Communication amid Uncertainty

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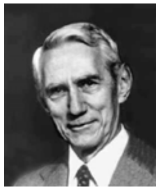
Based on:

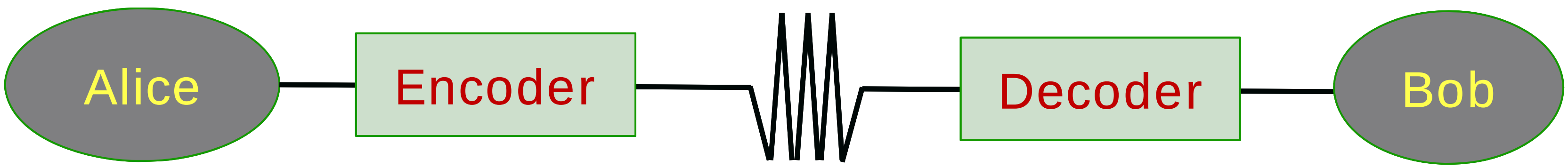
- -Universal Semantic Communication – Juba & S. (STOC 2008)
- -Goal-Oriented Communication – Goldreich, Juba & S. (JACM 2012)
- -Compression without a common prior … – Kalai, Khanna, Juba & S. (ICS 2011)
- -Efficient Semantic Communication with Compatible Beliefs – Juba & S. (ICS 2011)
- -Deterministic Compression with uncertain priors – Haramaty & S. (ITCS 2014)

Classical theory of communication



Shannon (1948)

- Clean architecture for reliable communication.



- Remarkable mathematical discoveries: Prob. Method, Entropy, (Mutual) Information
- Needs reliable encoder + decoder (two reliable computers).

Uncertainty in Communication?

- Always has been a central problem:
  - But usually focusses on uncertainty introduced by the channel
  - Standard Solution:
    - Use error-correcting codes
    - Significantly:
      - Design Encoder/Decoder jointly
      - Deploy Encoder at Sender, Decoder at Receiver

New Era, New Challenges:

- Interacting entities not jointly designed.
  - Can’t design encoder+decoder jointly.
  - Can they be build independently?
  - Can we have a theory about such?
    - Where we prove that they will work?

    - Hopefully:
      - YES
      - And the world of practice will adopt principles.

Example 1

- Printing in a new environment
  - Say, you are visiting a new university.
  - Printer is intelligent; so is your computer;
    - Can’t they figure out how to talk to each other?
- Problem (with current designs):
  - Computers need to know about the printer already to print on them.
  - Why can’t they also figure out how future printers will work?
    - Uncertainty (about printers of the future).

Example 2

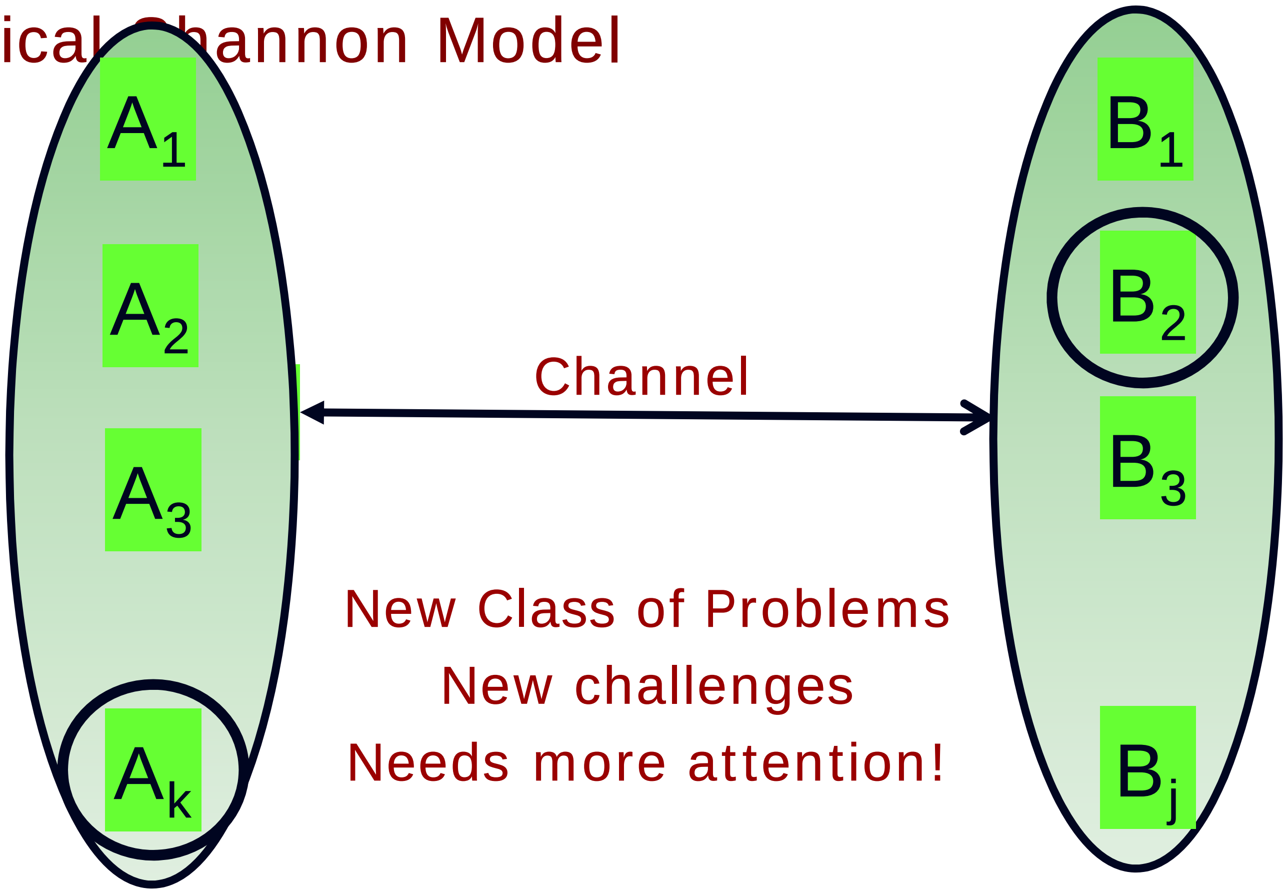
- Archiving data
  - Physical libraries have survived for 100s of years.
  - Digital books have survived for five years.
  - Can we be sure they will survive for the next five hundred?

- Problem: Uncertainty of the future.
  - What systems will prevail?
  - Why aren’t software systems ever constant?

Modelling uncertainty

Semantic Communication Model

Classical Shannon Model



New Class of Problems

New challenges

Needs more attention!

Nature of uncertainty

- A_i's, B_j's differ in beliefs, but can be centrally programmed/designed.
  - [Juba,Kalai,Khanna,S.'11] : Compression in this context has graceful degradation as beliefs diverge.
  - [Haramaty,S'13]: Role of randomness in this context.
- A_i's, B_j's differ in behavior:
  - Nothing to design any more (behavior already fixed).
  - Best hope: Can identify certain A_i's (universalists) that can interact successfully with many B_j's. Can eliminate certain B_j's on the grounds of "limited tolerance".
  - [Juba,S'08; Goldreich,J,S'12; J,S'11]: "All is not lost, if we keep goal of communication in mind"
  - [Leshno,S'13]: "Communication is a Coordination Game"
  - Details don't fit in margin …

II: Compression under uncertain beliefs/priors

Motivation

- New era of challenges needs new solutions.
  - Most old solutions do not cope well with uncertainty.
  - The one exception?
    - Natural communication (Humans ↔ Humans)
- What are the rules for human communication?
  - “Grammar/Language”
  - What kind of needs are they serving?
  - What kind of results are they getting? (out of scope)
  - If we were to design systems serving such needs, what performance could they achieve?

Role of Dictionary (/Grammar/Language)

$$\begin{aligned} M_1 &= w_{11}, w_{12}, \ldots \\ M_2 &= w_{21}, w_{22}, \ldots \\ M_3 &= w_{31}, w_{32}, \ldots \\ M_4 &= w_{41}, w_{42}, \ldots \\ &\ldots \end{aligned}$$

- Dictionary: maps words to meaning
  - Multiple words with same meaning
  - Multiple meanings to same word
- How to decide what word to use (encoding)?
- How to decide what a word means (decoding)?
  - Common answer: Context
- Really Dictionary specifies:
  - Encoding: context × meaning → word
  - Decoding: context × word → meaning
- Context implicit; encoding/decoding works even if context used not identical!

Context?

- In general complex notion …
  - What does sender know/believe
  - What does receiver know/believe
  - Modifies as conversation progresses.

- Our abstraction:
  - Context = Probability distribution on potential "meanings".
  - Certainly part of what the context provides; and sufficient abstraction to highlight the problem.

The problem

- Wish to design encoding/decoding schemes (E/D) to be used as follows:
  - Sender has distribution P on $M = \{1,2,\ldots,N\}$
  - Receiver has distribution Q on $M = \{1,2,\ldots,N\}$
  - Sender gets $X \in M$
  - Sends $E(P,X)$ to receiver.
  - Receiver receives $Y = E(P,X)$
  - Decodes to $\hat{X} = D(Q,Y)$
  - Want: $X = \hat{X}$ (provided P,Q close),
    - While minimizing $Exp_{X \leftarrow P} |E(P,X)|$

Closeness of distributions:

- P is Δ-close to Q if for all $X \in M$,

$$\frac{1}{2^{\Delta}} \leq \frac{P(X)}{Q(X)} \leq 2^{\Delta}$$

- P Δ-close to Q $\quad \Rightarrow \quad D(P||Q), D(Q||P) \leq \Delta$.

Dictionary = Shared Randomness?

- Modelling the dictionary: What should it be?

- Simplifying assumption – it is shared randomness, so …

- Assume sender and receiver have some shared randomness R and X, P, Q independent of R.
  - $Y = E(P, X, R)$
  - $\hat{X} = D(Q, Y, R)$

- Want $\forall X,\ \Pr_{R}\left[\hat{X} = X\right] \geq 1 - \epsilon$

Solution (variant of Arith. Coding)

- Use R to define sequences
  - $R_1\ [1], R_1\ [2], R_1\ [3], \ldots$
  - $R_2\ [1], R_2\ [2], R_2\ [3], \ldots$
  - …
  - $R_N\ [1], R_N\ [2], R_N\ [3], \ldots.$
- $E_\Delta(P, x, R) = R_x[1 \ldots L]$, where L chosen s.t. $\forall z \neq x$

  Either $R_z[1 \ldots L] \neq R_x[1 \ldots L]$

  Or $P(z) < \frac{P(x)}{4^\Delta}$
- $D_\Delta(Q, y, R) = \text{argmax}_{\hat{x}}\ \{Q(\hat{x})\}$ among $\hat{x} \in \{\, z \mid R_z[1 \ldots L] = y\}$

Performance

- Obviously decoding always correct.
- Easy exercise:
  - $\mathrm{Exp}_X\ [E(P,X)] = H(P) + 2\,\Delta$
- Limits:
  - No scheme can achieve $(1-\epsilon)\cdot[H(P)+\Delta]$
  - Can reduce randomness needed.

Implications

- Reflects the tension between ambiguity resolution and compression.
  - Larger the Δ ((estimated) gap in context), larger the encoding length.
  - Entropy is still a valid measure!
- Coding scheme reflects the nature of human process (extend messages till they feel unambiguous).
- The “shared randomness’’ assumption
  - A convenient starting point for discussion
  - But is dictionary independent of context?
    - This is problematic.

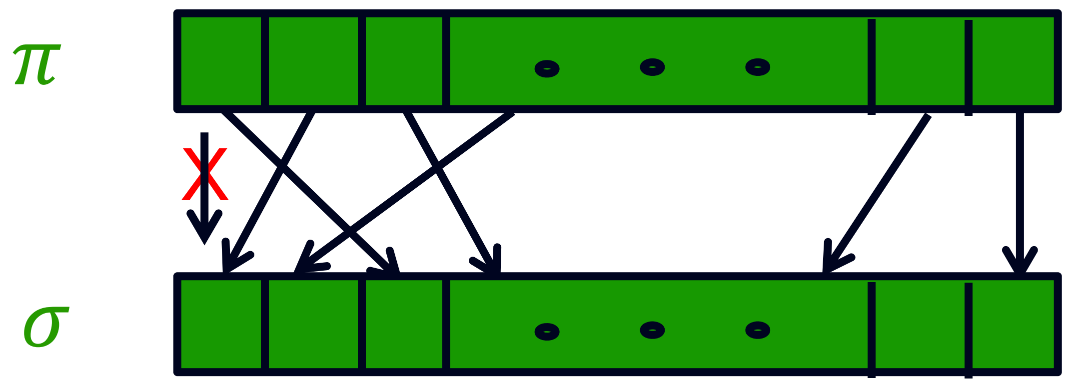
III: Deterministic Communication Amid Uncertainty

A challenging special case

- Say Alice and Bob have rankings of N players.
  - Rankings = bijections $\pi, \sigma : [N] \rightarrow [N]$
  - $\pi(i)$ = rank of i^{th} player in Alice's ranking.
- Further suppose they know rankings are close.
  - $\forall\, i \in [N]: |\pi(i) - \sigma(i)| \leq 2.$
- Bob wants to know: Is $\pi^{-1}(1) = \sigma^{-1}(1)$
- How many bits does Alice need to send (non-interactively).
  - With shared randomness – $O(1)$
  - Deterministically?
    - $O(1)$? $O(\log N)$? $O(\log\log\log N)$?

Model as a graph coloring problem

- Consider family of graphs $U_{N,\ell}$:
  - Vertices = permutations on $[N]$
  - Edges = ℓ-close permutations with distinct messages. (two potential Alices).



- Central question: What is $\chi(U_{N,\ell})$?

Main Results [w. Elad Haramaty]

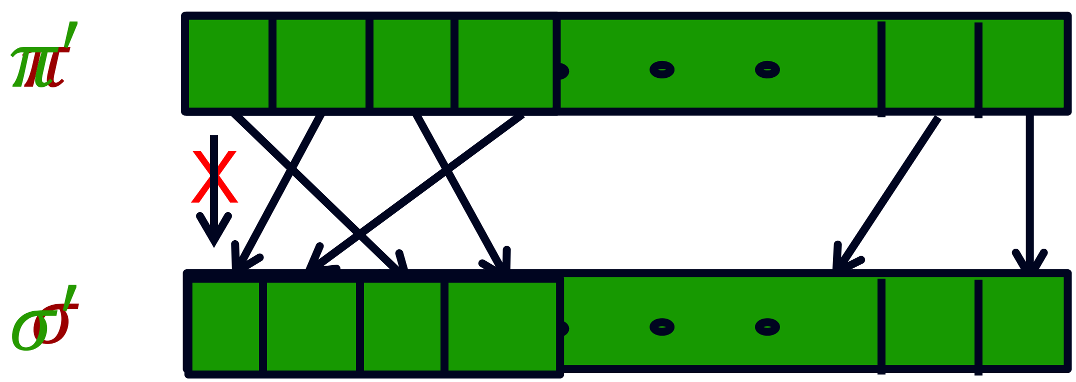
- Claim: Compression length for toy problem
  $\in \left[\log \chi\left(U_{N,2}\right), \log \chi\left(U_{N,4}\right)\right]$
- Thm 1: $\chi\left(U_{N,\ell}\right) \leq \ell^{O(\ell \log^* N)}$
  - $\log^{(i)} N \equiv \log\log \ldots N$ (i times)
  - $\log^* N \equiv \min\{i \mid \log^{(i)} N \leq 1\}$.
- Thm 2: $\exists$ uncertain comm. schemes with
  1. $\mathrm{Exp}_m[\,|E(P,m)|] \leq O(H(P) + \Delta + \log\log N)$ (0-error).
  1. $\mathrm{Exp}_m[\,|E(P,m)|] \leq \ell^{O(\epsilon^{-1}(H(P)+\Delta+\log^* N))}$ (ϵ -error).

- Rest of the talk: Graph coloring

Restricted Uncertainty Graphs

- Will look at $U_{N,\ell,k}$
  - Vertices: restrictions of permutations to first k coordinates.
  - Edges: $\pi' \leftrightarrow \sigma'$

$$\Leftrightarrow \exists\, \pi \text{ extending } \pi' \text{ and } \sigma \text{ extending } \sigma' \text{ with } \pi \leftrightarrow \sigma$$



Homomorphisms

- G homomorphic to H ($G \to H$) if
  $\exists\, \phi : V(G) \to V(H)$ s.t. $u \leftrightarrow_G v \Rightarrow \phi(u) \leftrightarrow_H \phi(v)$
- Homorphisms?
  - G is k-colorable $\Leftrightarrow G \to K_k$
  - $G \to H$ and $H \to L \Rightarrow G \to L$
- Homomorphisms and Uncertainty graphs.
  - $U_{N,\ell} = U_{N,\ell,N} \to U_{N,\ell,N-1} \to \cdots \to U_{N,\ell,\ell+1}$
- Suffices to upper bound $\chi\left(U_{N,\ell,k}\right)$

Chromatic number of $U_{N,\ell,\ell+1}$

- For $f:[N] \to [2\ell]$, Let
  $I_f = \{\pi \mid f(\pi_1) = 1,\ f(\pi_i) \neq 1,\ \forall\, i \in [2\ell] - \{1\}\}$

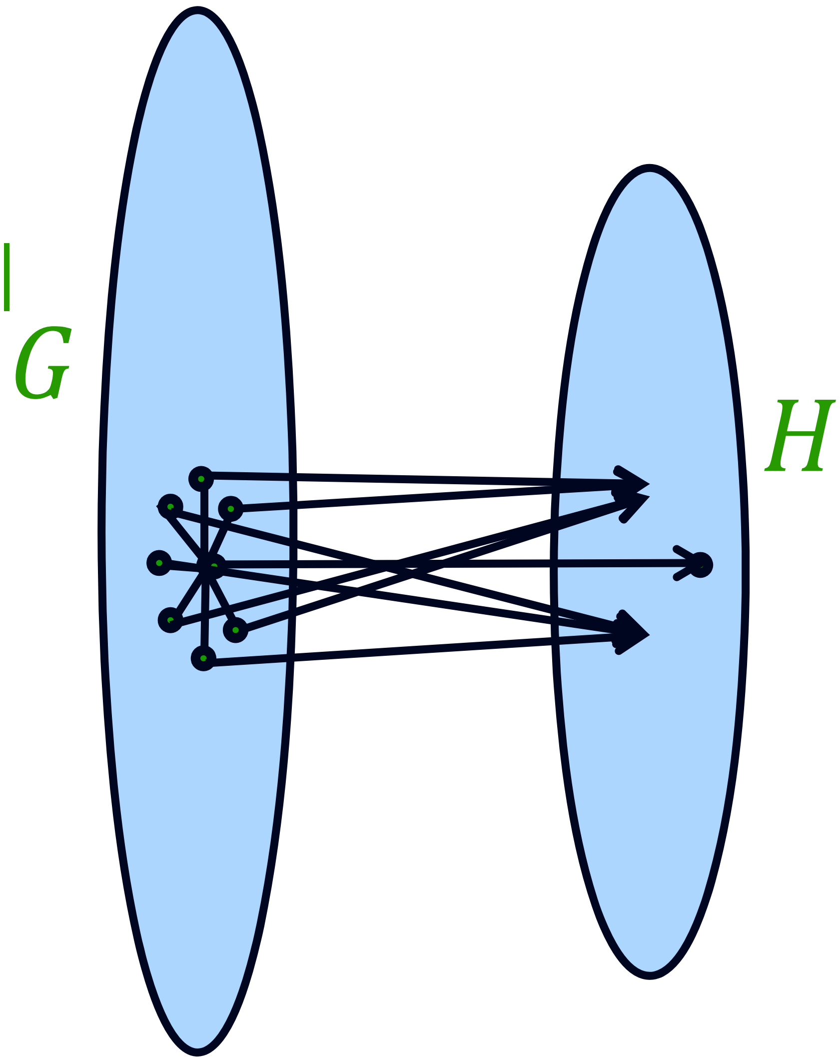
- Claim: $\forall f, I_f$ is an independent set of $U_{N,\ell,\ell+1}$

- Claim: $\forall \pi,\ \Pr_f\left[\pi \in I_f\right] \geq \frac{1}{4\ell}$

- Corollary: $\chi\left(U_{N,\ell,\ell+1}\right) \leq O(\ell^2 \log N)$

Better upper bounds:

- Say $\phi : G \rightarrow H$
- $d_\phi(u) \equiv |\{\ \phi(v) \mid v \leftrightarrow_G u\ \}|$
  $$d_\phi \equiv \max_u \{d_\phi(u)\}$$
- Lemma:
  $$\chi(G) \leq O(d_\phi^2 \log \chi(H))$$
- For $\phi_k : U_{N,\ell,k} \rightarrow U_{N,\ell,k-\ell}$
  $$d_{\phi_k} = \ell^{O(k)}$$



Better upper bounds:

- $d_\phi \equiv \max_u \{ |\{\phi(v) | v \leftrightarrow_G u\}| \}$
- Lemma: $\chi(G) \leq O\left(d_\phi^2 \log \chi(H)\right)$
- For $\phi_k : U_{N,\ell,k} \to U_{N,\ell,k-\ell}, \quad d_{\phi_k} \leq \ell^{O(k)}$
- Corollary: $\chi\left(U_{N,\ell,k}\right) \leq \ell^{O(k)} \log^{\left(\frac{k}{\ell}\right)} N$

- Aside: Can show: $\chi\left(U_{N,\ell,k}\right) \geq \log^{\Omega\left(\frac{k}{\ell}\right)} N$
  - Implies can't expect simple derandomization of the randomized compression scheme.

Future work?

- Open Questions:
  - Is $\chi(U_{N,\ell}) = O_\ell(1)$?
  - Can we compress arbitrary distributions to $O(H(P) + \Delta)$? $O(H(P) + \Delta + \log^* N)$? or even $O(H(P) + \Delta + \log\log\log N)$?
- On conceptual side:
  - Better understanding of forces on language.
    - Information-theoretic
    - Computational
    - Evolutionary
    - Game-theoretic
- Design better communication solutions!

Thank You