Notes for 18.312, Algebraic Combinatorics

Corina Tarniţă, Mihai Pătraşcu

1 Walks on Graphs

Theorem 1. *The number of closed walks of length l on a graph G is $\sum_{i=1}^{n} \lambda_i^l$, where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of* $\mathrm{Adj}(G)$*.*

Theorem 2. *If the number of closed walks of length l is $\lambda_1^l + \cdots + \lambda_n^l$ for any l, and G has n vertices, the eigenvalues of G must be $\lambda_1, \dots, \lambda_n$.*

Theorem 3. *The eigenvalues of the cube in n dimensions are $n - 2i$ with multiplicity $\binom{n}{i}$ for every $i = 0, \dots, n$.*

2 Sets and Intersections

2.1 The Sunflower Lemma

Definition 4. *A family of sets $A_1, \dots, A_m$ is called a sunflower if $(\forall)i, j : A_i \cap A_j = \bigcap_{k=1}^{m} A_k$.*

Theorem 5 (sunflower lemma, Erdős-Rado). *Let F be a family of distinct sets, all of cardinality k. If $|F| > k!(s-1)^k$, then F contains a sunflower with s petals.*

Proof: By induction on k. For $k = 1$, the sunflower has $|F|$ petals and has empty center. For general k, consider a maximal family H of disjoint sets from F. If $|H| \geq s$, then H is the sunflower. Otherwise, these sets contain at most $(s-1)k$ elements. Any set of F must contain at least one of them (H is maximal), so one element x is contained in $\geq |F|/(s-1)k > (k-1)!(s-1)^{k-1}$ sets. Remove x from these sets, inductively find a sunflower in these sets of cardinality $k-1$, and add x back to the center. $\square$

Theorem 6. *Let F be a family of (not necessarily distinct) sets of cardinalities $\leq k$. If $|F| > k!(s-1)^{k+1}$, then F contains a sunflower with s petals.*

Proof: Only the base case changes from the previous version. For $k = 1$, we have $|F| > (s-1)^2$, and sets have zero or one elements (so they are either identical or disjoint). Then either some set appears s times, or there are at least s distinct sets. $\square$

2.2 Pairwise Intersecting Families

Theorem 7 (Erdős-Ko-Rado). *Let $A_1, \dots, A_m \subset [n]$, with $|A_i| = k \leq n/2$ and $A_i \cap A_j \neq \emptyset$. Then $m \leq \binom{n-1}{k-1}$.*

Proof: Consider all $(n-1)!$ circular arrangements for $[n]$. For each A_i, the number of arrangements in which its elements appear as a contiguous circular segment is $k!(n-k)!$. On the other hand, in each circular arrangement at most k sets can appears as segments (consider that all the segments must overlap and that $k \leq n/2$). So $mk!(n-k)! \leq k(n-1)!$. $\square$

Theorem 8. *Let $A_1, \dots, A_m \subset [n]$, with $|A_i| \leq k \leq n/2, A_i \cap A_j \neq \emptyset, A_i \not\subseteq A_j$. Then $m \leq \binom{n-1}{k-1}$.*

Proof: By reduction to the previous theorem. Pick all sets of the smallest cardinality s, and add one element to each of them. There are $n-s$ choices to extend any set of cardinality s, so we have a regular bipartite graph and we can find a perfect matching $\binom{[n]}{s} \to \binom{[n]}{s+1}$ – so we map our sets to distinct bigger sets. Repeat this step until all sets have cardinality k. Note that the sets remain pairwise intersecting (we only add elements), and since no original $A_i \subset A_j$, we cannot create identical sets. $\square$

Theorem 9. *Let $A_1, \ldots, A_m \subset [n], |A_i| = r$ and $B_1, \ldots, B_m \subset [n], |B_i| = s$. If $A_i \cap B_i = \emptyset$ and $A_i \cap B_j \neq \emptyset, (\forall) i < j$, then $m \leq \binom{r+s}{r}$.*

Proof: For every $x \in [n]$, choose $v_x \in \mathbb{R}^{r+1}$ in general position (every $r+1$ vectors are linearly independent). One way to achieve this is to take vectors $(1, t, t^2, \ldots)$ for any t (the matrix formed by any $r+1$ of them is van der Monde, and has nonzero determinant). The elements of every set A_i span a hyperplane of dimension r. We associate A_i with the vector a_i normal to this hyperplane. Now for all B_j define:

$$f_j : \mathbb{R}^{r+1} \to \mathbb{R}, \quad f_j(\overline{x}) = \prod_{y \in B_j} \overline{v_y} \cdot \overline{x}$$

Note that $A_i \cap B_j \neq \emptyset \Rightarrow (\exists) x \in B_j, x \in A_i \Rightarrow (\exists) x \in B_j, a_i \perp v_x \Rightarrow f_j(a_i) = 0$. One the other hand, $A_i \cap B_i = \emptyset \Rightarrow (\forall) x \in B_i, x \notin A_i$. But then v_x is not in the hyperplane of A_i (since any $r+1$ vectors are linearly independent), so $v_x \cdot a_i \neq 0 \Rightarrow f_i(a_i) \neq 0$. These conditions imply that the f_i's are linearly independent.

Note that any f_i contains only monomials of degree s in $r+1$ variables. There are $\binom{r+s}{r}$ distinct monomials ($r+1$ multichoose s), so the dimension of the space is $\binom{r+s}{r}$. $\square$

Note: One can also give a proof based on exterior algebras, which is omitted here.

2.3 L-Intersecting Families

We begin with a special case of the main theorem:

Theorem 10. *Let $A_1, \ldots, A_m \subset [n]$, with $|A_i \cap A_j| = 1, (\forall) i \neq j$ and $|A_i| \geq 2$. Then $m \leq n$.*

Proof: Consider a matrix M with $M_{ij} = [j \in A_i]$. Then $(M \cdot M^t)_{ij} = |A_i \cap A_j|$. This matrix has 1 everywhere except the diagonal, and every diagonal element is ≥ 2. It follows that $\text{rank}(M \cdot M^t) = m$. But $\text{rank}(M \cdot M^t) \leq \text{rank}(M) = \text{rank}(M_t) \leq n$. $\square$

Corollary 11. *Consider n points, not all on a line. Then these points determine at least n distinct lines.*

Proof: Every point is the intersection of at least two lines determined by the points. For each point, consider the set of lines going through the point. Then the previous theorem applies (in particular $|A_i \cap A_j| = 1$ because any two points determine a unique line). $\square$

Theorem 12 (L-intersecting families). *Let L be a set of s integers. Also let $A_1, \ldots, A_m \subset [n]$ with $|A_i \cap A_j| \in L, (\forall) i \neq j$. Then $m \leq \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{0}$.*

Proof: Let $a_i \in \mathbb{R}^n$ be the incidence vector of A_i. Arrange the sets such that $|A_i| \leq |A_j|$. Define:

$$f_i : \mathbb{R}^n \to \mathbb{R}, \quad f_i(\overline{x}) = \prod_{\ell \in L, \ell < |A_i|} (\overline{a_i} \cdot \overline{x} - \ell)$$

Clearly $f_i(a_i) \neq 0$ because we only consider $\ell < |A_i|$. For every $j < i$, we have $|A_j \cap A_i| < |A_i|$. So $f_i(a_j) = 0 \iff \overline{a_i} \cdot \overline{a_j} \in L \iff |A_i \cap A_j| \in L$; hence, $f_i(a_j) = 0, (\forall) j < i$. This implies that the f_i's are independent.

Now every f_i has degree $\leq |L| = s$ and has n variables. We only considered inputs in $\{0,1\}^n$, and for $x \in \{0,1\}, x^k = x, (\forall) k > 1$. So we can linearize the polynomial, i.e. transform every power to the first power. With this linearization, there are only $\binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{0}$ independent monomials of degree $\leq s$. $\square$

Theorem 13. *Let L be a set of s integers. Also let $A_1, \ldots, A_m \subset [n]$ with $|A_i| = k$ and $|A_i \cap A_j| \in L, (\forall) i \neq j$. Then $m \leq \binom{n}{s}$.*

Proof: Let f_i be the linearized polynomials from the previous proof. For all $I \subset [n], |I| < s$, define:

$$g_I : \mathbb{R}^n \to \mathbb{R}, \quad g_I(x_1, \ldots, x_n) = \prod_{j \in I} x_j \left(\sum_{j=0}^{n} x_j - k \right)$$

Clearly, there are $\binom{n}{s-1} + \cdots + \binom{n}{0}$ such I's. Also $g_I(\overline{a_j}) = 0, (\forall) I, j$, because $|A_j| = k$. For every I, let u_I be the incidence vector of I. Now $g_I(u_I) \neq 0$ and $g_I(u_J) = 0$. This means that all vectors $\{g_I\} \cup \{f_i\}$ are linearly independent – first apply the a_i's to the sum, and deduce that the coefficients of the f_i's are zero, then apply the u_I's to show the coefficients for the g_I's are zero. By the dimension count from the previous theorem, we are done. $\square$

2.4 Modular Constraints

Theorem 14. *Let $A_1, \ldots, A_m \subset [n]$, with $|A_i \cap A_j|$ even and $|A_i|$ odd. Then $m \leq n$.*

Proof: We will work in $\mathbb{F}_2$. Consider the matrix M with $M_{ij} = [j \in A_i]$. Now $M \cdot M^t = I_m$ because $|A_i \cap A_j| \equiv 0, |A_i \cap A_i| \equiv 1$ (in $\mathbb{F}_2$). Now $\operatorname{rank}(I_m) = m \leq \operatorname{rank}(M) = \operatorname{rank}(M^t) \leq n$. $\square$

Proof (alternative): The incidence vectors $a_i \in (\mathbb{F}_2)^n$ are orthonormal, hence independent. $\square$

Theorem 15. *Let $A_1, \ldots, A_m \subset [n]$, with $|A_i \cap A_j|$ even and $|A_i|$ even. Then $m \leq 2^{\lfloor n/2 \rfloor}$.*

Proof: Consider the incidence vectors $a_i \in (\mathbb{F}_2)^n$. Let $U = \operatorname{span}(\{a_i\})$. We have $a_i \perp a_j, (\forall) i, j$ so $U \subset U^\perp \Rightarrow \dim U \leq \dim U'$. But $\dim U^\perp = n - \dim U$, because impose $\dim U$ independent linear constraints on the vectors from $U^\perp$ (perpendicularity to a basis of U). So $\dim U \leq n/2$. $\square$

Theorem 16. *Let p be a prime, and L a set of s integers in $\{0, \ldots, p-1\}$. Also let $A_1, \ldots, A_m \subset [n]$, with $|A_i| \bmod p \notin L$ and $|A_i \cap A_j| \bmod p \in L$. Then $m \leq \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{0}$.*

Proof: We will work in $\mathbb{F}_p$. For each A_i, consider its incidence vector a_i and define:

$$f_i : (\mathbb{F}_p)^n \to \mathbb{F}_p, \quad f_i(\overline{x}) = \prod_{\ell \in L} (\overline{a_i} \cdot \overline{x} - \ell)$$

Clearly $f_i(\overline{a_j}) = 0$ because $|A_i \cap A_j| \bmod p \in L$ and $f_i(\overline{a_i}) \neq 0$ because $|A_i| \bmod p \notin L$. Then the f_i's are independent. Each has degree $|L| = s$ and n variables. We can linearize the polynomials as before, and we get $\binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{0}$ independent monomials. $\square$

Corollary 17. *Let p be a prime, $n = 4p-1$ and $A_1, \ldots, A_m \subset [n]$, with $|A_i| = 2p-1$ and $|A_i \cap A_j| \neq p-1$. Then $m \leq 2\binom{4p-1}{p-1}$.*

Proof: Note that $2p - 1 \equiv p - 1 (\bmod p)$, and this is the only element excluded from L. So we have $L = \{0, \ldots, p-2\}$, and by the previous theorem $m \leq \binom{4p-1}{p-1} + \binom{4p-1}{p-2} + \cdots + \binom{4p-1}{0}$. The bound follows by induction. $\square$

2.5 Application: Disproving the Borsuk Conjecture

Borsuk's conjecture states that any set $B \subset \mathbb{R}^d$ can be broken into $d+1$ parts, all having diameter strictly smaller than $\mathrm{diam}(B)$. The conjecture is true for dimension 2 and 3, and when B is a body with smooth boundary. However, the conjecture is false in general, for large enough d.

Theorem 18 (Kahn-Kalai). *There exists a set $B \subset \mathbb{R}^d$ such that breaking B into parts of diameter strictly smaller than $\mathrm{diam}(B)$ requires at least $2^{\Omega(\sqrt{d})}$ parts.*

Proof: Let p be a prime. Consider all sets $A_i \in \binom{[4p-1]}{2p-1}$. For every A_i, let $F_i = \{\{x,y\} \mid x \in A_i, y \in [4p-1] \setminus A_i\}$. The F_i's are subsets of a $\binom{4p-1}{2}$-dimensional set, so we can view them as points in this dimension (by taking the incidence vector) – B will consist exactly of these points. Then $d^2(F_i, F_j) = |F_i \setminus F_j| + |F_j \setminus F_i|$. Let $r = |A_i \cap A_j|$. We have:

$$\begin{aligned} & F_i \setminus F_j = \{\{x,y\} \mid x \in A_i, y \notin A_i, x, y \in A_j\} \cup \{\{x,y\} \mid x \in A_i, y \notin A_i, x, y \notin A_j\} \\ \Rightarrow \quad & |F_i \setminus F_j| = |A_i \cap A_j| \cdot |\overline{A_i} \cap A_j| + |A_i \cap \overline{A_j}| \cdot |\overline{A_i} \cap \overline{A_j}| \\ \Rightarrow \quad & |F_i \setminus F_j| = r((2p-1)-r) + ((2p-1)-r)(r+1) = -2r^2 + (4p-3)r + (2p-1) \end{aligned}$$

By symmetry, $|F_j \setminus F_i|$ has the same value. So the distance from F_i to F_j is maximum when $r \approx p - 3/4$, which means $r = p-1$ (because r is an integer). Thus the diameter of B is defined by F_i, F_j when $|A_i \cap A_j| = p-1$. If we break B into parts of smaller diameter, a part must not contain any sets that intersect in $p-1$ places. By the previous corollary, each part can then have at most $2\binom{4p-1}{p-1}$ points, so there must be at least $\binom{4p-1}{2p-1}/2\binom{4p-1}{p-1}$ parts. This is roughly $(2/1.8)^{4p-1} = 2^{\Omega(\sqrt{d})}$. $\square$

2.6 Application: The Chromatic Number of $\mathbb{R}^d$

Definition 19. *The chromatic number of $\mathbb{R}^d$ is the minimum number of colors needed to color every point in $\mathbb{R}^d$ such that any segment of unit length has its ends colored by different colors.*

The chromatic number of a plane is known to be between 4 and 7. For high dimensions, we can prove exponential bounds:

Theorem 20. *The chromatic number of $\mathbb{R}^d$ is at most $2^{O(d \lg d)}$.*

Proof: Cover $\mathbb{R}^d$ with cubes of side $\frac{1}{2\sqrt{d}}$. Note the that diameter of such cubes is $1/2$. It suffices to color with different colors all such cubes from inside a cube of side 2. So $(4\sqrt{d})^d$ colors suffices. $\square$

Theorem 21. *The chromatic number of $\mathbb{R}^d$ is at least $2^{\Omega(d)}$.*

Note: A lower bound of $2^{\Omega(\sqrt{d})}$ follows trivially from the disproof of the Borsuk conjecture.

Proof: Let p be prime, and $d = 4p-1$. For each set $A_i \in \binom{[4p-1]}{2p-1}$, consider the point $x_i \in \mathbb{R}^d$ given by the incidence vector. The distance $d(x_i, x_j) = \sqrt{|A_i \setminus A_j| + |A_j \setminus A_i|}$. If $|A_i \cap A_j| = r$, then $d(x_i, x_j) = \sqrt{2((2p-1)-r)}$. Rescale everything so that $\sqrt{2((2p-1)-(p-1))}$ is the "unit" distance. A coloring of $\mathbb{R}^d$ must now color sets that intersect in $p-1$ places with different colors. By the known corollary, there can be at most $2\binom{4p-1}{p-1}$ sets for each color, so at least $\binom{4p-1}{2p-1}/2\binom{4p-1}{p-1} = 2^{\Omega(d)}$ colors are needed. $\square$

3 Partially Ordered Sets

poset − a set X with a partial order $\preceq$ which is transitive ($a \preceq b, b \preceq c \Rightarrow a \preceq c$), anti-symmetric ($a \preceq b, b \preceq a \Rightarrow a = b$) and reflexive ($a \preceq a$), but possibly incomplete ($(a \npreceq b) \wedge (b \npreceq a)$);

chain − a list of elements $x_0 \precneqq x_1 \precneqq x_2 \ldots$;

saturated chain − $(\forall)i, (\nexists)y : x_i \precneqq y \precneqq x_{i+1}$;

maximal chain − saturated chain, whose ends are minimal and maximal;

antichain − a list of elements, no two of which are comparable;

graded poset − elements are on levels according to some rank; every saturated chain goes through consecutive levels;

$(B_n, \subseteq)$ − the Boolean poset; $B_n = 2^{[n]}$ = all subsets of $[n]$;

3.1 Decompositions

Theorem 22. *If the longest chain has n elements, the poset can be decomposed into n disjoint antichains.*

Proof: Put all minimal elements in an antichain; remove them and repeat. □

Corollary 23. *In any sequence of $(n-1)(m-1)+1$ numbers, either there exists an increasing subsequence of length n or a decreasing subsequence of length m.*

Theorem 24 (Dillworth). *If the longest antichain has n elements, the poset can be decomposed into n disjoint chains.*

Proof: By induction on the size of the poset and n. Trivial for $n = 1$ and any size (the poset must be a chain). For the general case, let S be an antichain of length n. It follows that any element of the poset is comparable to some element of S (otherwise S is not maximum) and no element is less than some element of S and greater than some other (otherwise, two elements of S are comparable by transitivity). So the poset is decomposed into two parts S^+ and S^- which are above and below S (by definition $S \subset S^+$ and $S \subset S^-$). If both S^+ and $S-$ are nontrivial (not equal to S), then both are smaller than the original poset, so by the induction hypothesis, they can be decomposed into n chains. The elements of S are minimal in S^+ and maximal in S^- so there is one chain starting/ending in each one. By gluing these together, we get n chains that cover the original poset.

The only remaining case is when one cannot choose S such that S^+ and S^- are both nontrivial; in other words, the longest antichains consist either of only minimal or only maximal elements. Then take a maximal chain from the poset, and remove it. Now the longest antichain has size $n-1$ and we can apply the induction hypothesis. Indeed, assume that the longest antichain still had size n. Then there is an antichain which is not formed entirely of minimal or maximal elements (because we removed one minimal and one maximal elements as part of the maximal chain, and so there are $\leq n-1$ of either one). □

Corollary 25 (Hall's matching theorem). *In a bipartite graph $G = L \cup R$, there is a perfect matching from L to R iff $(\forall)T \subseteq L, |N(T)| \geq |T|$, where $N(T)$ denotes the neighbors of vertices in T.*

Proof: Create a poset with two levels. The lower one is R and the higher one is L. The order relations correspond to edges in the graph. Any antichain consists of some $T \subseteq L$ and $S \subseteq R$, such that no elements of T is comparable to an elements of S. This means that $N(T) \cap S = \emptyset$. Thus $|T| + |S| \leq |N(T)| + |S| \leq |R|$. Then the poset can be decomposed into $|R|$ disjoint chains, each one of which must contain an element of R. Then every element in L can be matched to the other element of its chain. □

3.2 The Möbius Function

Consider a poset $(P, \preceq)$. Let's say we have two functions f and g defined on the poset, related by $f(x) = \sum_{x \preceq y} g(y)$. We would like to find an expression for g in terms of f. To do that, we define the matrix Z with entries $(Z)_{x,y \in P} = \zeta(x,y) = [x \preceq y]$. Viewing f and g as vectors $(f(x))_{x \in P}, (f(y))_{y \in P}$, we can rewrite the relation as $f = Zg$. Thus, $g = Z^{-1} f$, so what we want is to find the inverse of Z. We will denote $M = Z^{-1}$, and the entries $(M)_{x,y \in P} = \mu(x,y)$, where μ is called the Möbius function of P.

Theorem 26. *The inverse M always exists and has integer entries.*

Proof: By topological sorting, we can arrange the elements of P in a line such that $x \preceq y$ implies x comes before y. Then Z will be upper-triangular, and it's diagonal will consist only of ones. So $\det Z = 1$, which proves the theorem. □

Proof (constructive): Since M is upper-triangular, its inverse should also be upper-triangular. So we let $\mu(x,y) = 0$ whenever it's not the case that $x \preceq y$. It is also easy to see that the entries on the diagonal should be one: $\mu(x,x) = 1$. These conditions alone guarantee that the diagonal of $M \cdot Z$ contains only ones. It remains to guarantee that the other entries are zero:

$$(\forall) x \neq y : \sum_c \mu(x,c)\zeta(c,y) = 0 \quad \Rightarrow \sum_{x \preceq c \preceq y} \mu(x,c) \cdot 1 = 0 \quad \Rightarrow \mu(x,y) = -\sum_{x \prec c \precneqq y} \mu(x,c)$$

This gives a recurrence for μ, and finishes the construction. Note that the recursion is well-defined, since $\mu(x,y)$ depends only on values of $\mu(x,c)$ where $c \precneqq y$. □

Theorem 27. *Let $p_k(x,y)$ be the number of chains of length k from x to y. Then $\mu(x,y) = \sum_{k=0}^{\infty} (-1)^k p_k(x,y)$.*

Note: Viewed as a definition for μ, this is yet another constructive proof of the previous theorem.

Proof: Let $N = Z - I$. Since N is upper-triangular and has zeros on the diagonal, it is nilpotent of degree n (i.e. $N^n = 0$). Remember the classical power series expansion $\frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k$. This will also hold for finding $M = (I+N)^{-1}$. Indeed, consider the expression: $(N+I)(I - N + N^2 - N^3 + \dots)$; note that the first parenthesis is precisely Z. Since $N^n = 0$, we can ignore all terms after N^{n-1} in the second parenthesis. Then the expression will become $I \pm N^n = I$. So we have $M = Z^{-1} = \sum_{k=0}^{\infty} (-1)^k N^k$. But $\left(N^k\right)_{x,y}$ is precisely the number of chains of length k from x to y, because we are taking a walk in a (directed) graph. □

3.3 Application: The Inclusion-Exclusion Principle

Theorem 28 (the inclusion-exclusion principle). *For any sets $A_1, \dots, A_n$ we have:*

$$|A_1 \cup \dots \cup A_n| = \sum_{j=1}^{n} (-1)^j \sum_{|J|=j} \left| \bigcap_{i \in J} A_i \right| = \sum_{J \subseteq [n], J \neq \emptyset} (-1)^{|J|} \left| \bigcap_{i \in J} A_i \right|$$

Proof: The Möbius function of B_n is $\mu(A,B) = (-1)^{|B|-|A|}$ if $A \subseteq B$ and 0 otherwise. We need only verify the identity:

$$0 = \sum_{A \subseteq C \subseteq B} \mu(A,C) = \sum_{A \subseteq C \subseteq B} (-1)^{|B|-|C|} = \sum_{r=0}^{|B|-|A|} \binom{|B|-|A|}{r} (-1)^{|B|-|A|-r}$$

This is true by the binomial theorem. Now define:

$$(\forall) J \subseteq [n] : \quad f(J) = \left| \bigcap_{i \in J} A_i \right| \qquad g(J) = \left| \left(\bigcap_{i \in J} A_i \right) \cap \left(\bigcap_{i \notin J} \overline{A_i} \right) \right|$$

Every element x is in some sets $A_j, j \in J$ and in the complement of the others – in which case it is counted in $g(J)$. For every x counted in $f(I)$, we must have $I \subseteq J$ (x is at least in the intersection of all sets I). We get $f(I) = \sum_{I \subseteq J} g(J)$. This means that $f = Zg$, so $g = Mf$. Thus:

$$0 = g(\emptyset) = \sum_J (-1)^{|J|} f(J) = \sum_J (-1)^{|J|} \left| \bigcap_{i \in J} A_i \right| = |A_1 \cup \cdots \cup A_n| + \sum_{J \neq \emptyset} (-1)^{|J|} \left| \bigcap_{i \in J} A_i \right|$$

$\square$

3.4 Sperner's Property

Definition 29. *A graded poset has Sperner's property if the largest level is a largest antichain.*

Theorem 30. *The poset $(B_n, \subseteq)$ has Sperner's property, and the largest antichain consists of all sets of cardinality $\lfloor n/2 \rfloor$ (the same is true for cardinality $\lceil n/2 \rceil$).*

Proof: Consider an antichain of B_n. A set of cardinality k is contained in exactly $k!(n-k)!$ maximal chains (we must remove k elements in some order to go down, and add $n-k$ elements to go up). We know $k!(n-k)! \geq \lfloor n/2 \rfloor ! \lceil n/2 \rceil !$. For the sets of the antichain, these chains must be distinct, or otherwise two sets are comparable. In total there are $n!$ maximal chains, so the size of the antichain must be at most $\frac{n!}{\lfloor n/2 \rfloor ! \lceil n/2 \rceil !} = \binom{n}{\lfloor n/2 \rfloor} = \binom{n}{\lceil n/2 \rceil}$. $\square$

We now consider a special case of group action, namely the action of a subgroup G of S_n on the set B_n. A permutation $\pi \in G$ "acts" on $A \in B_n$ by producing $\pi(A) = \{\pi(a) \mid (\forall) a \in A\}$. The orbit of A is the family $\{\pi(A) \mid (\forall)\pi \in G\}$. Note in particular that all sets of an orbit have the same cardinality.

The quotient poset B_n/G is the set of orbits of G on B_n. The order relation is defined by $O \preceq O' \iff (\exists) A \in O, B \in O' : A \subseteq B$. It is easy to check that this is a partial order, and the poset is graded:

$O \preceq O$ **because:** $A \in O, A \subseteq A$;

$O \preceq O', O' \preceq O \Rightarrow O = O'$ **because:** the sets in O and O' must have the same cardinality; now if there exist $A \in O, B \in O' : A \subseteq B$, it must be the case that $A = B$ so $O = O'$;

$O \preceq O', O' \preceq O'' \Rightarrow O \preceq O''$ **because:** by definition, $(\exists) A \in O, B \in O' : A \subseteq B$ and $(\exists) B' \in O', C \in O'' : B' \subseteq C$; also $(\exists)\pi \in G : B' = \pi(B)$. But then $\pi(A) \in O$ and $\pi(A) \subseteq \pi(B) = B' \subseteq C$;

Theorem 31. *The poset B_n/G has Sperner's property.*

The proof is quite long and will be given in the next section; we first give an application:

Corollary 32. *Consider graphs on n vertices. The largest family of graphs, none of which is isomorphic to a subgroup of another, consists of all nonisomorphic graphs with $\binom{n}{2}/2$ edges.*

Proof: A graph is given by it's set of edges, so it is an element of $B_{\binom{n}{2}}$. Now for $\pi \in S_n$, consider the action $\pi((a,b)) = (\pi(a), \pi(b))$; this gives a subgroup G of permutations on the $\binom{n}{2}$ edges that is isomorphic to S_n. A graph is isomorphic to another if they are in the same orbit under this group, so the isomorphic graphs form the poset B_n/G. $\square$

3.5 Proof of Sperner's Property for B_n/G

We begin with a general method that can be used to show a poset P is Sperner. Let $P_0, \ldots, P_n$ be the levels of P; let the maximum level be P_k. A collection of functions $\{\mu_i\} \cup \{\xi_i\}$ is called an order matching if it satisfies the following conditions:

$$P_0 \xrightarrow{\mu_0} P_1 \xrightarrow{\mu_1} \ldots \xrightarrow{\mu_{k-1}} P_k \xleftarrow{\xi_k} P_{k+1} \xleftarrow{\xi_{k+1}} \ldots \xleftarrow{\xi_{n-1}} P_n$$

$$\mu_i, \xi_i \text{ are injective}, \quad (\forall) i < k, A \in P_i : A \preceq \mu_i(A), \quad (\forall) i > k, A \in P_{i+1} : \xi_i(A) \preceq A$$

Lemma 33. *If an order matching exists, P is Sperner.*

Proof: Consider all elements $a \in P_k$. Start building a chain containing a: as long as the first element has an inverse through μ, extend the chain down; as long as the last element has an inverse through ξ, extend the chain up. Since μ and ξ preserve order, this will be a chain. Since μ and ξ are injective, this extension is unique, so every preimage of a through successive applications of μ's or ξ's will be in a's chain. Thus, these chains cover the poset, because every element can be mapped to something in P_k through successive applications of μ's or ξ's. We have found $|P_k|$ chains covering the poset, so any antichain can contain at most $|P_k|$ elements, and thus P_k is a maximal antichain. $\square$

In what follows, we will ignore the ξ functions, and consider only the "lower half" of the poset; results for the other half follow symmetrically. For any finite set S, let $\mathbb{R}S$ be the real vector space of dimension $|S|$, with a basis given by the elements of S. In other words, this is a set of formal sums of elements from S: $\mathbb{R}S = \{\sum_{A\in S} c_A A \mid (\forall)(c_A) \in \mathbb{R}^{|S|}\}$.

Definition 34. *A linear transformation $U_i : \mathbb{R}P_i \to \mathbb{R}P_{i+1}$ is said to be raising, if $(\forall)A \in P_i : U_i(A) = \sum_{A\prec B\in P_{i+1}} c_B B$. In other words, the coefficient in $U_i(A)$ of any element incomparable to A is zero.*

Lemma 35. *Assume there exists a linear transformation $U_i : \mathbb{R}P_i \to \mathbb{R}P_{i+1}$ which is injective and raising. Then there exists an order matching μ_i as defined above.*

Proof: Consider the matrix of U_i in the standard bases P_i and P_{i+1}. Since U_i is injective, the rank of the matrix must be $|P_i|$. Consider a minor of size $|P_i|$ which has nonzero determinant. Let $S \subset P_{i+1}$ be the elements corresponding to the rows of this minor. Denote the entries of the minor by $(c_{A,B})_{A\in P_i, B\in S}$. Since the determinant is nonzero, at least one term in the sum-of-products expansion must be nonzero, so there exists a permutation $\pi : P_i \to S$ such that $c_{A,\pi(A)} \neq 0, (\forall)A \in P_i$. But this is precisely an order matching, since it is injective on P_i, and each A can only be mapped to $\pi(A) \succ A$, because $c_{A,B} = 0$ whenever $A \not\prec B$. $\square$

We now switch to the special case of $P = B_n$. Define the following linear maps:

$$U_i : \mathbb{R}(B_n)_i \to \mathbb{R}(B_n)_{i+1}, \quad U_i(A) = \sum_{A\prec B\in(B_n)_{i+1}} B,\ (\forall)A \in (B_n)_i$$
$$D_i : \mathbb{R}(B_n)_i \to \mathbb{R}(B_n)_{i-1}, \quad D_i(A) = \sum_{B\prec A, B\in(B_n)_{i-1}} B,\ (\forall)A \in (B_n)_i$$

Note that U_i and D_i are completely defined, since they are defined on a basis of $\mathbb{R}(B_n)_i$. Also note that $[U_i] = [D_{i+1}]^t$, where $[U_i]$ is the matrix of U_i in the standard bases. Finally, U_i is trivially raising.

Lemma 36. *For $i < n/2$, the map U_i is injective.*

Note: Combined with the previous lemmas, this gives another proof that B_n is Sperner.

Proof: We will first prove that $D_{i+1}U_i - U_{i-1}D_i = (n-2i)I_{|P_i|}$. It suffices to prove the functional identity for elements of the basis, i.e. when applied to all $A \in P_i$. Consider the coefficient of B in $(D_{i+1}U_i - U_{i-1}D_i)(A)$. This corresponds to the number of paths from A to B, taken through up or down moves. We have three cases:

- $B = A$: the map U_i could add any one of $n - i$ elements, and the map D_i must remove this element; similarly, the map D_i must remove one of the i elements, and U_{i-1} must add it back. So the coefficient is $n - 2i$.
- $|B \cap A| = i - 1$: one element must be removed from A and another added to B. So the coefficient in both $D_{i+1}U_i$ and $U_{i-1}D_i$ is 1, and the final coefficient is zero.
- $|B \cap A| < i - 1$: the coefficient is zero, because there is no way to reach B from A by two changes.

We have thus proved that $D_{i+1}U_i - U_{i-1}D_i = (n - 2i)I_{|P_i|}$. This can be rewritten as $U_i^t U_i = (n - 2i)I + U_{i-1}U_{i-1}^t$. Now note that $U_{i-1}U_{i-1}^t$ is positive semidefinite. Since $i < n/2$, we have that $(n-2i)I$ is positive definite, so $U_i^t U_i$ must be positive definite. This means that $U_i^t U_i$ has full rank, so U_i is injective. $\square$

The rest of the proof that B_n/G is Sperner is omitted.

4 Young Diagrams

- a partition λ of n is a decreasing sequence of non-negative integers which sum to n: $(\lambda_1 \geq \lambda_2 \geq \cdots \geq 0), \sum \lambda_i = n$.
- let $L(m,n)$ be the set of partitions with at most m nonzero parts, in which each part is at most n. The diagram associated with a partition in $L(m,n)$ fits in a rectangle of $m \times n$.
- we can organize $L(m,n)$ as a poset with the partial order: $\lambda \preceq \mu \iff (\forall) i : \lambda_i \leq \mu_i$. From a graphical point of view, $\lambda \preceq \mu$ iff we can get from μ to λ by removing some squares.
- $L(m,n)$ is isomorphic to $L(n,m)$ – just transpose the rectangle.
- $|L(m,n)| = \binom{m+n}{n}$ because the boundary of the Young diagram is a walk from $(0,0)$ to (m,n).
- $L(m,n)$ is a graded poset, with the rank being the sum of the partition (the number of filled boxes).
- the longest chain has length $m+n$ (add a new box each time).

Theorem 37. *The poset $L(m,n)$ is Sperner, and the largest antichain consists of partitions of $\lfloor mn/2 \rfloor$.*

Proof: We show that $L(m,n) \cong B_{mn}/G$ for a certain group G. The group G is generated by the following permutations: the transposition of any two rows (this is actually the product of n transpositions), and the transposition of any two elements in the same row. Given any subset of $[mn]$, which is a subset of the cells in the $m \times n$ rectangle, we can sort it using permutations in G to reach a Young diagram: first arrange all filled cells in each row at the beginning of the row, and then permute the rows in decreasing order of the number of filled cells. So there is at least one Young diagram in each orbit. Also, this rearrangement is uniquely specified, so there is exactly one Young diagram in each orbit, so $B_{mn}/G \cong L(m,n)$. □

Let $P_i(m,n)$ be the size of the i-th level of $L(m,n)$. We now develop a way to find the P_i's, based on q-nomial coefficients. Let q be a nondeterminate; define:

- $[j] = 1 + q + q^2 + \cdots + q^{j-1}$;
- $[k]! = [1] \cdot [2] \cdot \cdots \cdot [k]$;
- $\begin{bmatrix} k \\ j \end{bmatrix} = \frac{[k]!}{[j]![k-j]!}$;
- plugging $q = 1$, we get $[k]! = k!$, $\begin{bmatrix} k \\ j \end{bmatrix} = \binom{k}{j}$;

Theorem 38 (Pascal's formula). *The q-nomial coefficients satisfy $\begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} m-1 \\ n \end{bmatrix} + \begin{bmatrix} m-1 \\ n-1 \end{bmatrix} \cdot q^{m-n}$.*

Note: This shows that q-nomial coefficients are polynomials in q with integer coefficients.

Proof: Computation. □

Theorem 39. *We have: $\sum P_i(m,n) q^i = \begin{bmatrix} m+n \\ m \end{bmatrix}$.*

Proof: Let $P(m,n) = \sum P_i(m,n) q^i$. If we prove $P(m,n) = P(m,n-1) + q^n P(m-1,n)$, we are done because this is also the recursion for the q-nomial coefficients. To establish this relation, compare the coefficient of q^i on both sides. We get $P_i(m,n) = P_i(m,n-1) + P_{i-n}(m-1,n)$. The left-hand side counts the number of partitions of i which fit in an $m \times n$ rectangle. A partition either has all parts strictly smaller than n, in which case we can remove the last column and get $P_i(m,n-1)$, or has at least one part equal to n, in which case we can remove this part and get $P_{i-n}(m-1,n)$. □

5 Application: Subsets With The Same Sum

For given n and k, we want to find a set $S \subset \mathbb{R}^+$, $|S| = n$ and a number α, such that the number of k-subsets of S of sum α is maximized.

Assume $S = [n]$; we now seek to compute the best α. Consider a subset $\{i_1 < \cdots < i_k\} \subset S$ with $\sum i_j = \alpha$. We can consider the set $\{i_1 - 1 \le i_2 - 2 \le \cdots \le i_k - k\}$ which will have sum $\sum(i_j - j) = \alpha - \binom{k+1}{2}$. Such subsets are actually partitions of $\alpha - \binom{k+1}{2}$ that are in $L(k, n-k)$. By the previous result, the number of partitions is maximized when:

$$\alpha - \binom{k+1}{2} = \left\lfloor \frac{k(n-k)}{2} \right\rfloor \Rightarrow \alpha = \left\lfloor \frac{k(n+1)}{2} \right\rfloor$$

We now prove that one cannot have more subsets with the same sum even for arbitrary S. Consider some $S = \{a_1 < a_2 < \cdots < a_n\}$ and some α. A subset $\{a_{i_1}, \ldots, a_{i_k}\}$ of sum α can be described by the partition $(i_1 - 1, i_2 - 2, \ldots, i_k - k)$ which is a partition in $L(k, n-k)$. Now consider another subset of sum α, giving the partition $(j_1 - 1, j_2 - 2, \ldots, j_k - k)$. Observe that these two partitions cannot be comparable. Indeed, if $(\forall)t : i_t \le j_t$ (and the inequality is strict at least once), then $a_{i_t} \le a_{j_t}$ (and the inequality is strict at least once) so $\sum a_{i_t} < \sum a_{j_t} \Rightarrow \alpha < \alpha$. So the subsets give an antichain of $L(k, n-k)$. By the Sperner property, no antichain can be larger than the largest level, and for $S = [n]$ we managed to achieve exactly the size of the largest level of $L(k, n-k)$, so $S = [n]$ is optimal.

6 Polya Theory

We want to count the number inequivalent colorings of a set X under the action of a subgroup G of the permutation group on X. Two colorings are equivalent if a permutation can take one to the other (so inequivalent colorings are orbits under G). Let $K(i_1, i_2, \ldots)$ be number of colorings using the first color i_1 times, the second color i_2 times and so on; this is only defined for $\sum i_k = |X|$. Let $c(\sigma, \ell)$ be the number of cycles of σ of length ℓ. Define:

$$Z_G(z_1, z_2, \ldots) = \frac{1}{|G|} \sum_{\sigma \in G} \prod_{\ell=0}^{|X|} z_k^{c(\sigma,\ell)}$$

Theorem 40 (Polya). *Let $r_1, r_2, \ldots$ be indeterminates. We have:*

$$\sum_{i_1+i_2+\cdots=|X|} K(i_1, i_2, \ldots)\ r_1^{i_1} r_2^{i_2} \ldots = Z_G(r_1 + r_2 + \ldots,\ r_1^2 + r_2^2 + \ldots,\ \ldots)$$

Corollary 41. *The number of inequivalent colorings using n colors (and no restriction on the number of times each color is used) is $\frac{1}{|G|} \sum_{\sigma \in G} n^{c(\sigma)}$, where $c(\sigma)$ is the number of cycles of σ.*

7 Counting Spanning Trees of Graphs

Theorem 42 (Cauchy-Binet formula). *If $A \in \mathrm{Mat}(m \times n)$ and $B \in \mathrm{Mat}(n \times m)$, with $n \le m$, then:*

$$\det AB = \sum_{S \subset \{1,\ldots,m\}, |S|=n} \det A_s \det B_s.$$

Proof: Use the identity:

$$\left| \begin{pmatrix} I_n & A_{nm} \\ 0_{mn} & I_m \end{pmatrix} \right| \cdot \left| \begin{pmatrix} A_{nm} & 0_n \\ -I_m & B_{mn} \end{pmatrix} \right| = \left| \begin{pmatrix} 0_{nm} & (AB)_n \\ -I_m & B_{mn} \end{pmatrix} \right| = \pm \det AB$$

□

Definition 43. *Let G be a loopless graph on n vertices. The Laplacian of G is:*

$$L(G) = \begin{pmatrix} \deg v_1 & 0 & \dots & 0 \\ 0 & \deg v_2 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \deg v_n \end{pmatrix} - \mathrm{Adj}(G).$$

Give an arbitrary orientation to G, $|V| = n$, $|E| = m$. Then, let M be the $n \times m$ matrix which has the vertices on the rows and the edges on the columns. For instance, if $(v_3, v_7) = e$ is an oriented edge from v_3 to v_7, then column e of M will have a 1 on the row corresponding to v_3 and a -1 on the row corresponding to v_7 and will have 0's everywhere else.

Then, $L(G) = M \cdot M^t$. Let M_0 be the matrix M with the last row removed.

Lemma 44. *Let $S \subset \{1, ..., m\}, |S| = n - 1$. Then:*

$$det(M_0)_s = \begin{cases} \pm 1 & \textit{if the edges of } G \textit{ corresponding to the columns indicated by } S \textit{ form a spanning tree} \\ 0 & \textit{otherwise} \end{cases}$$

Proof: If the edges don't form a tree, then they must contain a cycle and so, there must exist a linear combination of ± 1 times the columns corresponding to them which equals $0 \Rightarrow$ the determinant is 0.

If they form a spanning tree, then we can prove by induction that the determinant is ± 1. Assume it is true for $|S| = n - 2$ and we will prove for $n - 1$. Vertex v_n must be covered by the spanning tree, so there exists an edge in this tree which enters it $\Rightarrow$ there is a -1 on that column on the final row. So, in M_0, there is only a 1 on that column, say on row v_k. Then, if we expand the determinant of M_0 by that column, the matrix that we're left with is the same as M_0 but for a graph without the vertex v_k, and for which the columns of S left form a spanning tree; and so we can apply the induction hypothesis. ☐

Corollary 45. *The number of spanning trees of G is exactly* $\det(M_0 \cdot M_0^t)$.

Proof: Using the Cauchy Binet formula and the previous result, we get:

$$\det M_0 \cdot M_0^t = \sum_{S \subset \{1,...,m\}, |S| = n-1} \det(M_0)_s \det(M_0^t)_s = \sum \det(M_0)_s^2$$

which is precisely the number of spanning trees. ☐

Corollary 46. *Let L_0 be L with the last column and last row removed. Then, the number of spanning trees is given by* $\det L_0$.

Proof: This is clear by just multiplying M_0 and M_0^t. ☐

Corollary 47. *The number of spanning trees of G is $\frac{1}{n}\mu_1 \dots \mu_{n-1}$, where $\{\mu_i\}$ are the non-zero eigenvalues of L.*

Proof: Let's compute $det(L - xI)$. We know that the sum of all rows and of all columns in L is equal to zero. Then, adding all rows to the last one and all columns to the last one will not change the determinant and we get:

$$\det(L - xI) = \det \begin{pmatrix} & & & -x \\ & L_0 - xI & & \vdots \\ & & & -x \\ -x & \dots & -x & -nx \end{pmatrix}$$

Then, the coefficient of x is $-n\det L_0$. However, if we think about the polynomial as $\det(L-xI) = (\mu_1 - x)\ldots(\mu_n - x)$, where the μ_i's are the eigenvalues of L, then the coefficient of x is the product of the non-zero eigenvalues of $L \Rightarrow$

$$detL_0 = \frac{1}{n}\mu_1 \ldots \mu_{n-1}.$$

□

7.1 Examples – Special Cases

G is a regular graph. Then, the degree of each vertex is the same and equals d and $L(G) = d \cdot I_n - \mathrm{Adj}(G)$. The eigenvalues of L are $\{d - \lambda_i\}$, where $\{\lambda_i\}$ are the eigenvalues of $\mathrm{Adj}(G)$.

G is a complete graph. Then, $\mathrm{Adj}(G) = J - I$ and so its eigenvalues are $\{-1, ... -1, n-1\}$. Also, $L(G) = (n-1)I - \mathrm{Adj}(G)$. Then, the number of spanning trees in G is:

$$\frac{1}{n}\prod_{i=1}^{n-1}(n-1-(-1)) = n^{n-2}.$$

$G = C_n = \{0,1\}^n$, the graph of the cube Then, $\mathrm{Adj}(G)$ has eigenvalues $n-2i$ with multiplicities $\binom{n}{i}$, and $L(G) = nI - \mathrm{Adj}(G)$, so the number of spanning trees in G is:

$$\frac{1}{2^n}\prod_{i=1}^{n}(n-(n-2i))^{\binom{n}{i}}.$$

8 Eulerian Cycles and Spanning Trees

Definition 48. *An Eulerian cycle in a general graph G is a closed path that uses every edge in G exactly once.*

Theorem 49. *Let G be a directed graph. If G, seen as a non-directed graph, is connected, and $\forall v \in V(G), \mathrm{outdeg}(v) = \mathrm{indeg}(v)$, then G has an Eulerian cycle.*

Proof: Start at a vertex v and walk in the graph, going through every edge once. The only place where we could get stuck (because G is balanced) is back at v. Now remove this entire cycle from the graph, removing the vertexes that have been completely used. We will be left with a balanced, connected graph in which, by induction, we can find an Eulerian cycle. But since the original graph was connected, then the Eulerian path must have at least a vertex in common with the cycle that we originally found. And thus, when we get to that vertex in the original cycle, we simply walk on the Eulerian cycle next and then return to complete the cycle. □

Definition 50. *An oriented tree T with a root v_0 is a directed tree such that from every vertex of T there is a unique path to v_0.*

Definition 51. *Let G be a directed graph, e an edge and v a vertex. Then, $\epsilon(G,e)$ is the number of Eulerian cycles in G that start with e. $\tau(G,v)$ is the number of oriented spanning trees of G, rooted at v.*

Theorem 52. *If G is a balanced and connected, directed graph, e and edge and v is the initial vertex of e, then:*

$$\epsilon(G,e) = \tau(G,v) \cdot \prod_{u \in V(G)} (\mathrm{outdeg}(u) - 1)!$$

Proof:

"$\Rightarrow$" Consider an Eulerian cycle starting with e. For every vertex $u \neq v$, let e_u be the last edge used in the Eulerian cycle, and for which $\text{init}(e_u) = u$.

We claim that the set of $n-1$ edges e_u forms an oriented tree. We just need to show that there are no oriented cycles made out of e_u's. Assume that there is a cycle. Then, when we walk on the Eulerian path and get to vertex u for the last time, so we must take e_u out; but since e_u is part of a cycle made only of edges that can be used the last time by the Eulerian cycle, we will return to vertex u and since we already took $e_u \Rightarrow$ we are stuck. Contradiction, because we can only get stuck at v.

Thus, we showed that when we take an Eulerian path starting at e, we must fix the edge we walk on the last time we leave $u \neq v$ and we can use all the other $(\text{outdeg}(u) - 1)$ edges in any order we want, which gives us the inclusion we want.

"$\Leftarrow$" Given an oriented tree T rooted at v, we start an Eulerian cycle by taking the edge e first. At every vertex we choose arbitrarily an outgoing edge, such that the edges of the T are used last in the Eulerian tour.

We claim that we always end up with an Eulerian graph. First, note that we can only get stuck at v because the graph is balanced (as before). Now, assume there is one edge, e', which is not used in the path and let $\text{init}(e') = u$. Then, by the way we construct our path, the edge in T outgoing from u will also not be used (because it was supposed to be used last); say this enters vertex w. Since this edge is not used, then this means that we entered w less than the maximum number of times and thus, the edge in the tree outgoing from w will also not be used. This way, we get to v and conclude that v is also visited less than the maximum number of possible times, which means that we couldn't have gotten stuck at v and that we can keep going. $\square$

Corollary 53. *$\tau(G, v)$ is independent of the choice of v.*

Proof: This follows from the fact that $\epsilon(G, e)$ does not depend on e. This is because an Eulerian tour has to go through all edges once and thus, instead at starting at e, we can just rotate it and start at any other edge. $\square$

Definition 54. *If G is a directed graph, define $\overline{L}(G) \in \text{Mat}(n \times n)$ such that:*

$$\overline{l_{ij}} = \begin{cases} \text{outdeg}(v_i) & \text{if } i = j \\ -\#\text{of edges from } v_i \text{ to } v_j & \text{if } i \neq j \end{cases}$$

Note that the sum of the rows and the sum of the columns of this matrix are the zero vector.

Theorem 55. *Let $v_1, \ldots, v_n$ be the vertices of G and let $\overline{L_0}(G)$ to be the matrix $\overline{L}(G)$ with the last row and column removed. Then: $\tau(G, v_n) = \det \overline{L_0}(G)$.*

Proof: By induction on the number of edges of G. If G has $n-1$ edges, then we have the following two cases:

- there exists a vertex $v' \neq v_n$ such that $\text{outdeg}(v') = 0$, in which case there are 0 trees rooted at v_n;
- for all vertices $v \neq v_n, \text{outdeg}(v) = 1$, in which case G is an oriented tree rooted at v_n.

In both cases, we get the desired results. Now, if the number of edges $\geq n$, we have the following cases:

- if $\text{outdeg}(v_n) > 0$, then we remove one of the edges going out of v_n and we don't change the number of spanning trees rooted at v_n; also, $\overline{L_0}$ doesn't change either because we only affect the row and columns deleted from $\overline{L}$. Thus, the determinant also doesn't change and we get the same answer.
- if $\text{outdeg}(v_n) = 0$, then there must exist $v \neq v_n$ such that $\text{outdeg}(v) \geq 2$. Let $e_1, \ldots, e_k$ be all the edges going out from v, and consider the two subgraphs $G \setminus \{e_1\}$ and $G \setminus \{e_2, \ldots, e_k\}$ whose numbers we know by induction hypothesis. For these two sets, the sum of the determinants of their corresponding matrices is equal to the sum of the determinant of the original matrix because it is simply a split of the row corresponding to v into two sums. And thus we get the result.

Note that here we assumed that this formula is true even for disconnected graphs. To show that this is indeed true, assume that the graph is disconnected – then, clearly, the number of spanning trees is equal to zero. In the matrix $\overline{L_0}$, consider the connected component which does not contain v_n. Then, it is unaffected by the fact that we removed the last row and last column, and is we add the columns corresponding to this connected component, we will get 0, which means that the determinant will be 0. ☐

Corollary 56. *If $\overline{L}(G)$ has eigenvalues $\mu_1, \dots \mu_n$, with $\mu_n = 0$, then:*

$$\tau(G, v_n) = \frac{1}{n}\mu_1 \dots \mu_{n-1}.$$

Proof: This follows from the same reasoning as for the number of trees in an undirected graph. ☐

Corollary 57. *The number of spanning trees of a graph G defined as in the previous corollary is $\frac{1}{n}\mu_1 \dots \mu_{n-1}$.*

9 Brower's Fixed Point Theorem

simplex − the convex hull of $n+1$ independent points in $\mathbb{R}^n$, called vertices;
face − the convex hull of a subset of the vertices;
facet − a face of dimension $n-1$ in $\mathbb{R}^n$;
simplicial complex − a family $\mathcal{F}$ of sets $F_1, \dots F_k \subset [n]$ such that if $A \subset F_i$, then $A \in \mathcal{F}$.

Lemma 58 (Sperner). *Let T be a triangulation of a simplex in $\mathbb{R}^n$. We have a labeling of the vertices of T by the numbers $[n+1]$ such that:*

- *the $n+1$ vertices of the original simplex get pairwise different labels;*
- *let F be a face of the original simplex and $x \in F$ a vertex of T; then, x is labeled by one of the labels of the vertices of F.*

Then, there is an odd number of n-dimensional simplices in the triangulation, such that their vertices are labeled by $n+1$ pairwise different labels.

Proof: By induction on the dimension n. Assume true for $n-1$ and count the number of $(n-1)$-dimensional simplices in T with n different labels (we have $n+1$ classes of such simplices, depending on which one of the $n+1$ colors is missing).

- every n-dimensional simplex of T, with less than $n+1$ different labels has on its boundary either 0 such $(n-1)$-simplices or 2 of the same class; so far, the cardinality of the classes is even;
- denote by X the number of n-dimensional simplices in T with $(n+1)$ different labels; then, X contributes with $|X|$ to each class; thus, the cardinality of each class has the same parity as $|X|$;
- when we did this count, we counted twice all interior $(n-1)$-dimensional simplices and once all those on the boundary;
- each facet ($(n-1)$-dimensional simplex) of the original simplex is part of only one of those classes, and all $(n-1)$-dimensional simplices inside it will have the same colors as itself (will belong to the same class) and by the induction hypothesis, the number of such simplices colored by all n colors of the class is odd; thus, the cardinality of the class is odd.
- finally, $|X|$ is odd.

☐

Theorem 59 (Brower's Fixed Point Theorem). *Given $f : B^n \to B^n$ continuous, there exists $x \in B^n$ such that $f(x) = x$.*

Proof: We will do the proof using a simplex Δ^n instead of B^n. Assume there exists such a function without a fixed point from Δ^n to Δ^n. Then, we can define a new function $g : \Delta^n \to \partial\Delta^n$ such that $g(x)$ is the point on the boundary where the line $(f(x), x)$ intersects it. Clearly, g is the identity on the boundary. So, assume such a function g exists.

Then, take an arbitrarily small triangulation. Label the vertices by $n+1$ different colors and label the interior vertices of the triangulation by the color of the points on the boundary to which they are mapped by g. Also, label the points on the boundary with the color of the closest vertex. This coloring satisfies the conditions of the Sperner lemma $\Rightarrow$ there must exist an $(n-1)$-dimensional simplex colored with $n+1$ distinct colors.

g is continuous on Δ^n, so g is uniformly continuous. Take $\epsilon = \frac{1}{10}$ and take such a fine triangulation that every two vertices of it are at distance at most δ_ϵ; there will be at least one vertex which is mapped very far from the others by $g \Rightarrow$ contradiction. $\square$

10 The Borsuk-Ulam Theorem

$B^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$
$S^n =$ boundary of $B^n = \{x \in R^{n+1} \mid \|x\| = 1\}$

Lemma 60 (Tucker's Lemma). *Let T be a triangulation of an n-dimensional Gross polytope ($G =$ the convex hull of $\{-1, +1\}^n$) and let $\lambda : V(T) \to \{\pm 1, \ldots, \pm n\}$ be a labeling of the vertices antipodal on the boundary ($\lambda(v) = -\lambda(-v)$, if $v \in \partial G$). Then, there exists a 1-dimensional simplex (an edge) labeled by opposite (antipodal) numbers.*

Theorem 61 (Borsuk-Ulam – Version IV). *There exists no $F : B^n \to S^{n-1}$, antipodal on the boundary and continuous.*

Proof: Take a fine triangulation of B^n, antipodal on the boundary, in the following way: if $v \in T \Rightarrow f(v) = (a_1, \ldots, a_n) \in S^{n-1} \Rightarrow \sum a_i^2 = 1 \Rightarrow$ there exists i such that $a_i \geq \frac{1}{\sqrt{n}}$. Then, the label of v will be:

$$\lambda(v) = \min\{i \mid |a_i| \geq \frac{1}{\sqrt{n}}\} \cdot \mathrm{sign}(a_i)$$

f is antipodal on the boundary, so if $v \in \partial B^n$, then $\lambda(v) = -\lambda(-v)$. Then, by Tucker's Lemma, we have an edge (v_1, v_2) with two opposite colors $(i, -i)$. Then, $\|f(x) - f(y)\| \geq \frac{2}{\sqrt{n}}$, contradiction with the continuity of f. $\square$

Theorem 62 (Version I.). *For any $f : S^n \to \mathbb{R}^n$ continuous, there exists $x \in S^n$ such that $f(x) = f(-x)$.*

Theorem 63 (Version II.). *For any $f : S^n \to \mathbb{R}^n$ continuous and antipodal ($f(x) = -f(-x), \forall x \in S^n$), there exists $x \in S^n$ such that $f(x) = 0$.*

Theorem 64 (Version III.). *There exists no antipodal, continuous function $f : S^n \to S^{n-1}$.*

Theorem 65 (Version IV.). *There is no continuous function $f : B^n \to S^{n-1}$ antipodal on the boundary of B^n.*

Theorem 66 (Version V.). *If $F_1, \ldots, F_{n+1}$ is a closed cover of S^n, then there exist x, i such that x and $-x$ are both in F_i.*

Lemma 67. *S^{n-1} can be covered by $n+1$ sets, none of which contains two antipodal points.*

10.1 Equivalence of the Five Versions

$I \Rightarrow II$. $f : S^n \to \mathbb{R}^n$ continuous $\Rightarrow$ by I, there must exist x_0 such that $f(x_0) = f(-x_0)$. Since f is also antipodal, this implies that $f(x_0) = -f(-x_0) \Rightarrow f(x_0) = 0$.

II $\Rightarrow$ *III*. assume there exists a function $f : S^n \to S^{n-1}$, continuous and antipodal. Since $S^{n-1} \subset \mathbb{R}^n$, then by II, we have that there must exist $x \in S^n$ such that $f(x) = 0$, but such x can't exist because $0 \notin S^{n-1}$.

III $\Rightarrow$ *I*. Assume there exists $f : S^n \to \mathbb{R}^n$ continuous and $f(x) \neq f(-x), \forall x$. Then, consider the function $g : S^n \to S^{n-1}$ defined as follows:

$$g(x) = \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|}$$

Clearly, g is continuous and is antipodal, contradiction with III.

I $\Rightarrow$ *V*. Let $F_1, \ldots, F_{n+1}$ be a closed cover of S^n and define $f : S^n \to \mathbb{R}^n$ as follows: $f(x) = (d(x, F_1), \ldots, d(x, F_n))$. Clearly, f is continuous and by I we have that there must exist an $x_0 \in S^n$ such that $f(x_0) = f(-x_0)$.

- if one of the coordinates of $f(x_0) = f(-x_0)$ is zero, then $d(x_0, F_i) = d(-x_0, F_i) = 0$ and so both x_0 and $-x_0$ belong to the same set, F_i;
- if no coordinate is 0, then $x_0, -x_0 \notin F_i$, for any $i \in \{1, \ldots, n\}$ and since $F_1, \ldots, F_{n+1}$ is a cover, then both x_0 and $-x_0$ must be in F_{n+1}.

V $\Rightarrow$ *III*. Assume there exists a function $f : S^n \to S^{n-1}$, antipodal and continuous. Then, we know by the previous lemma, that S^{n-1} can be covered with $n+1$ closed sets, say $F_1, \ldots, F_{n+1}$ such that none of them contain two antipodal points. Then, $f^{-1}(F_1), \ldots, f^{-1}(F_{n+1})$ will form a closed cover of S^n. But, by V, this means that there must exist $x_0 \in S^n$ and i such that $x_0, -x_0 \in f^{-1}(F_i) \iff f(x_0)$ and $f(-x_0) \in F_i$ and since f is antipodal, this means that $f(x_0)$ and $-f(x_0) \in F_i$ which is a contradiction with the choice of the sets.

III $\Rightarrow$ *IV*. Assume there exists $f : B^n \to S^{n-1}$, continuous and antipodal on the boundary of B^n, which is S^{n-1}. Then, take the projection $\pi : S^n \to B^n$. It is continuous and thus the composition $g = f \circ \pi : S^n \to S^{n-1}$ is continuous. Moreover, f is antipodal on the boundary and so, for $x \in S^n$, $g(-x) = f(\pi(-x)) = f(-\pi(x)) = -f(\pi(x)) = -g(x)$. So, g is continuous and antipodal, which contradicts III.

IV $\Rightarrow$ *III*. Assume there exists $f : S^n \to S^{n-1}$ antipodal and continuous. And now construct the function $g : B^n \to S^{n-1}, g(x) = f(\pi^{-1}(x))$, where π is defined as above. Then, g is continuous and for all $x \in S^{n-1}$, $g(-x) = f(\pi^{-1}(-x)) = f(-\pi^{-1}(x)) = -f(\pi^{-1}(x)) = -g(x)$, because f is antipodal. Thus, g is antipodal on the boundary, which contradicts III.

Brower's Fixed Point Theorem via Borsuk-Ulam. Brower's fixed point theorem states that: given $f : B^n \to B^n$ continuous, $(\exists) x \in B^n$ such that $f(x) = x$. One can give a different proof based on the Borsuk-Ulam theorem.

Proof: Assume there exists no x such that $f(x) = x$. Then, construct the function $g : B^n \to S^{n-1}$ as follows: for every $x \in B^n$, let $g(x)$ be the point where the line $(f(x), x)$ intersects S^{n-1}. Thus, for $x \in S^{n-1}$, clearly, $g(x) = x$. g is continuous and, on the boundary S^{n-1}, it is antipodal since it is the identity. This contradicts IV. $\square$

11 Intersection Graphs

Definition 68. *Given m sets $F_1, \ldots, F_m \in \mathbb{R}^n$, the corresponding intersection graph will have m vertices (corresponding to the sets); we connect two vertices i and j if and only if F_i and F_j intersect.*

11.1 Large Bipartite Structures

Lemma 69. *Let $U, V \in \mathbb{R}^d$ be sets of vectors. Then, there exist $U' \subset U$ and $V' \subset V$ such that:*

- $|U'| \geq \frac{|U|}{2^d}$ *and* $|V'| \geq \frac{|V|}{2^d}$*;*
- *either* $\langle u, v\rangle \leq 0$ *for every* $u \in U'$ *and* $v \in V'$*, or* $\langle u, v\rangle > 0$ *for every* $u \in U'$ *and* $v \in V'$*.*

Note: It can be shown that in general sets U', V' larger than $\frac{|U|}{2^{\Omega(n)}}$ and $\frac{|V|}{2^{\Omega(n)}}$ cannot be found.

Theorem 70. *Let G be the intersection graph of m disks in the plane. Then G contains either a complete bipartite graph of size $c \cdot m \times c \cdot m$ or an empty bipartite graph of size $c \cdot m \times c \cdot m$ (in the sense that there exist sets S, T of size $\Omega(m)$ such that either all disks from S intersect all disks from T, or no disk from S intersects a disk from T).*

Proof: Using the previous lemma, apply the fact that two discs (x_1, y_1, r_1) and (x_2, y_2, r_2) intersect if and only if:

$$\begin{aligned} & (x_1 - x_2)^2 + (y_1 - y_2)^2 \leq (r_1 + r_2)^2 \\ \iff \quad & \langle (x_1^2, -2x_1, 1, y_1^2, -2y_1, 1, -r_1^2, -2r_1, 1), (1, x_2, x_2^2, 1, y_2, y_2^2, 1, r_2, r_2^2)\rangle \leq 0 \end{aligned}$$

And if we assign a vector of the first type to U' and one of the second type to V', we obtain the desired result, with $|U'|, |V'| \geq \frac{m}{2^9}$. □

Note: A similar result can be shown for intersection graphs of segments, by applying the lemma multiple times. Using results from algebraic geometry, this type of results can be extended to the general class of semi-algebraic sets.

11.2 Large Cliques or Anticliques

Theorem 71 (Ramsey). *Consider a coloring of the edges of K_n (the complete graph on n vertices) by two colors, red and green. Then, if $n \geq \binom{a+b-2}{a-1}$, either we have a red clique of size a or a green clique of size b.*

Proof: Induction. □

Note: This implies that any graph on n vertices contains either a clique of size $\Omega(\log n)$ or an anticlique of size $\Omega(\log n)$. The bound can be shown to be tight by the probabilistic method, but no explicit construction is known. Below we see that geometric intersection graphs have cliques or anticliques of $\Omega(n^\rho)$, which partially explains why an efficient construction for $O(\lg n)$ is hard to find.

Theorem 72. *Let $F_1, \ldots F_m$ be m disks in the plane. Then, the intersection graph contains either $\Omega(\sqrt{m})$ vertices, every two of which are connected, or $\Omega(\sqrt{m})$ vertices, no two of which are connected.*

Note: The bound of $\Theta(\sqrt{m})$ is easily seen to be tight.

Proof: Start with the smallest disk, take all its neighbors and see if there are $\sqrt{m}$ discs connected with it. If not, delete all neighbors and keep just the smallest disc and put it in a separate list. Then, pick second smallest and do the same, and so on $\Rightarrow$ $\sqrt{m}$ steps. If we end up with $\sqrt{m}$ disconnected disks, we're done.

If there exists a disk with $\sqrt{m}$ neighbors, inflate it to radius ($3\times$ its radius) $\Rightarrow$ this disk has area $9\times$ that of the original disk. Since all the other disks intersecting the smallest one have area greater than it, then any of those disks covers at least 1/9 of the area of the inflated disk. So, the area covered by all neighboring disks is at least $\frac{1}{9}\sqrt{m}$ (considered only inside the big disk). However, the area of the entire disk is 1, so there is at least a point which is covered by $\frac{1}{9}\sqrt{m}$ disks, so they all intersect. □

Theorem 73. *Let G be the intersection graph of n segments in the plane. Then either G contains either a clique or an anticlique of size $\Omega(n^\rho)$, where ρ is a positive constant.*

Proof: Define a family $\mathcal{F}$ of graphs as follows:

- the graph on one vertex is in $\mathcal{F}$;
- if $G_1, G_2 \in \mathcal{F}$, then the graph formed by the disjoint union of these two graphs is also in $\mathcal{F}$ (by disjoint union, we mean that G_1 and G_2 are separate components of the new graph) – call this the union operation;
- if $G_1, G_2 \in \mathcal{F}$, then the graph with G_1 in one party, G_2 in another party, and all edges between the two parties, is in $\mathcal{F}$ – call this the join operation.

For each $G \in \mathcal{F}$, the chromatic number of G is equal to the size of the largest clique. This follows easily by induction: the union operation makes $\chi(G) = \max(\chi(G_1), \chi(G_2))$, and the largest clique is the max of those in G_1 and G_2; the join operation makes $\chi(G) = \chi(G_1) + \chi(G_2)$ and the largest clique is formed by the union of the cliques in G_1 and G_2 (since we add all possible edges from G_1 to G_2).

Now observe that a graph on n vertices from $\mathcal{F}$ contains either a clique or an anticlique of size $\Omega(\sqrt{n})$. Indeed, if the largest clique is $< \sqrt{n}$, then the graph can be colored with $< \sqrt{n}$ colors. Then a color class has $> \sqrt{n}$ nodes, which form an anticlique.

Finally, we show that the intersection graph contains a large induced subgraph (of size $\Omega(n^\rho)$) which is in $\mathcal{F}$. By a previous lemma, we can find two large sets S, T (of size $\Omega(n)$), such that either all vertices in S are connected to all in T, or all are not connected. Recursively, we can find large subgraphs from $\mathcal{F}$ with vertices in S and T. Now we either take the union or join of these two subgraphs, so we stay in $\mathcal{F}$. The recursion for the size of the subgraph is $S(n) = 2S(\Omega(n))$, which solves to $S(n) = n^{\Omega(1)}$. $\square$