Wrapping Spheres with Flat Paper*

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Abstract

We study wrappings of smooth (convex) surfaces by a flat piece of paper or foil. Such wrappings differ from standard mathematical origami because they require infinitely many infinitesimally small folds ("crumpling") in order to transform the flat sheet into a surface of nonzero curvature. Our goal is to find shapes that wrap a given surface, have small area and small perimeter (for efficient material usage), and tile the plane (for efficient mass production). Our results focus on the case of wrapping a sphere. We characterize the smallest square that wraps the unit sphere, show that a 0.1% smaller equilateral triangle suffices, and find a 20% smaller shape contained in the equilateral triangle that still tiles the plane and has small perimeter.

Keywords: folding, contractive mapping, sphere, Mozartkugel

1 Introduction

Traditional mathematical origami [DD01, DDMO04, DO07, Hul01] considers folding a flat polygon of paper along a finite collection of creases into a flat or three-dimensional origami. The resulting surface necessarily has zero (intrinsic) Gaussian curvature at every point—that is, every point of paper has a neighborhood that is isometric to a disk—because such folding preserves the intrinsic metric of the paper. If we imagine coalescing overlapping layers of paper, then the origami can have vertices of nonzero curvature at crease vertices, but only finitely many. (A simple example of this possibility is folding an origami box.) The remaining points of zero curvature need not be (extrinsically) flat—they can curve along ruled surfaces—and even the creases themselves can curve [DO07, chapter 20]. But it is impossible, for example, to fold a sphere, where every point has nonzero curvature.

This paper studies a kind of folding that does not preserve the intrinsic metric and can change the Gaussian curvature at all points. In particular, it becomes possible to fold a sphere from a square. This type of folding is perhaps best described physically as wrapping with foil. Indeed, our original motivation for this type of folding (detailed below) is the wrapping of spherical confectioneries by foil. Confectioners foil is relatively easy to "crinkle", i.e., infinitesimally wrinkle or

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crumple around the desired surface. Crinkling enables the effective formation of positive curvature throughout the wrapping.

We model the intuitive notion of crinkling with the mathematical notion of a *contractive mapping*. We still prevent the material from stretching, but allow it to shrink arbitrarily; in other words, the intrinsic metric can be contracted but not expanded by the wrapping. A theorem of Burago and Zalgaller [BZ96] justifies this definition: it shows, roughly, that such a contractive wrapping of a smooth surface can be approximated arbitrarily closely (in an extrinsic sense) by infinitesimal wiggling of a metric-preserving folding. We formalize these notions and connections in Section 2. This model opens up the study of wrapping a smooth convex surface by a flat shape.

We consider three objectives in such wrappings: minimizing area, minimizing perimeter, and the shape tiling the plane. Minimizing area naturally minimizes the material usage. Minimizing perimeter results in a minimum amount of cutting from a sheet of material. Minimizing perimeter is also good if we take the Minkowski sum of the shape with an ε-radius disk, which increases the area by roughly ε times the perimeter for small ε. This provides a simple approach for ensuring an ε overlap where the boundary of the shape meets itself (which the crinkling then locks together). Minimizing perimeter relative to area also encourages a relatively fat shape; a large enclosed disk also enables the accurate presentation of an image on the wrapping. Finally, the shape tiling the plane enables efficient use of large sheets of material when the wrapping is mass-produced.

Minimization of both area and perimeter, with or without the tiling constraint, is a bicriterion optimization problem. An ideal solution is a full characterization of the *Pareto curve*, that is, the minimum area possible for each perimeter and vice versa. Just minimizing the area is not interesting: starting from an arbitrarily thin rectangular strip, it is possible to wrap any surface using material arbitrarily close to the surface area, by a modification of the method in [DDM00]. Minimizing perimeter alone remains an interesting open problem, however.

In this paper, we focus on wrappings of the simplest smooth convex surface, the unit sphere. We study some of the most natural wrappings based on the idea of "petals", and analyze their performance. Figure 1 plots the performance of some of these wrappings in the area–perimeter plane; the lower envelope is our best approximation of the Pareto curve. Our results in particular characterize the smallest square that wraps the unit sphere, and reveal a wrapping equilateral triangle of smaller area than this square. These shapes are of particular interest because they tile the plane and are simple. We also discover several more sophisticated wrappings that are better in some of the metrics.

Motivation. The Mozartkugel ("Mozart sphere") [Wika, Wikb] is a famous fine Austrian confectionery: a sphere with marzipan in its core, encased in nougat or praline cream, and coated with dark chocolate. It was invented in 1890 by Paul Fürst in Salzburg (where Wolfgang Amadeus Mozart was born), six years after he founded his confectionery company, Fürst. Fürst (the company) still to this day makes Mozartkugeln by hand, about 1.4 million per year, under the name "Original Salzburger Mozartkugel" [Für]. At the 1905 Paris Exhibition, Paul Fürst received a gold medal for the Mozartkugel.

Many other companies now make similar Mozartkugeln, but Mirabell is the market leader with their "Echte (Genuine) Salzburger Mozartkugeln" [Mir]. Mirabell has made over 1.5 billion, about 90 million per year, originally by hand but now by industrial methods. Mirabell claims their product to be the only Mozartkugel that is perfectly spherical. They are also the only Mozartkugel to have been taken into outer space, by the first Austrian astronaut Franz Viehböck as a gift to the Russian cosmonauts on the MIR space station. Despite industrial techniques, each Mozartkugel still takes about 2.5 hours to make.

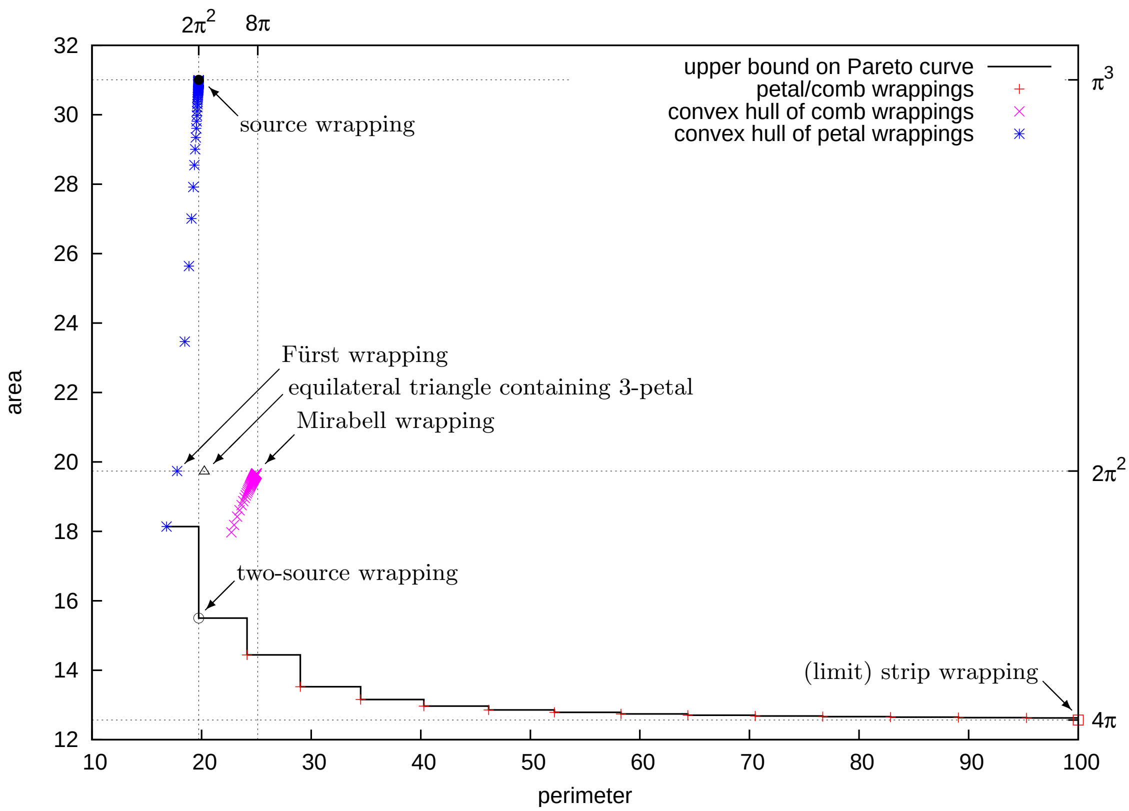


Figure 1: Area–perimeter plot of the families of sphere wrappings we discuss.

Although most of a Mozartkugel is edible, each sphere is individually wrapped in a square of aluminum foil. To minimize the amount of this wasted, inedible material, it is natural to study the smallest piece of foil that can wrap a unit sphere. Because the pieces will be cut from a large sheet of foil, we would also like the unfolded shape to tile the plane. All studied confectionery wrappings also expand their boundaries slightly, allowing the edges to overlap and thus safely securing the chocolate even in an approximate folding. Minimizing the perimeter of the shape will approximately minimize the material wastage from this process.

We begin our exploration with the actual wrappings by Fürst and Mirabell. Our extensive experiments throughout 2007 have revealed that each company wraps their Mozartkugeln consistently, but the companies differ. Ignoring the slight overlap, Fürst uses a square of side length $\pi\sqrt{2}$, while Mirabell uses a $\pi \times 2\pi$ rectangle. It may seem surprising that these two different wrappings have identical area, but we show that this fact stems from a certain bijection between the shapes' underlying "petals". Fürst's wrapping, however, is clearly superior in that it has smaller perimeter. On the other hand, our equilateral-triangle wrapping is simple enough that it might be considered by either company. Its area savings of 0.1% may prove significant on the many millions of Mozartkugel consumed each year. Even better, some portions of this shape can be cut away while still wrapping the sphere and tiling the plane, achieving a 20% savings.[1]

[1]In addition to direct savings in material costs for Mozartkugel manufacturers, the reduced material usage also indirectly cuts down on CO_2 emissions, and therefore partially solves the global-warming problem and consequently the little-reported but equally important chocolate-melting problem.

2 Wrapping in General

In standard mathematical origami [DDMO04, DO07], a *piece of paper* P is a two-dimensional manifold, typically a flat polygon, and a *folding* $f : P \to \mathbb{R}^3$ is a noncrossing isometric mapping of this piece of paper into Euclidean 3-space. The *isometric* constraint means that the mapping preserves the intrinsic metric, i.e., the distances as measured by shortest paths on the piece of paper before and after mapping via the folding.

But there is no isometric folding of a square into a sphere, or any surface with infinitely many points of nonzero Gaussian curvature. Therefore we define a new, less restrictive type of folding that allows changing curvature but still prevents stretching of the material.

Definition of wrapping. A *wrapping* is a noncrossing contractive mapping of a piece of paper into Euclidean 3-space. The *contractive* constraint means that every distance either decreases or stays the same, as measured by shortest paths on the piece of paper before and after mapping via the folding. Contractive mappings are called *short* or *contracting* by Burago and Zalgaller [BZ96] and *submetry maps* by Pak [Pak06].[2] Here we do not specify the precise noncrossing constraint, because the details are involved even for isometric foldings [DDMO04, DO07]. For our purposes, we can use a simple but strong condition that the mapping is one-to-one—an *embedding*—except at pairs of points whose shortest-path distance is contracted down to zero.

Why contraction? To match reality, the contractive definition of wrapping effectively assumes that length contraction can be achieved by continuous infinitesimal crinkling of the material. This assumption is justified for smooth mappings by the following theorem of Burago and Zalgaller [BZ96]:

Theorem 1 [BZ96] *Every contractive C^2-immersion f of a polyhedral metric P (e.g., a polygon) admits a C^0-approximation by isometric piecewise-linear C^0-immersions. In other words, for any $\varepsilon > 0$, there is an isometric piecewise-linear C^0-immersion f_ε of P for which $\|f_\varepsilon(p) - f(p)\| < \varepsilon$ for all points p of P. Furthermore, if f is an embedding (one-to-one), then so are the approximations f_ε.*

This theorem says that, if we allow an ε variation of the target shape, then any smooth contractive mapping can in fact be achieved by a regular isometric folding. The embedding clause says that even our simple noncrossing condition can be preserved: the portion P' of the paper P that is mapped one-to-one remains so, while the remaining points contracted down to P' can come along for the ride.

Wrapping motions. This connection to isometric foldings allows us to apply existing technology for transforming the folded states given by a wrapping into an entire continuous folding motion from the piece of paper to the surface—modulo the ε approximation from Theorem 1. Specifically, we can apply the following result:

Theorem 2 [DDMO04] *Every isometric folded state of a simple polygon can be obtained by a continuous isometric folding motion from the unfolded polygon, while avoiding crossings.*

[2] In fact, Burago and Zalgaller [BZ96] require that every distance shrinks by some factor $C < 1$. This discrepancy does not affect our statement of Theorem 1, however: their theorem applies for all $C < 1$ scalings of the target metric, and we can take the limit as $C \to 1$. The same argument is implicit in [Pak06].

We conjecture that the exact image of the contractive mapping can also be obtained by a continuous contraction (called a *shrinking* by Pak [Pak06]) while avoiding crossings. One natural approach is to continuously interpolate the intrinsic metric of the paper, compute the resulting curvature at every point, and construct a corresponding convex surface (and prove its existence). We suspect that this approach avoids crossings for the wrappings described in this paper but not in general.

Stretched paths and wrappings. We can model one family of wrappings by expressing which distances are preserved isometrically. An optimal wrapping should be isometric along some path, for otherwise we could uniformly scale up the object and the wrapping function while keeping the piece of paper fixed and the wrapping contractive. We call a path *taut* if the wrapping is isometric along it. A *taut wrapping* has the property that every point is covered by some taut path. Such a wrapping can be specified by a set of taut paths on the piece of paper, and their mapping on the target surface, whose union covers the entire paper and the entire surface. Although not all such specifications are valid—we need to check that all other paths are contractive—the specification does uniquely determine a wrapping.

We specify all of our wrappings in this way, under the belief that taut wrappings are generally the most efficient. It would be interesting to formalize this belief, say, for all wrappings on the area–perimeter Pareto curve. We also conjecture that such a wrapping is described by a tree of taut curves: that the space of taut curves is locally one-dimensional, connected, and acyclic.

Convex chains. A useful lemma for proving that sphere wrappings are contractive is the following:

Lemma 3 [ACC+08] *Given a open chain on a sphere that is convex together with the closing edge, if we embed the chain into the plane with matching edge lengths and angles, then the closing edge increases in length.*

Here the edges of the chain represent a chain of taut paths in the wrapping, whose lengths and angles are preserved between the unfolded piece of paper in the plane and the folding on the sphere. The lemma tells us that the shortest-path (closing) distance contracts when mapped to the sphere.

Source wrapping. A special case of taut wrapping is when the taut paths consist of the shortest paths from one point x to every other point y. We call this a *source wrapping* by analogy to source unfoldings of polyhedra [MMP87, DO07]. Equivalently, a source wrapping is a taut wrapping in which the taut paths form a star, i.e., a tree of depth 1. In this case, we are rolling geodesics in the piece of paper onto geodesics of the target surface. The reverse of this situation corresponds to continuous unfoldings of smooth polyhedra as considered by Benbernou, Cahn, and O'Rourke [BCO04]. Although perhaps the most natural kind of wrapping, this special case is too restrictive for our purposes: it essentially forces the sphere to be wrapped by a disk of radius π in order for the geodesics from x to reach around to the pole opposite x. The source folding is indeed a wrapping by Lemma 3 because every two points are connected by a (convex) chain of two taut paths. The wrapping has area π^3 and perimeter $2\pi^2$, as shown by the topmost point in Figure 1. As we will see, this wrapping is not on the Pareto curve, and in particular the area can be substantially improved.

A simple variation of the source wrapping uses two antipodal sources, connected by a taut path, with every other point connected by a taut path to the nearest source. The corresponding unfolding is then two disks of radius $\pi/2$ attached at one point. The wrapping has area $\pi^3/2$ and perimeter

$2\pi^2$, i.e., the same perimeter and half the area of the source unfolding. This wrapping is called the *two-source wrapping* in Figure 1.

Strip wrapping. If we start with an arbitrarily long and narrow rectangle, we can wrap a unit sphere using paper area arbitrarily close to the surface area 4π. We take an arbitrarily close (in surface area) polyhedral approximation circumscribing the sphere, and wrap the polyhedron using standard origami techniques of [DDM00]. Then we centrally project that wrapping onto the circumscribed sphere. This projection is a contractive mapping because $\tan \alpha \geq \alpha$ for $0 \leq \alpha < \pi/2$. The rectangle of paper even tiles the plane. The perimeter, however, increases drastically as the area approaches optimality. It remains open to compute the exact area–perimeter trade-off achieved by this method. In Figure 1 we simply show the limit point of 4π area and infinite perimeter. This point is on the Pareto curve.

3 Petal Wrapping

Our first family of nontrivial sphere wrappings are the *k-petal wrappings*. To define them, we construct a depth-2 tree of taut paths that cover all points on the sphere; refer to Figure 2. First we construct k taut paths $p_1, p_2, \ldots, p_k$ along great circular arcs from the south pole to the north pole (meridians), dividing the 2π angle around each pole into k equal parts of $2\pi/k$. To each path p_i we assign an *orange peel* with apex angles $2\pi/k$, centered on the path p_i and bounded by the great-circle angular bisectors between p_{i-1} and p_i and between p_i and p_{i+1}. In other words, these orange peels form the Voronoi diagram of sites p_1, p_2, $\ldots$, p_k on the sphere. These orange peels partition the surface of the sphere into k equal pieces.

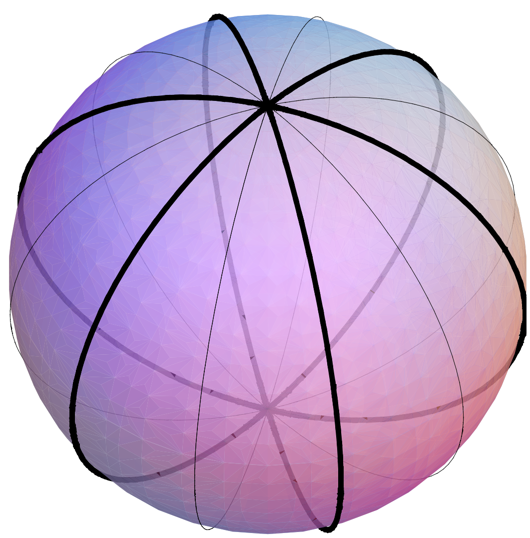


Figure 2: The taut paths $p_1, p_2, \ldots, p_k$ are bold meridians; the boundaries of the orange peels are thin meridians. In this example, $k = 6$.

Then we construct a continuum of taut paths to cover each orange peel. Specifically, for every point q along each path p_i, we construct two taut paths emanating from q, proceeding along geodesics perpendicular to p_i in both directions, and stopping at the boundary of p_i's orange peel.

These taut paths cover every point of the sphere (covering boundary points twice). It remains to find a suitable piece of paper that wraps according to these taut paths. The main challenge is to unfold the half of an orange peel left of a path p_i. Then we can easily glue the two halves together along the (straight) unfolded path p_i, resulting in what we call a *petal*, and finally join these petals at the unfolded south pole. The resulting shape is the k-petal wrapping.

To unroll half of a petal, we parameterize as shown in Figure 3. Here $B = \pi/k$ is the half-petal angle; $c \in [0, \pi]$ is a given amount that we traverse along the center path p_i starting at the south-pole endpoint; $A = \pi/2$ specifies that we turn perpendicular from that point; and b is the distance that we travel in that direction. Our goal is to determine b in terms of c.

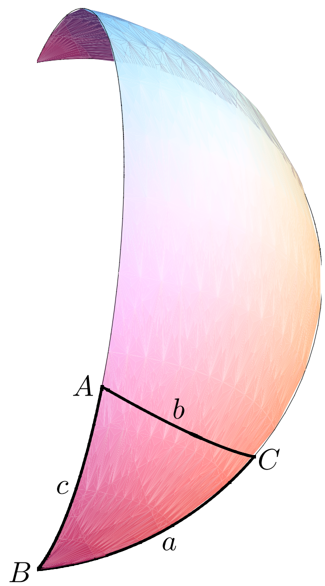


Figure 3: Half of a petal, labeled in preparation for spherical trigonometry.

By the spherical law of cosines,

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c.$$

Now $\cos A = \cos(\pi/2) = 0$ and $\sin A = \sin(\pi/2) = 1$, so this equation simplifies to $\cos C = \sin B \cos c$. Hence, $\sin C = \sqrt{1 - \sin^2 B \cos^2 c}$. By the spherical law of sines,

$$\frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$

Substituting $\sin C = \sqrt{1 - \sin^2 B \cos^2 c}$, we obtain

$$\frac{\sin B}{\sin b} = \frac{\sqrt{1 - \sin^2 B \cos^2 c}}{\sin c},$$

i.e.,

$$\sin b = \frac{\sin B \sin c}{\sqrt{1 - \sin^2 B \cos^2 c}}.$$

Using the fact that $\tan^2 b = 1/(1/\sin^2 b - 1)$, we obtain that $\tan b = (\sin B \sin c)/\cos B$. Taking arctan of both sides, we determine the value of b in terms of the parameter c and the known quantity $B = \pi/k$:

$$b = b(c) = \arctan\Big(\sin c \tan(\pi/k)\Big).$$

To summarize, one half of a petal is given by the curve $\{(c, b(c)) \mid 0 \le c \le \pi\}$. The area of the k-petal wrapping is thus given by the integral $\int_{c=0}^{\pi} b(c)\, dc$, multiplied by the number $2k$ of half petals. The perimeter is given by the arc-length integral $\displaystyle\int_{c=0}^{\pi} \sqrt{1 + \left(\frac{db(c)}{dc}\right)^2}\, dc$, again multiplied by $2k$. We are not aware of a closed-form solution for either integral. Also, the second derivative of $b(c)$ is always negative, except at $c = 0$ and $c = \pi$ where it is zero, proving that the petal is convex.

Figure 4 shows the resulting petal wrappings for $k = 3, 4, 5, 6$. It remains to show that these are actually wrappings, i.e., contractive. First, any two points within the same half of a petal are connected by a convex ($90°$) chain of taut paths, where the middle path is along one of the paths p_i. Thus, by Lemma 3, their distance contracts when mapped to the sphere. Now consider any two points q and q' in different half petals. Their shortest path on the piece of paper visits two half petals (either within the same petal or between two petals). Decompose the shortest path at the transition point t between the two half petals. By the previous argument, each of the two paths $q\,t$ and $t\,q'$ contracts when mapped to the sphere. By the triangle inequality, the distance between q and q' on the sphere is at most the sum of the distances between q and t and between t and q' on the sphere, each of which is contracted relative to the corresponding distances measured on the piece of paper, which sum to exactly the distance between q and q' in the piece of paper. Thus the wrapping is contractive.

The bottom curve in Figure 1 plots the performance of the petal wrappings. To our knowledge, all of these points may be on the Pareto curve. In the limit $k \to \infty$, we obtain the same 4π area and infinite perimeter of the strip wrapping.

We can also compute the area and perimeter of the convex hull of the k-petal wrapping, for $k > 3$, which is a regular k-gon. (The special case $k = 3$ is considered later.) The circumcircle of the regular polygon has radius π, so the area is $\frac{1}{2} k\pi^2 \sin(2\pi/k)$ and the perimeter is $2k\pi \sin(\pi/k)$. The left curve in Figure 1 plots the performance of these wrappings. None of these points lie on the Pareto curve: they are all dominated by the $k = 3$ case considered later, In the limit $k \to \infty$, we obtain the circular source unfolding at the top of the plot.

The convex hull of the k-petal wrapping is also a wrapping by a similar contractiveness argument. Now we decompose the shortest path between two points on the piece of paper into possibly several subpaths, namely, maximal paths that are within a single half petal or exterior to all petals. The former type has already been handled. It remains to consider two points q and q' on the boundary of two facing half petals. Assume by possible relabeling that q is farther than q' from the center of the wrapping (the base of both petals). Point q' contracts against a corresponding point $\tilde{q}$ on q's petal. Because the angle $q'\,\tilde{q}\,q$ is obtuse, the hypotenuse $q\,q'$ is longer than the length $q\,\tilde{q}$. The points q and $\tilde{q}$ are in a common half petal, so contract when mapped to the sphere. Thus so do q and q'.

4 Comb Wrapping

Our second family of nontrivial sphere wrappings are the *k-comb wrappings*. To define them, we construct a depth-3 tree of taut paths that cover all points on the sphere. We start with a backbone path taut around the equator. Then we add k taut paths extending up and k taut paths extending down perpendicularly from the equator to the north and south poles, respectively. Finally, every other point on the sphere connects perpendicularly to one of these paths along a taut path.

Interestingly, these wrappings are almost identical to k-petal wrappings: the piece of paper for the k-comb wrapping can be constructed by gluing k vertical petals side by side, joining at their middles. Figure 5 shows the examples $k = 3, 4, 5, 6$. As a consequence, both the area and the

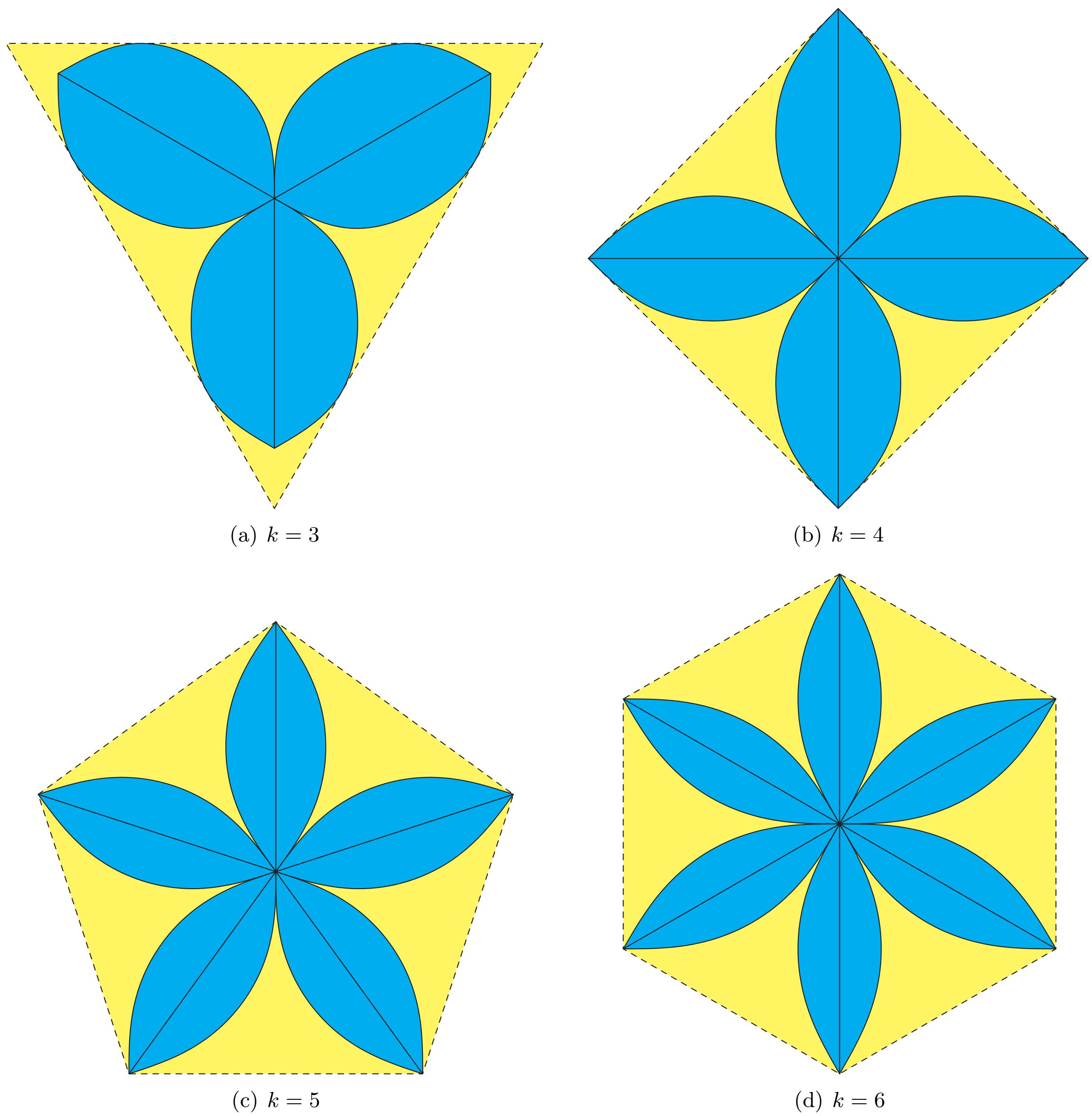
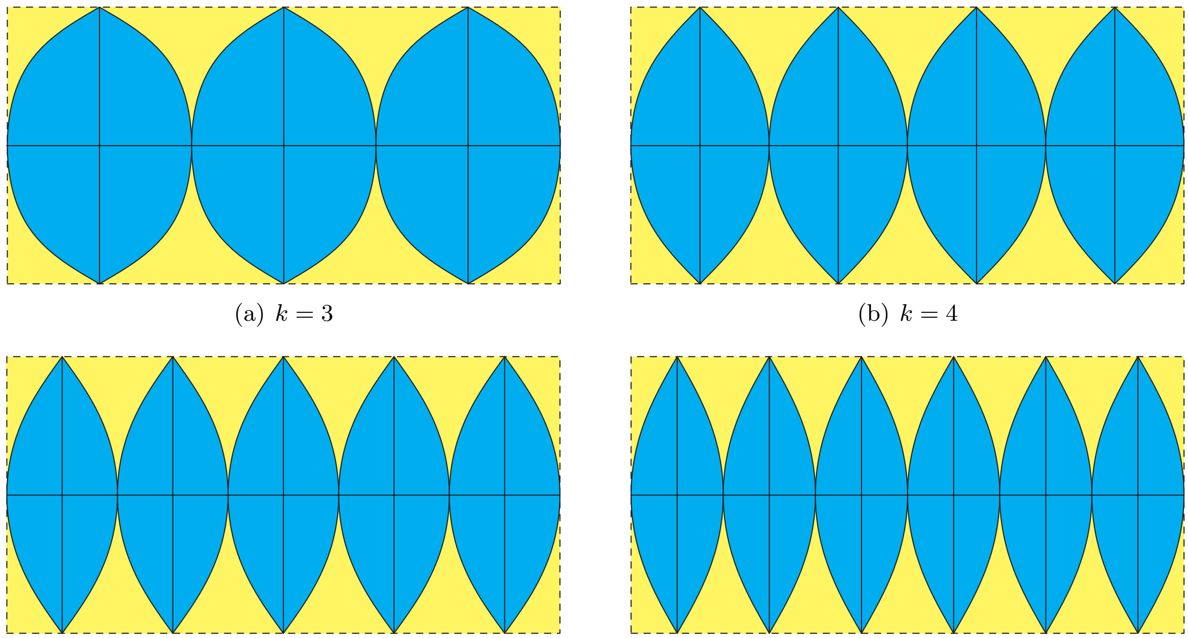


Figure 4: k-petal wrappings.

perimeter of the k-comb wrapping are identical to that of the k-petal wrapping, and thus they are plotted by the same lower curve in Figure 1.

On the other hand, the convex hull of the k-comb wrapping is quite different: a middle $2\pi(k-1)/k \times \pi$ rectangle with a half petal glued on two of the sides. Thus the area is $2\pi^2(k-1)/k$ plus the area of one petal ($1/k$th the area of the k-petal wrapping), and the perimeter is $2\pi + 4\pi(k-1)/k$ plus the perimeter of one petal. The middle curve in Figure 1 plots the performance of these wrappings. None of these wrappings are on the Pareto curve, being dominated by the two-source wrapping.

The k-comb wrapping and its convex hull are contractive by analogous arguments to the k-petal



(a) $k = 3$ (b) $k = 4$ (c) $k = 5$ (d) $k = 6$

Figure 5: k-comb wrappings.

wrappings.

The comb wrappings in Figure 5 look similar to the classic map projections known in cartography as *interrupted sinusoidal projections*. The difference is that the map projection unfolds along latitudinal lines, while we unfold along great circular arcs.

5 Mozartkugel Wrapping

The angle at the tip (or base) of a half petal is intuitively π/k, because that is the corresponding angle on the sphere. This fact can be verified by taking the derivative $\frac{db(c)}{dc}$ at $c = 0$. For $k = 4$, the derivative is 1 which implies a half angle of $\pi/4$. Because the petals are convex, the convex hull of the petal wrapping for $k = 4$ is exactly the square of diagonal 2π; see Figure 4(b). This square has area $2\pi^2$ and perimeter $8\pi/\sqrt{2} \approx 5.656854\pi$. The second point from the left of Figure 1 plots the performance of this wrapping. This square wrapping is precisely that used by Fürst's Original Salzburger Mozartkugel [Für] (except that in practice it is expanded a bit to ensure overlap).

No smaller square could wrap the unit sphere because the length of the path connecting the center of the square to the point mapped to the antipodal point must have length at least π. Thus we obtain the smallest square that wraps the unit sphere. This optimality result mirrors one of the few optimal wrapping results from the isometric folding literature, namely, the smallest square that wraps the unit cube [CJL01].

On the other hand, consider the smallest rectangle enclosing the k-comb wrapping for any k. The backbone reaches around the equator, so has length 2π. The perpendicular paths connect the north and south poles, so have total extent π. Therefore the rectangle is $\pi \times 2\pi$, which has the same area $2\pi^2$; see Figure 5(b). The perimeter is somewhat larger, 6π. This wrapping can be seen in Figure 1 as the limit of the convex hulls of the comb wrappings. Mirabell's Echte Salzburger

Mozartkugeln [Mir] uses precisely the 4-comb wrapping expanded out to this containing rectangle (and expanded a bit further to ensure overlap).

Note that neither of the two Mozartkugel wrappings are on the Pareto curve, illustrating the wastefulness of our industrial society.

6 Triangle Wrapping

For $k = 3$, the angle at the tip of the petals can be computed similarly to obtain $2\pi/3$, which is natural as the three petals meet at the north pole, their angles summing to 2π. However, the convex hull of the 3-petal wrapping is not a triangle. We compute its smallest enclosing equilateral triangle. Each supporting line of the triangle must be tangent to two of the petals. The tangent point on the petal can be computed by finding the point (c, b) on its boundary that maximizes the direction $(-\cos(\pi/3), \sin(\pi/3))$. Plugging this into the formula for $b = b(c)$, we obtain

$$c = \arccos\left(\tfrac{\sqrt{57}}{6} - \tfrac{1}{2}\right) \approx 0.7100861,$$

$$b = \arcsin \frac{\sqrt{\sqrt{57}-5}}{\sqrt{\sqrt{57}-3}} \approx 0.8459698.$$

Thus the supporting line is at a distance

$$h \;=\; \tfrac{\pi}{2} - \tfrac{1}{2}\arccos\left(\tfrac{\sqrt{57}}{6} - \tfrac{1}{2}\right) + \tfrac{\sqrt{3}}{2}\arcsin \frac{\sqrt{\sqrt{57}-5}}{\sqrt{\sqrt{57}-3}} \;\approx\; 0.6201901\,\pi$$

from the center. The equilateral triangle has an inscribed disk of radius h. The area is therefore $3h^2 \tan(\pi/3) \approx 1.998626\,\pi^2$, about 0.1% less than the $2\pi^2$ area of the smallest wrapping square. The side length of the triangle is $s = 2h\tan(\pi/3) \approx 2.148401\,\pi$, so the perimeter is $3s \approx 6.445204\,\pi$. The triangle point in Figure 1 plots the performance of this wrapping.

Using this triangle, we can also compute the area and perimeter of the convex hull of the 3-petal wrapping. The area of the hull is the area of the equilateral triangle, minus the area of the three small equilateral triangles bounded by two tangency points and a corner of the large triangle, plus the area of the six partial half petals contained in those small triangles. The small triangles have side length $2b$, so each has area $\frac{1}{4}(2b)^2\sqrt{3} = b^2\sqrt{3} \approx 1.239568$. Each partial half petal has area $\int_{\hat{c}=0}^{c} b(\hat{c})\, d\hat{c} \approx 0.03597917\,\pi^2$.[3] The hull area is thus $\approx 1.837717\,\pi^2$. Similarly, the perimeter of the hull is the perimeter of the large triangle, minus two sides of each of the three small triangles, plus the perimeter of the six partial half petals. The perimeter of each partial half petal is $\approx 0.3560733\,\pi$, so the hull perimeter is $\approx 5.350277\,\pi$. The leftmost point in Figure 1 plots the performance of this wrapping. This is the smallest perimeter we have seen for a shape that wraps the unit sphere, and one of the only two (together with the two-source unfolding) that dominates the triangle wrapping.

7 Tiling

Instead of expanding the petal wrappings to tilable regular polygons, we can pack the petal wrappings directly and expand them just to fill the extra space. Figure 6 shows what we believe to be the best tiling from the k-petal wrapping. The area of the resulting tile shape is $\approx 1.603304\,\pi^2$, a substantial improvement over the equilateral triangle of area $\approx 1.998626\,\pi^2$. In the center, we show how the tile leaves room for a large inscribed disk for a confectionery logo.

[3] This approximation was computed using Maple at 50 digits of precision.

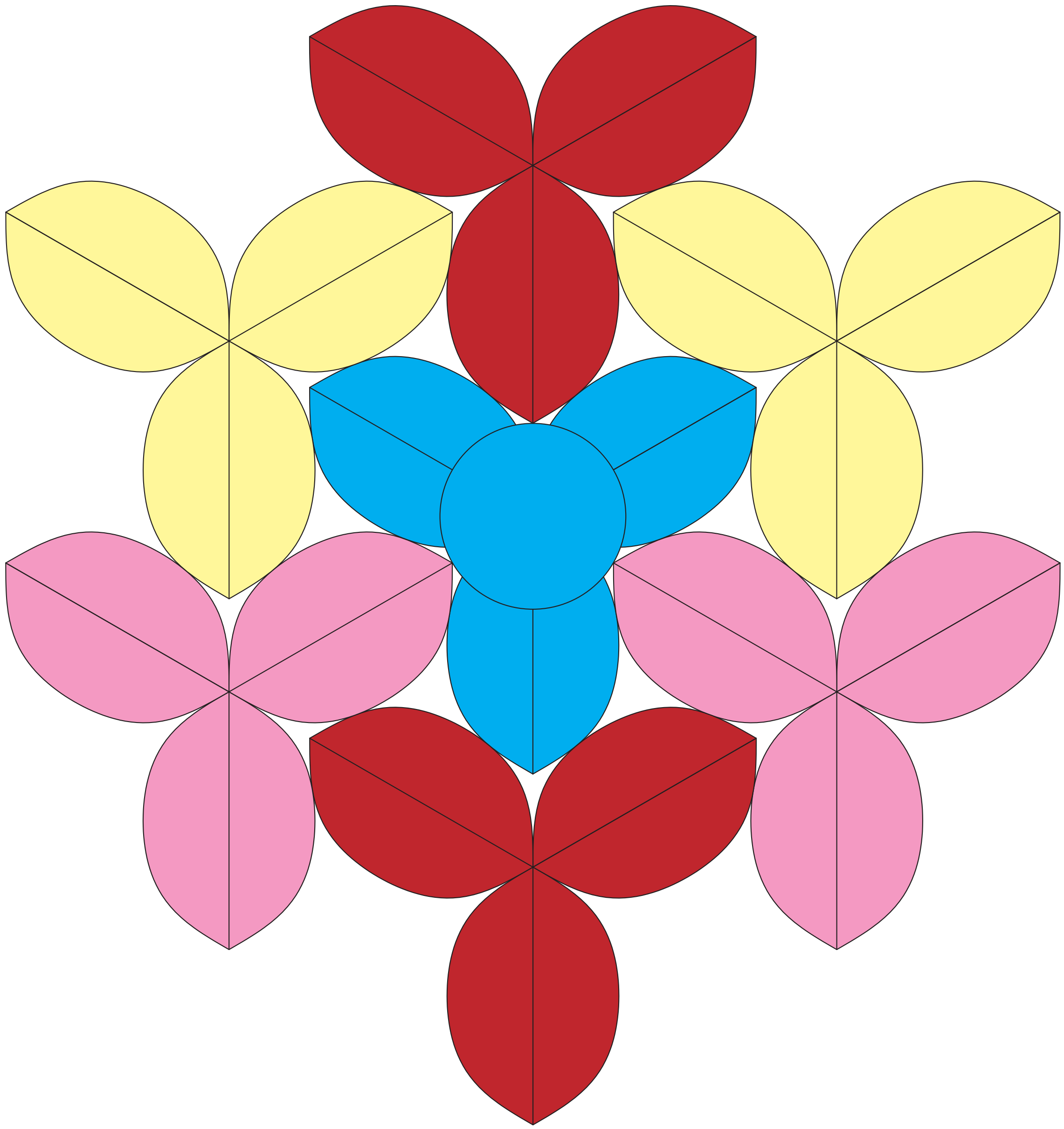
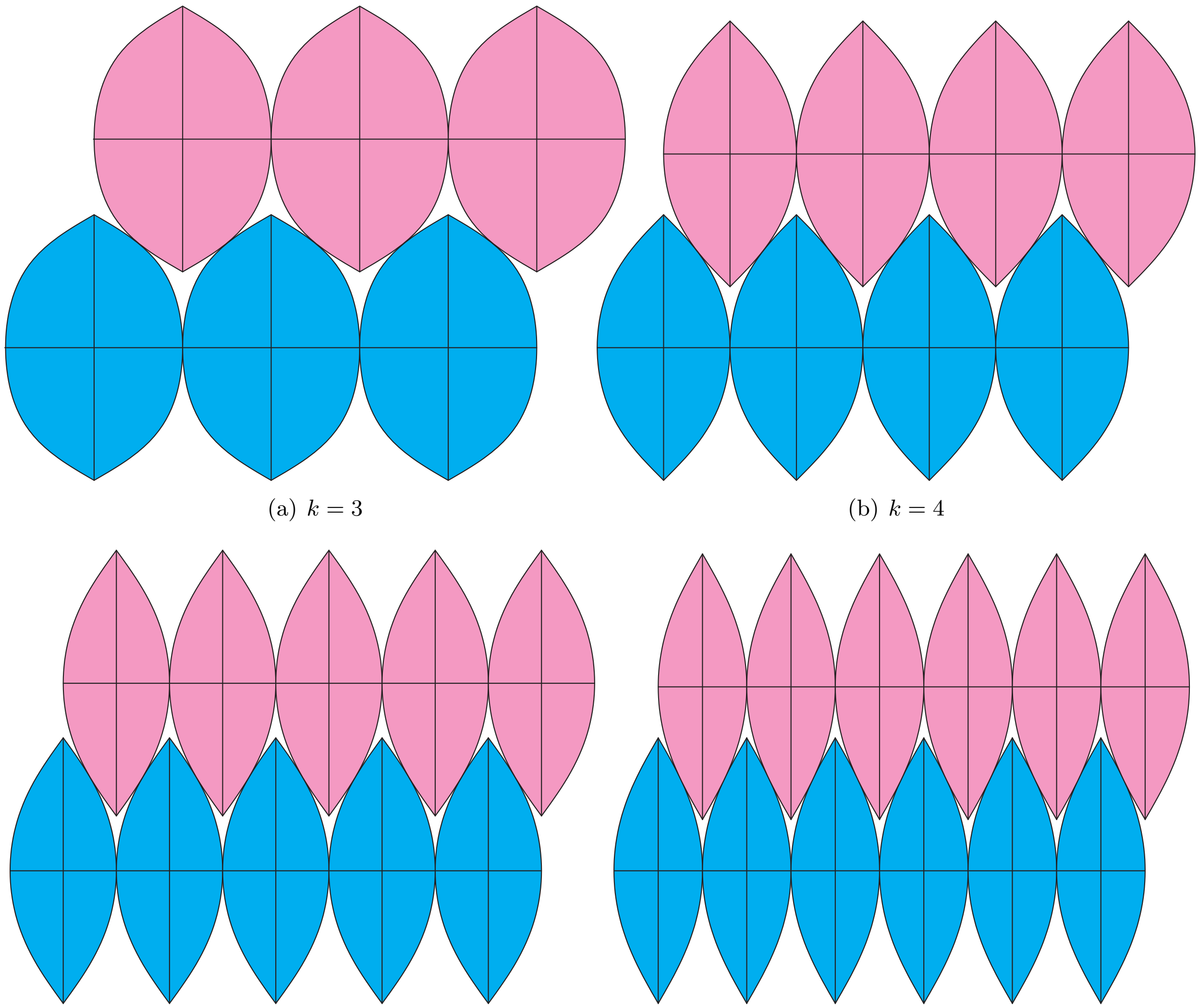


Figure 6: Packing the 3-petal wrapping.

Starting from the comb wrapping, we obtain even better tilings, as shown in Figure 7. As $k \to \infty$, the area usage approaches $4\pi^2/3 \approx 4.188790\,\pi$, less than 5% larger than the optimal $4\,\pi$. Of course, these wrappings have increasingly large perimeter.

8 Conclusion

This paper initiates a new research direction in the area of *computational confectionery*. We leave as open problems the study of wrapping other geometric confectioneries, or further improving our wrappings of the Mozartkugel. What is the complete shape of the Pareto curve? What is the shape of minimum perimeter that can wrap a unit sphere? What if the shape must tile the plane? What



(a) $k = 3$ (b) $k = 4$

(c) $k = 5$ (d) $k = 6$

Figure 7: Packing the k-comb wrapping.

about smooth surfaces other than the sphere?

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