Tetramonohedron Development with Minimum Cut Length

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1 Introduction

In this paper, we investigate the following problem. You are given a large sheet of material, from which you would like to make n packages of the same shape Q. The sheet is tough so cutting is expensive (such as with a waterjet), so you would like to minimize the total length of cut. To minimize the total length of cut and to reduce waste of material, we focus on *tiling* shapes P. The sheet is large and n is huge, so the boundary shape of the sheet itself is insignificant. What is the best shape of P?

Minimizing the length of cut when we develop a given polyhedron is a natural question. Surprisingly, however, there is little research on this topic. The only results we know are by Akiyama et al. [1], who investigate the minimum length of cutting to develop each of five regular polyhedra.

In this paper, we focus on the case that P folds into a *tetramonohedron* Q, that is, a tetrahedron made from four congruent triangles. It is known that any development of a tetramonohedron tiles the plane [2]. Therefore, we can focus on minimizing the cut length without worrying about whether the unfolding P will tile. In this paper, we generalize the result for a regular tetrahedron from [1] to a family of tetramonohedra.

Precisely, our problem is formalized as follows: tile an infinite sheet with a single polygon P, while minimizing the perimeter of P, such that P folds into a target tetramonohedron Q, As a secondary goal, we are interested in maximizing the volume of T, as motivated by useful packaging. When we only focus on the volume, T has the maximum volume if it is a regular tetrahedron. On the other hand, when we focus only on the minimum length of cut, P is a regular hexagon, and then the folded tetramonohedron Q is in fact a doubly covered rectangle; i.e., the package has volume 0. Hence, there is a trade-off between volume and cut length.

We investigate this trade-off for tetramonohedra whose faces are isosceles triangles. When we focus on these target polyhedra, four isosceles triangles of size $1 : \sqrt{14}/2 : \sqrt{14}/2 = 1 : 1.87\cdots : 1.87\cdots$ form a reasonable solution from the viewpoints of volume and cut-length.

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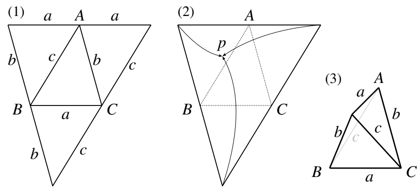

Figure 1: Folding a tetramonohedron by four copies of a unit triangle ABC.

2 Preliminaries

The vertices of a triangle T are A, B, and C, and let a, b, c be the lengths of three edges of T. The angles $\angle A$, $\angle B$, and $\angle C$ denote the corresponding angles at the vertices A, B, and C. We assume that the triangle inequality always holds.

Based on this triangle T, we may fold a tetramonohedron that consists of four copies of T as shown in Figure 1.

We first point out that this construction works only for acute triangles:

Theorem 1 *Let T be a unit-area triangle with three edge lengths $a \leq b \leq c$. Then, by the procedure above, four copies of T (1) form a tetramonohedron if T is an acute triangle, (2) form a doubly covered rectangle if T is a right triangle, and (3) do not form any polyhedron otherwise.*

Hence, we consider only the case that the unit-area triangle T is an acute triangle.

Let Q be any tetramonohedron folded from a triangle T. Let P be the net of Q obtained by minimum total cut length. Applying the analysis in [1], we can observe that the minimum cut lines form a Steiner tree on the surface of Q that spans the four vertices of Q. This spanning tree as follows; refer to Figure 2(a). Take some two triangular faces of Q. Their edge-length set $\{w, x, y\}$ is the set $\{a/2, b/2, c/2\}$. Then two points p and p' are the Fermat point, and the point q is at the center of the edge of length w. When we cut along the Steiner tree, we obtain a net as shown in Figure 2(b). When Q is a tetramonohedron, it is easy to see that the

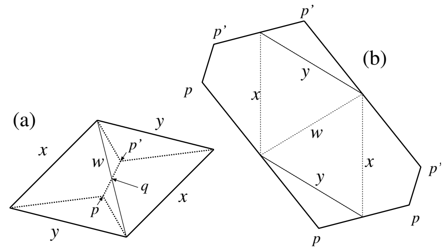

Figure 2: Minimum-cut Steiner tree, and resulting hexagon.

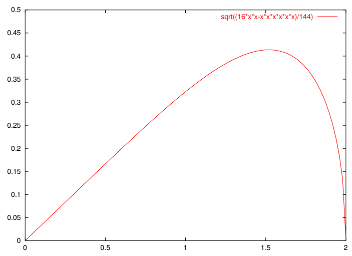

Figure 3: Volume

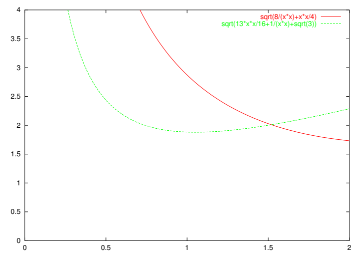

Figure 4: Minimum cut length

resulting net P is a hexagon such that (1) each angle is 120°, and (2) two parallel edges are of the same length.

3 Minimum Cut of Isosceles Tetramonohedron

Let Q be any tetramonohedron of surface area 4 that consists of four congruent isosceles unit triangles. That is, Q can be specified by two parameters a, b such that a unit triangle made by an edge of length a and two edges of length b. Because the triangle is acute, we have b/a is in $(1/\sqrt{2}, \infty)$. Based on case analysis, we obtain the following theorem:

Theorem 2 *Let Q be a tetramonohedron of surface area 4 that consists of four congruent isosceles unit triangles. Let a be the length of the base of the isosceles unit triangle. Then the volume of Q is given by $\frac{16a^2-a^6}{144}$, and the minimum cut length is given by* $\min\{\sqrt{\frac{8}{a^2}+\frac{a^2}{4}}, \sqrt{\frac{13a^2}{16}+\frac{1}{a^2}+\sqrt{3}}\}$.

Note that Q is a regular tetrahedron when $a = (16/3)^{1/4} = 1.51967\cdots$. Figures 3 and 4 give the volumes and the minimum cut lengths, respectively, for $0 \le a \le 2$. The volume takes its maximum, and the cutting-length function changes terms, when Q is a regular tetrahedron. The cutting length takes minimum value when $a = (16/13)^{1/4} = 1.053\cdots$ and $b/a = \sqrt{14}/2 = 1.87\cdots$.

4 Conclusion

Another interesting open problem is cutting unit cubes. It is known that each of the eleven edge unfoldings of a unit cube forms a tiling. Unfortunately, the development with the shortest cut length in [1] does not form a tiling (Figure 5(a)). When we pack its copies as in Figure 5(b), (1) two unit edges (top and bottom) are shared, and (2) four parts of edges (left and right) are shared. In case (2), the length of each shared part is at most $\frac{\sqrt{3}-1}{2}$ by simple geometric analysis. Therefore, for sufficiently large n, the total length of cut per cube is $(1+1)/2+16\times\ldots = 6+3\sqrt{3} = 11.196\ldots$. We conjecture that the development shown in Figure 6(a) is the best development in our criteria; in the tiling of this pattern (Figure 6(b)), the total length of cut per cube is $(8\sqrt{2}+2)/2 = 6.656\ldots$.

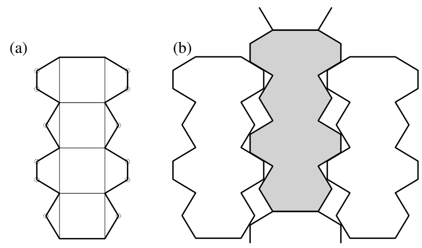

Figure 5: (a) A net of a cube with minimum perimeter in [1]. It is not a tiling, and one possible efficient packing is shown in (b).

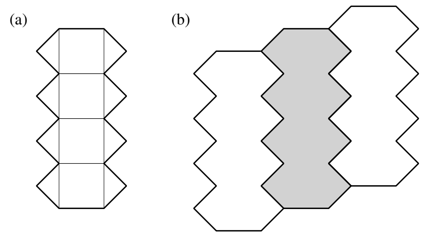

Figure 6: (a) A net of a cube, and (b) its tiling.

Conjecture 3 *When we make many unit cubes from a large sheet, the tiling pattern Figure 5(b) is the best way since (a) it has no waste, and (b) the total cutting is the shortest.*

References

[1] J. Akiyama, X. Chen, G. Nakamura, and M.-J. Ruiz. Minimum Perimeter Developments of the Platonic Solids. *Thai Journal of Mathematics*, 9(3):461–487, 2011.

[2] J. Akiyama and C. Nara. Developments of Polyhedra Using Oblique Coordinates. *J. Indones. Math. Soc.*, 13(1):99–114, 2007.