Locally Decodable Codes from Lifting

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Error-Correcting Codes

- (Linear) Code $C \subseteq \mathbb{F}_q^n$.
  - $n \stackrel{\text{def}}{=}$ block length
  - $k = \dim(C) \stackrel{\text{def}}{=}$ message length
  - $R(C) \stackrel{\text{def}}{=} k/n$: Rate of C (want as high as possible)
  - $\delta(C) \stackrel{\text{def}}{=} \min_{x \neq y \in C} \{\delta(x, y) \stackrel{\text{def}}{=} \Pr_i [x_i \neq y_i]\}$.
- Basic Algorithmic Tasks
  - Encoding: map message in $\mathbb{F}_q^k$ to codeword.
  - Testing: Decide if $x \in C$
  - Correcting: If $x \notin C$, find nearest $y \in C$ to x.

Locality in Algorithms

- "Sublinear" time algorithms:
  - Algorithms that run in time o(input), o(output).
  - Assume random access to input
  - Provide random access to output
  - Typically probabilistic; allowed to compute output on approximation to input.
- LTCs: Codes that have sublinear time testers.
  - Decide if $x \in C$ probabilistically.
  - Allowed to accept x if $\delta(x, C)$ small.
- LCCs: Codes that have sublinear time correctors.
  - If $\delta(x, C)$ is small, compute y_i , for $y \in C$ closest to x.

LTCs and LCCs: Formally

- C is a (ℓ, ϵ)-LTC if there exists a tester that
  - Makes $\ell(n)$ queries to x.
  - Accept $x \in C$ w.p. 1
  - Reject x w.p. at least $\epsilon \cdot \delta(x, C)$.
- C is a (ℓ, ϵ)-LCC if exists decoder D s.t.
  - Given oracle access x close to $y \in C$, and i
  - Decoder makes $\ell(n)$ queries to x.
  - Decoder $D^x(i)$ usually outputs y_i.
    - $\Pr_i[D^x(i) \neq y_i] \leq \delta(x, y)/\epsilon$
- Often: ignore ϵ and focus on ℓ

Example: Multivariate Polynomials

- Message = multivariate polynomial; encoding = evaluations everywhere.
  - $\mathrm{RM}[m, d, q] \stackrel{\text{def}}{=}$
    $$\{\langle f(\alpha)\rangle_{\alpha \in \mathbb{F}_q^m} \mid f \in \mathbb{F}_{\mathrm{q}}[x_1, \dots, x_m], \deg(f) \le d\}$$
- Locality?
  - Restrictions of low-degree polynomials to lines yield low-degree (univariate) polynomials.
  - Random lines sample $\mathbb{F}_q^m$ uniformly (pairwise independently).

LDCs and LTCs from Polynomials

- Decoding ($d \le q$):
  - Problem: Given $f \approx p$, $\alpha \in \mathbb{F}_q^m$, compute $p(\alpha)$.
  - Pick random β and consider $f|_L$ where $\mathrm{L} = \{\alpha + t\beta \mid t \in \mathbb{F}_q\}$ is a random line $\ni \alpha$.
  - Find univ. poly $h \approx f|_L$ and output $h(\alpha)$
- Testing ($d \le q$):
  - Verify $\deg(f|_L) \le d$.
- Parameters:
  - $n = q^m; \ell = q = n^{\frac{1}{m}}; R(C) \approx \left(\frac{1}{m}\right)^m$

Decoding Polynomials

- $d < q$
  - Correct more errors (possibly list-decode)
  - can correct $\approx 1 - \sqrt{d/q}$ fraction errors [STV].
- $d > q$
  - Distance of code $\delta \approx q^{-d/(q-1)}$
  - Decode by projecting to $\approx \frac{d}{q-1}$ dimensions. “decoding dimension”.
  - Locality $\approx 1/\delta$.
  - Lots of work to decode from $\approx \delta$ fraction errors [GKZ,G].
  - Open when $q = d = 3$ [Gopalan].

Testing Polynomials

- $d \ll q$:
  - Even slight advantage on test implies correlation with polynomial.[RS, AS]
- $d > q$:
  - Testing dimension $t = \frac{d}{q - \frac{q}{p}}$; where $q = p^s$;
  - Project to t dimensions and test.
  - $\left(q^t, \min\left\{\epsilon_q, q^{-2t}\right\}\right)$-LTC.

Testing vs. Decoding dimensions

- Why is decoding dimension $d/(q-1)$?
  - Every function on fewer variables is a degree d polynomial. So clearly need at least this many dimensions.
- Why is testing dimension $d/(q-q/p)$?
  - Consider $q = 2^s$ and $f = x^{\frac{q}{2}} y^{\frac{q}{2}}$.
  - On line $y = ax + b$,
  - $f = x^{\frac{q}{2}} (ax+b)^{\frac{q}{2}} = x^{\frac{q}{2}} \left(a^{\frac{q}{2}} x^{\frac{q}{2}} + b^{\frac{q}{2}} \right) = a^{\frac{q}{2}} x + b^{\frac{q}{2}} x^{\frac{q}{2}}$.
  - So $\deg(f) = q$, but f has degree $\leq \frac{q}{2}$ on every line!
  - In general if $q = p^s$ then powers of p pass through (…)
  - Aside: Using more than testing dimension has not paid dividend with one exception [RazSafra]

Other LTCs and LDCs

- Composition of codes yields better LTCs.
  - Reduces $\ell(\cdot)$ (to even 3) without too much loss in $R(C)$.
  - But till recently, $R(C) \leq \frac{1}{2}$
- LDCs
  - Till 2006, multivariate polynomials almost best known.
  - 2007+ [Yekhanin, Raghavendra, Efremenko] – great improvements for $\ell(n) = O(1); n = \text{ superpoly}(k)$.
  - 2010 [KoppartySarafYekhanin] Multiplicity codes get $R(C) \to 1$ with $\ell(n) = n^{\epsilon}$
  - For $\ell(n) = \log n$; multiv. Polys are still best known.

Today

- New Locally Correctible and Testable Codes from "Lifting".
  - $R(C) \to 1; \ell(n) = n^{\epsilon}$ for arbitrary $\epsilon > 0$.
  - First "LCCs + LTCs" to achieve this.
  - Only the second "LCCs" with this property
    - After Multiplicity codes [KoppartySarafYekhanin]

The codes

- Alphabet: $\mathbb{F}_q$
- Coordinates: $\mathbb{F}_q^m$
- Parameter: degree d
- Message space:

  $\{f: \mathbb{F}_q^m \to \mathbb{F}_q \mid \deg(f|_L) \leq d, \forall \text{ lines } L\}$
- Code: Evaluations of message on all of $\mathbb{F}_q^m$
- And oh … $q = 2^s; d = (1 - \epsilon)q; m = O(1)$

Recall: Bad news about $\mathbb{F}_{2^s}$

- Functions that look like degree d polynomials on every line $\neq$ degree d m-variate polynomials.
- But this is good news!
  - Message space includes all degree d polynomials.
  - And has more.
  - So rate is higher!
  - But does this make a quantitative difference?
    - As we will see … **YES!** Most of the dimension comes from the ``illegitimate'' functions.

Generalizing: Lifted Codes

- Consider $B \subseteq \{\mathbb{F}_Q^t \to \mathbb{F}_q\}$.
  - $\mathbb{F}_Q$ extends $\mathbb{F}_q$
  - Preferably B invariant under affine transformations of $\mathbb{F}_Q^t$.
- Lifted code $C \stackrel{\text{def}}{=} \text{Lift\_}m(B) \subseteq \left\{\mathbb{F}_Q^m \to \mathbb{F}_q\right\}$
  - $C = \{f \mid f|_A \in B, \forall\, t$-dim. affine subspaces $A\}$.
- Previous example:
  - $B = \left\{f : \mathbb{F}_q \to \mathbb{F}_q \,\middle|\, \deg(f) \leq d\right\}$

Properties of lifted codes

- Distance:
  - $\delta(C) \geq \delta(B) - Q^{-t} + \mathrm{Q}^{-m} \approx \delta(B)$
- Local Decodability:
  - Same decoding algorithm as for RM codes.
  - B is (ℓ, ϵ)-LDC implies C is $(\ell, \Omega(\epsilon))$-LDC.
- Local Testability?

Local Testability of lifted codes

- Local Testability:
  - Test: Pick A and verify $f|_A \in B$.
  - "Single-orbit characterization": (Q^t, Q^{-2t})-LTC [KS]
  - (Better?) analysis for lifted tests: (Q^t, ϵ_Q)-LTC [HRS] (extends [BKSSZ,HSS])
- Musings:
  - Analyses not robust (test can't accept if $f|_A \approx B$.)
  - Still: generalizes almost all known tests … [Main exceptions – [ALMSS,PS,RS,AS]].
  - Key question: what is min K s.t. $f|_{A_1}, \ldots, f|_{A_K} \in B \Rightarrow$ there exists an interpolator $g \in C$ s.t. $g|_{A_i} = f|_{A_i}$

Returning to (our) lifted codes

- Distance ✓
- Local Decodability ✓
- Local Testability ✓
- Rate?
  - No generic analysis; has to be done on case by case basis.
  - Just have to figure out which monomials are in C.

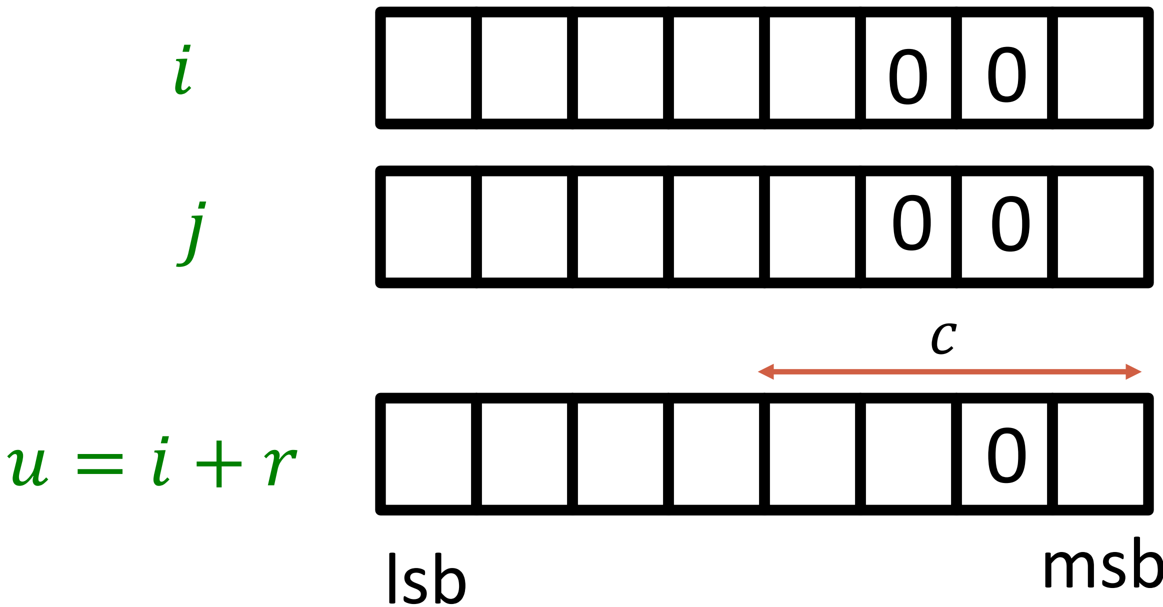
Rate of bivariate Lifted RS codes

- $B = \{f \in \mathbb{F}_q[x] \mid \deg(f) \leq d = (1-\epsilon)q\}$; $q = 2^s$
  - Will set $\epsilon = 2^{-c}$ and let $c \to \infty$.
- $C = \{f: \mathbb{F}_q[x,y] \mid f|_{y=ax+b} \in B, \forall a, b\}$
  - When is $x^i y^j \in C$?
  - Clearly if $i + j \leq d$; But that is at most $\frac{q^2}{2}$ pairs.
  - Want $\approx \frac{q^2}{2}$ more such pairs.
  - When is every term of $x^i(ax+b)^j \mathrm{mod}(x^q - x)$ of degree at most d?

Lucas's theorem & Rate

- Notation: $r \leq_2 j$, if $r = \sum_i r_i 2^i$ and $j = \sum_i j_i 2^i$ ($r_i, j_i \in \{0,1\}$) and $r_i \leq j_i$ for all i.
- Lucas's Theorem: $x^r \in \operatorname{supp}\left((ax+b)^j\right)$ iff $r \leq_2 j$.
- $\Rightarrow \operatorname{supp}\left(x^i(ax+b)^j\right) \ni x^{i+r}$ iff $r \leq_2 j$
- So given i, j; $\exists r \leq_2 j$ s. t. $i + r \pmod{' q} > d$?

Binary addition etc.



$$i_{k-1} = 0 \ \& \ i_k = 0 \ \& \ j_{k-1} = 0 \ \& \ j_k = 0 \Rightarrow u_k = 0$$

$$u_k = 0 \Rightarrow u \leq d$$

$$\Pr_{i,j}[i_{\{k-1\}} \dots j_k \neq 0000] \leq 15/16$$

$$\Pr_{i,j}[i + r \ (\mathrm{mod}' q) > d] \leq \left(\frac{15}{16}\right)^{\frac{c}{2}} \rightarrow 0 \text{ as } c \rightarrow \infty$$

Other lifted codes

- Best LCC with O(1) locality.
  - $B = \{f : \mathbb{F}_{2^s} \to \mathbb{F}_2 \mid \sum_a f(a) = 0\};$
  - $s = \log_2 \ell = O(1)$
  - $C = \mathrm{Lift}_m(B);$
  - $n = 2^{sm}$; ℓ-LCC; $\dim(C) = (\log n)^{\ell}$
- Alternate codes for BGHMRS construction:
  - $B = \left\{f : \mathbb{F}_4^{m - \log 1/\epsilon} \to \mathbb{F}_2 \middle| \sum_a f(a) = 0\right\}$
  - $C = \mathrm{Lift}_m(B);$
  - $\ell = \epsilon n$; $\dim(C) = n - \mathrm{polylog}(n)$

Nikodym Sets

- $N \subseteq \mathbb{F}_q^m$ is a Nikodym set if it almost contains a line through every point:
  - $\forall a \in \mathbb{F}_q^m, \exists\, b \in \mathbb{F}_q^m$ s.t. $\{a + tb \mid t \in \mathbb{F}_q\} \subseteq N \cup \{a\}$
- Similar to Kakeya Set (which contain line in every direction).
  - $\forall\, b \in \mathbb{F}_q^m, \exists\, a \in \mathbb{F}_q^m$ s.t. $\{a + tb \mid t \in \mathbb{F}_q\} \subseteq K$
- [Dvir], [DKSS]: $|K|, |N| \geq \left(\frac{q}{2}\right)^m$

Proof ("Polynomial Method")

- Find low-degree poly $P \neq 0$ s.t. $P(b) = 0, \forall\, b \in N$.
- $\deg(P) < q - 1$ provided $|N| < \binom{m+q-2}{m}$.
- But now $P|_{L_a} = 0, \forall$ Nikodym lines $L_a \Rightarrow P(a) = 0\ \forall a,$
  contradicting $P \neq 0$.
- Conclude $|N| \geq \binom{m+q-2}{m} \approx \frac{q^m}{m!}$.
- Multiplicities, more work, yields $|N| \geq \left(\frac{q}{2}\right)^m$.
- But what do we really need from P?
  - P comes from a large dimensional vector space.
  - $P|_L$ is low-degree!
  - Using P from lifted code yields $|N| \geq \big(1 - o(1)\big)q^m$ (provided q of small characteristic).

Conclusions

- Lifted codes seem to extend "low-degree polynomials" nicely:
  - Most locality features remain same.
  - Rest are open problems.
  - Lead to new codes.
- More generally: Affine-invariant codes worth exploring.
  - Can we improve on multiv. poly in polylog locality regime?

Thank You