$
  - Query complexity: $q = F^{m'}$; Time = poly(q); $m' = o(m) \Rightarrow$ sublinear!
- Local Testing:
  - Verify w restricted to subspace is of degree t.
  - Same complexity; Analysis much harder.

Polynomial Codes

- Many parameters: m, t, F

- Many tradeoffs possible:
  - Locality $(\log k)^2$ with $n = k^4$;
  - Locality $\epsilon.k$ with $n = O(k)$;
  - Locality (constant) q, with $n = \exp(k^{(1/q-1)})$

Are Polynomial Codes (Roughly) Best?

- No! [Ambainis97] [GoldreichS.00] …
- No!! [Beimel,Ishai,Kushilevitz,Raymond]
- Really … Seriously … No!!!!

  [Yekhanin07,Raghavendra08,Efremenko09]

  [Kopparty-Saraf-Yekhanin '10]

Recent LDCs - I

[Kopparty-Saraf-Yekhanin '10] s

The Concern

- Poor rate of polynomial codes:
  - Best rate (for any non-trivial locality): ½ (bivariate polynomials, $\sqrt{n}$ locality).
  - Locality n^{ϵ} : Rate $\epsilon^{(1/\epsilon)}$ (use $1/\epsilon$ variables).
- Practical codes use high rates (say 80%)

Bivariate Polynomials

- Use $t = (1 - \rho).F$; $\rho \to 0$
- Yields $\delta(C) \approx \rho$.
- # coefficients: $k < ½.(1- \rho)^2.F^2$
- Encoding length: $n = F^2$.
- Rate $\approx ½.(1 - \rho)^2$

- Can't use degree > F; Hence Rate < ½ !

Mutliplicity Codes

- Idea:
  - Encode polynomial $P(x,y)$ by its evaluations, <u>and evaluations of its (partial) derivatives</u>!
- Sample parameters:
  - $n = 3F^2$ (F^2 evaluations of $\{P + P_x + P_y\}$).
  - However, degree can now be larger than F.
  - $t = 2.(1 - \rho).F \Rightarrow \delta(C) = \rho$.
  - $k = 2 . (1 - \rho)^2 . F^2$; Rate $\approx 2/3$.
  - Locality = $O(F) = O(\sqrt{k})$
- Getting better:
  - With more multipicity, rate goes up.
  - With more variables, locality goes down.

Multiplicity Codes: The Theorem

- Theorem:

  $\forall\ \alpha, \beta > 0$,

  $\exists\ \delta > 0$ and LDC C: $\{0,1\}^k \rightarrow \{0,1\}^n$ with

  Rate $\geq 1 - \alpha$,

  Distance $\geq \delta$,

  Locality $\leq k^\beta$ (decodable with k^β queries).

Recent LDCs - II

[Yekhanin ‘07, Raghavendra ‘08, Efremenko ‘09]

Other end of spectrum

- Minimum locality possible?
  - $q = 2$: Hadamard codes achieve $n = 2^k$;
    - [Kerenedis, deWolf]: $n \geq \exp(k)$.
  - $q = 3$: Best possible = ?.
    - Till 2006: Widely held belief: $n \geq \exp(k^{.1})$
    - [Yekhanin '07]: $n \leq \exp(k^{.0000001})$
    - [Raghavendra '08]: Clarified above.
    - [Efremenko '09]: $n \leq \exp(\exp(\sqrt{(\log k)}))$ ...

Essence of the idea:

- Build "good" combinatorial matrix over Z_m (integers modulo m).
- Embed Z_m in multiplicative subgroup of F.
- Get locally decodable code over F.

“Good” Combinatorial matrix

$$A = \left(\begin{array}{ccccc|c} 0 & \dots & & & & \\ \dots & 0 & \dots & & & \\ \dots & & 0 & \dots & & \\ \dots & & & 0 & \dots & \\ \dots & & & & 0 & \end{array}\right)$$

arbitrary

- k x n matrix over Z_m
- Zeroes on diagonal
- Non-zero off-diagonal
- Columns closed under addition

Embedding into a field

- Let $A = [a_{ij}]$ be good over Z_m .
- Let $\omega \in F$ be primitive m^{th} root of unity.
- Let $G = [\ \omega^{a_{ij}}\ ]$.
- Thm [Y, R, E]:
  G generates an m query LDC over F !!!

Highly non-intuitive!

Improvements

- Let $A = [a_{ij}]$ be good; Let $G = [\ \omega^{a_{ij}}\ ]$.
- Off-diagonal entries of A from S

  $\Rightarrow$ code is (|S|+1)-locally decodable.

  (suffices for [Efremenko]).
- ω^S roots of t-sparse polynomial

  $\Rightarrow$ code is t-locally decodable.

  (critical for [Yekhanin]).

"Good" Matrices?

- [Yekhanin]:
  - Picked m prime.
  - Hand-constructed matrix.
  - Achieved $n = \exp(k^{(1/|S|)})$
  - Optimal if m prime!
  - Managed to make S large (10^6) with $t=3$.
- [Efremenko]
  - m composite!
  - Achieves $|S| = 3$ and $n = \exp(\exp(\sqrt{(\log k)}))$
    ([Beigel,Barrington,Rudich];[Grolmusz])
  - Optimal?

Limits to Local Decodability: Katz-Trevisan

- q queries $\Rightarrow$ $n = k^{1 + \Omega(1/q)}$
- Technique:
  - Recall D(x) computes C(x) whp for all x.
  - Can assume (with some modifications) that query pattern uniform for any fixed x.
  - Can find many random strings such that their query sets are disjoint.
  - In such case, random subset of $n^{1-1/q}$ coordinates of codeword contain at least one query set, for most x.
  - Yields desired bound.

Some general results

- Sparse, High-Distance Codes:
  - Are Locally Decodable and Testable
    - [KaufmanLitsyn, KaufmanS]

- 2-transitive codes of small dual-distance:
  - Are Locally Decodable
    - [Alon,Kaufman,Krivelevich,Litsyn,Ron]

- Linear-invariant codes of small dual-distance:
  - Are also Locally Testable
    - [KaufmanS]

Summary

- Local algorithms in error-detection/correction lead to interesting new questions.

- Non-trivial progress so far.

- Limits largely unknown
  - O(1)-query LDCs must have Rate(C) = 0
    - [Katz-Trevisan]

Questions

- Can LTC replace RS (on your hard disks)?
  - Lower bounds?
  - Better error models?

- Simple/General near optimal constructions?
- Other applications to mathematics/computation? (PCPs necessary/sufficient)?
- Lower bounds for LDCs?/Better constructions?

Thank You!