The Geometry of Binary Search Trees

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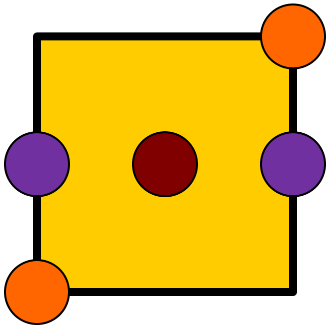
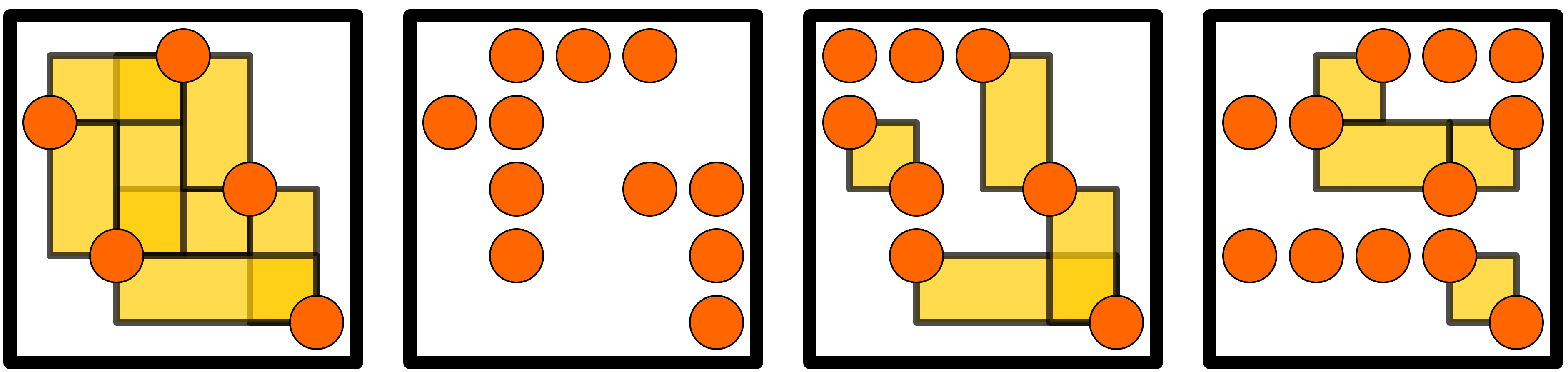
ASS

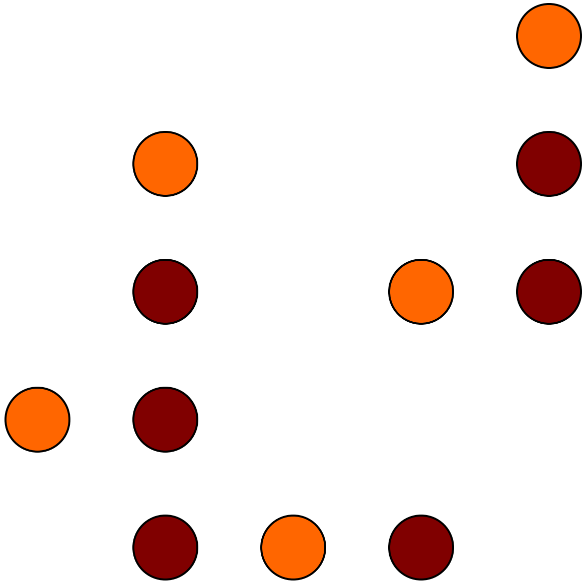
- Point set in the plane
- **ASS** = any nontrivial rectangle spanned by two points contains another point (interior or on boundary)





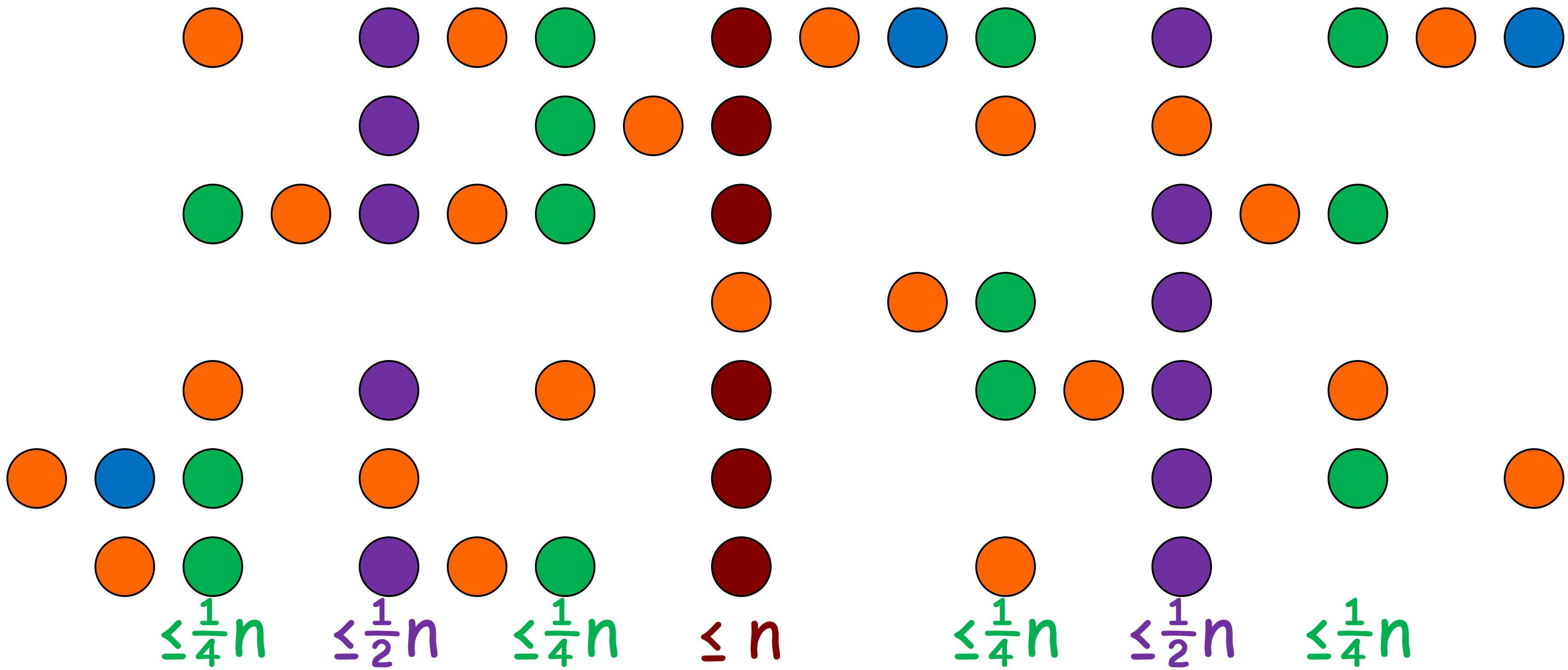
MinASS Problem

- **Problem:** Given a point set, find the minimum ASS superset
  - Up to constant factors, can assume input points are in **general position** (no two horiz/vert aligned)



MinASS in Worst Case

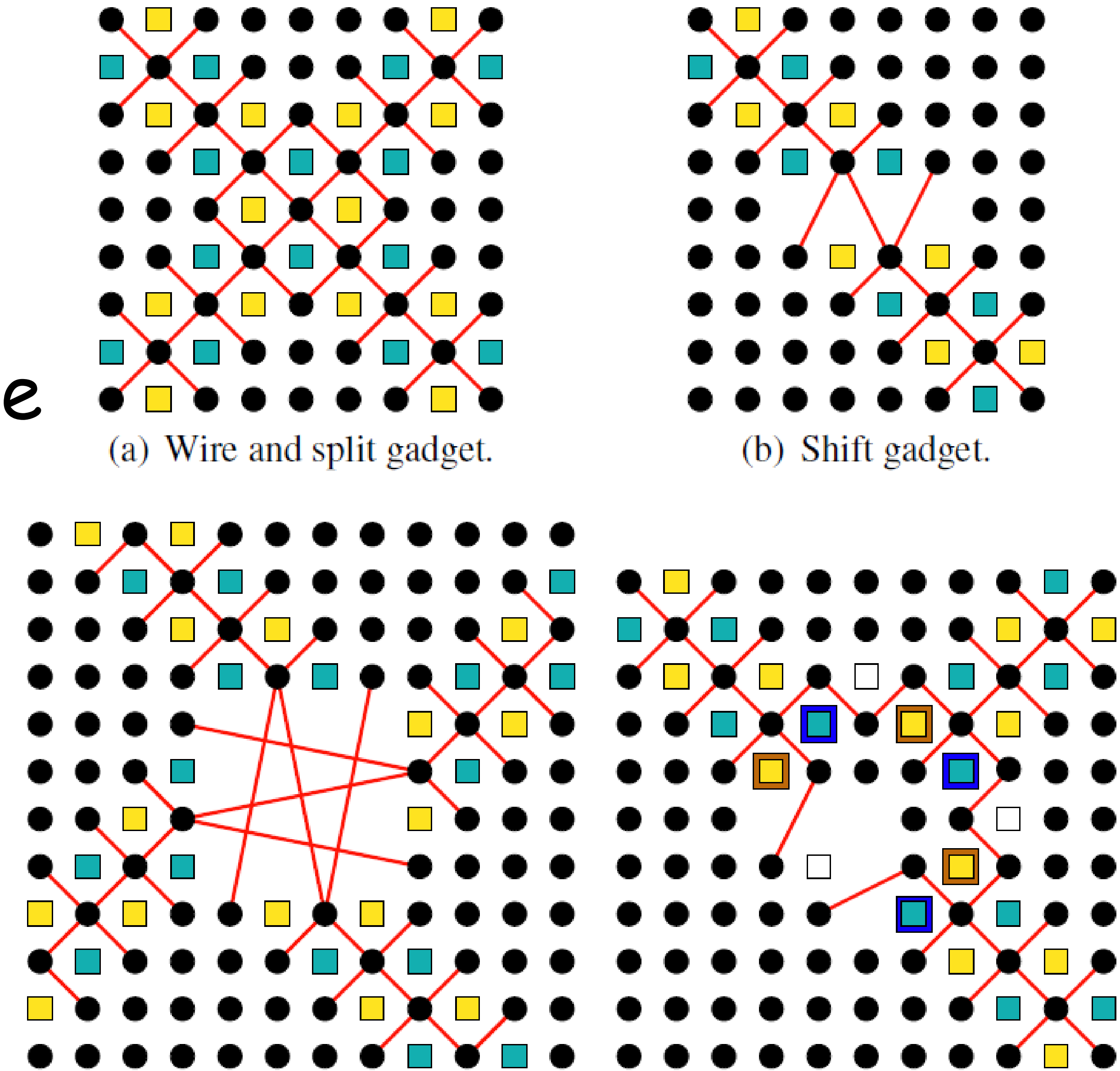
- $O(n \lg n)$ points always suffice



- $\Omega(n \lg n)$ points are sometimes necessary (random; bit-reversal permutation matrix)

NP-completeness

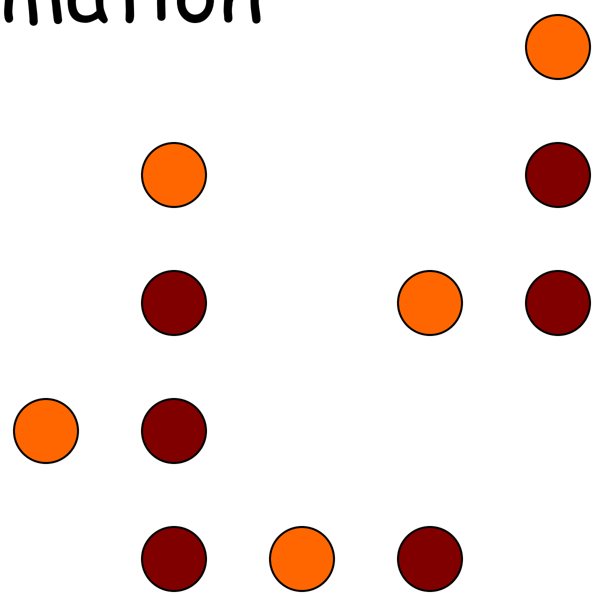
- **Theorem:** MinASS is NP-complete
- **OPEN:** General position MinASS



(a) Wire and split gadget. (b) Shift gadget. (c) Crossover gadget. (d) Not-All-Equal clause gadget.

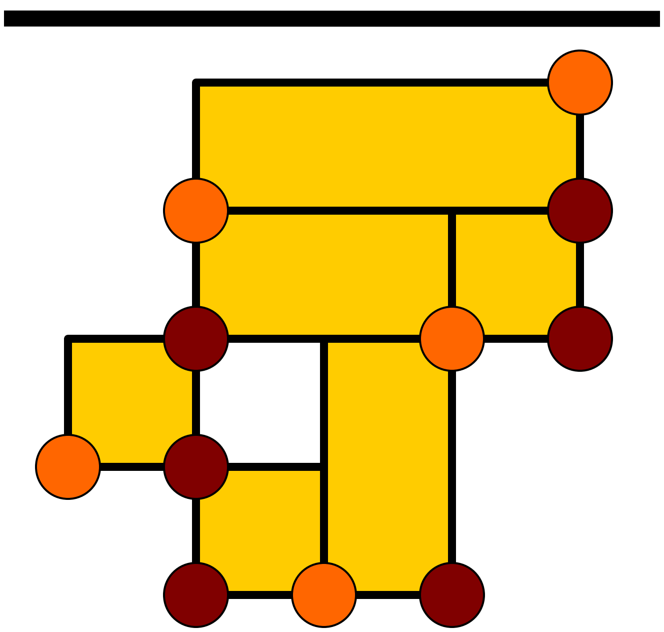
Approximating MinASS

- **OPEN:** O(1)-approximation?
- **Known:** O(lg lg n)-approximation
  [and it's not easy]



GreedyASS

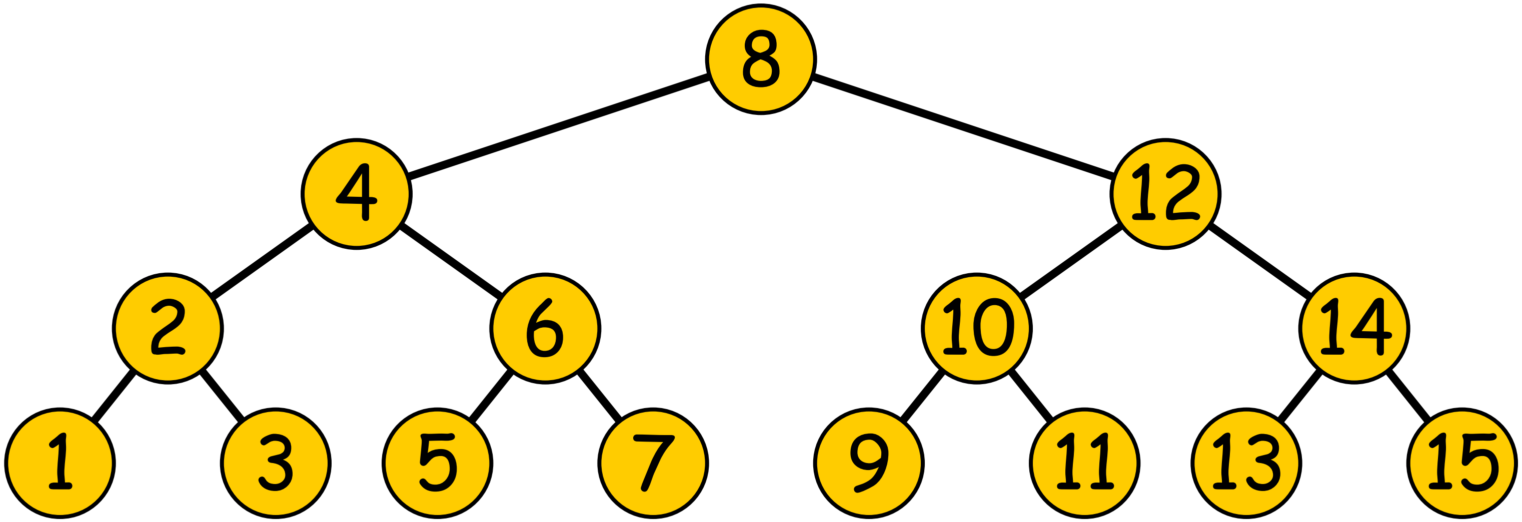
- Imagine points arriving row by row
- Add necessary points on the new row to remain ASS



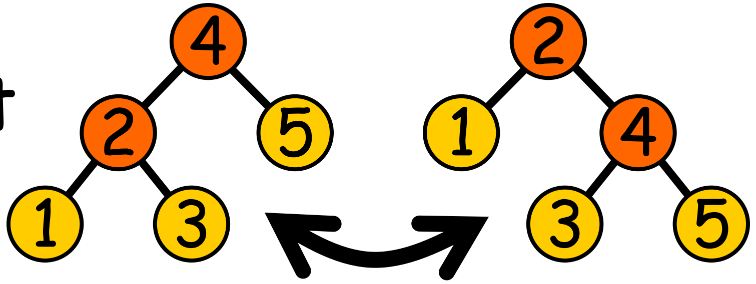
- **Conjecture:** O(1)-approximation

Binary Search Tree (BST)

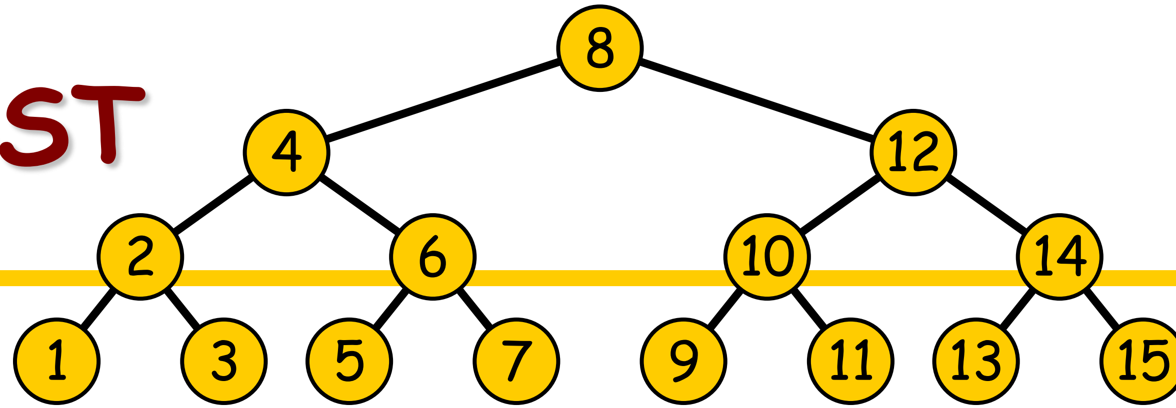
- Recall our good old friends:



- Unit-cost operations:
  - Move finger up/left/right
  - Rotate left/right



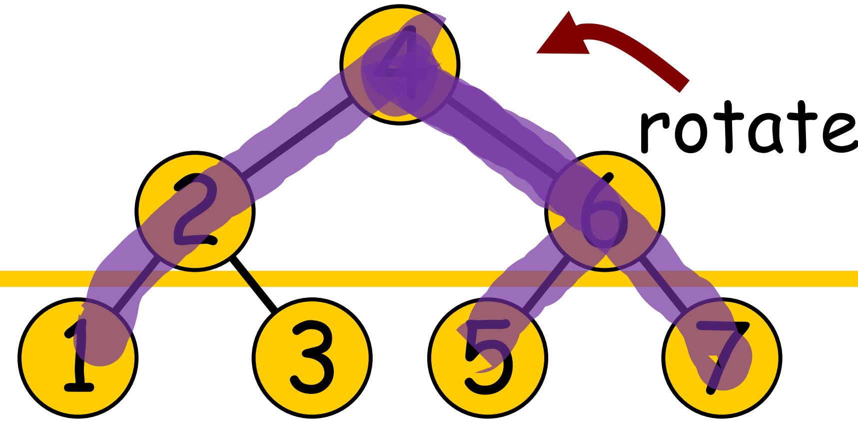
The Best BST

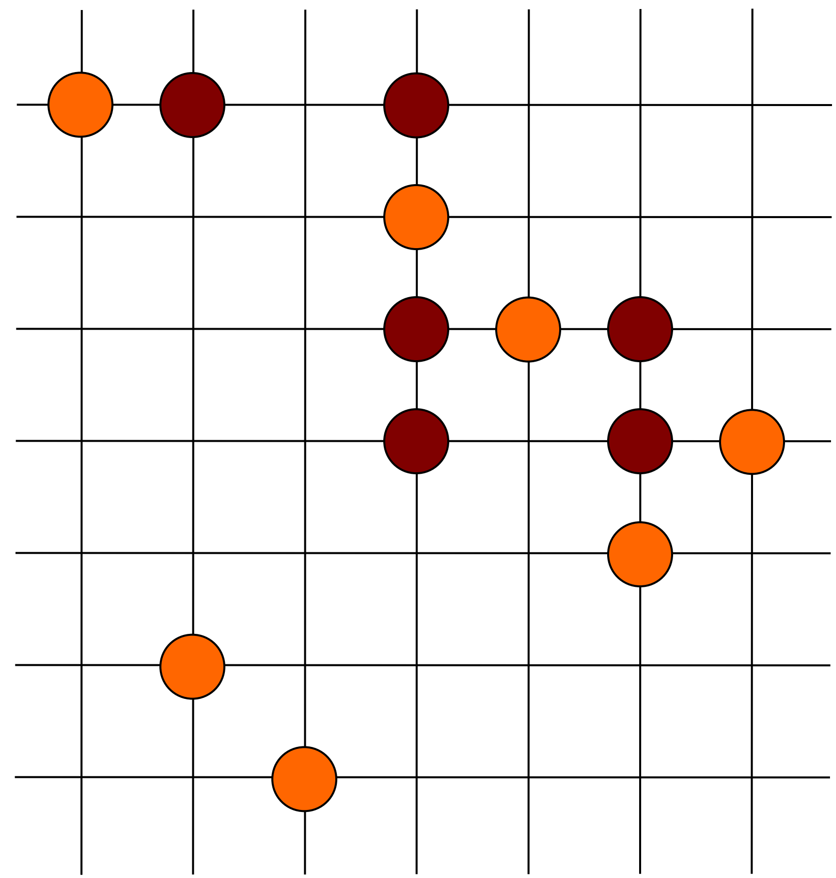


- **Problem:**
  - Given (offline) access sequence $S = x_1, x_2, \ldots, x_m$
  - OPT(S) = minimum sequence of unit-cost ops in BST to touch $x_1, x_2, \ldots, x_m$ in order
- Without rotations, this problem is solved by (static) optimal BSTs of Knuth [1971]
- **Ultimate goal:** O(1)-competitive online algorithm

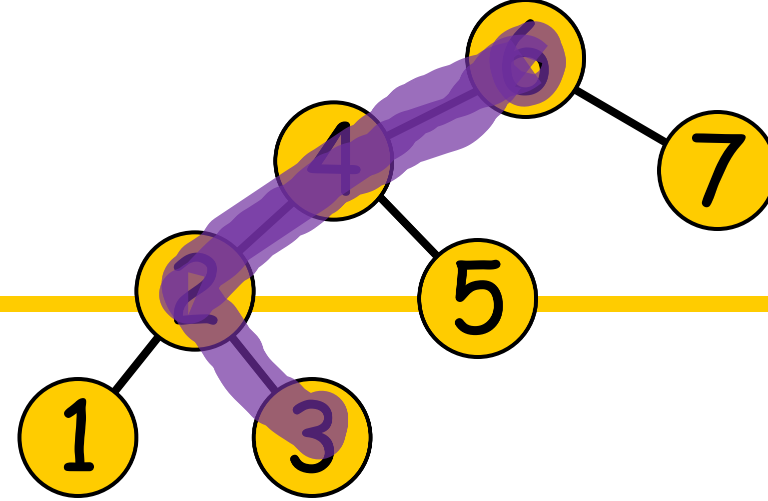
Example of BST Execution

- Access sequence: 1, 4, 5, 7, 6, 2, 3





Example of BST Execution



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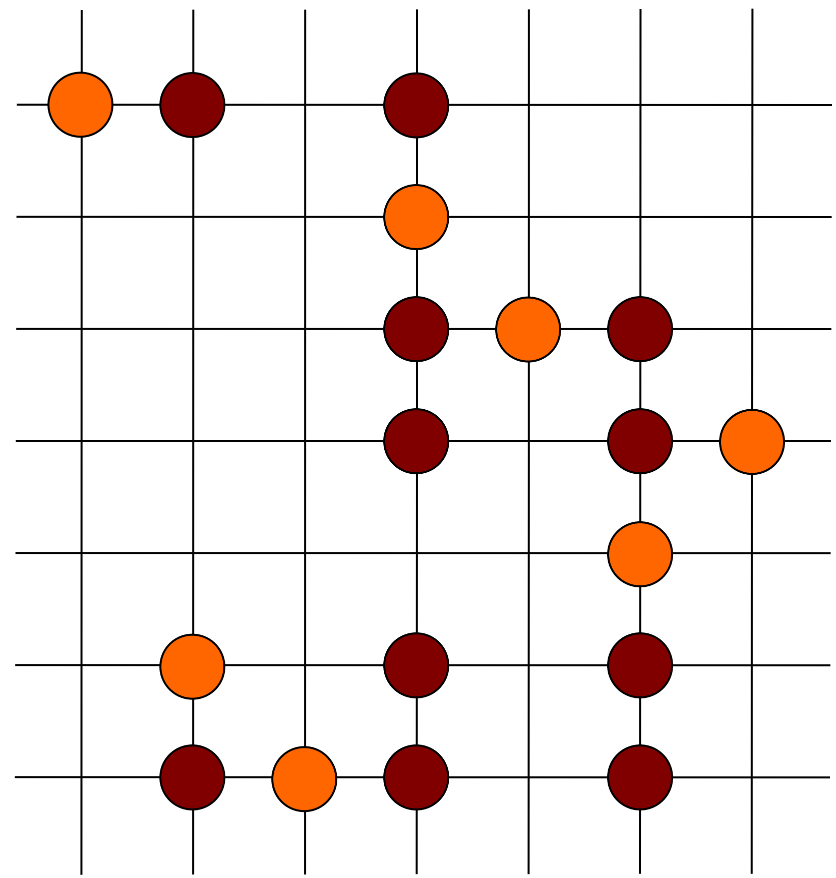
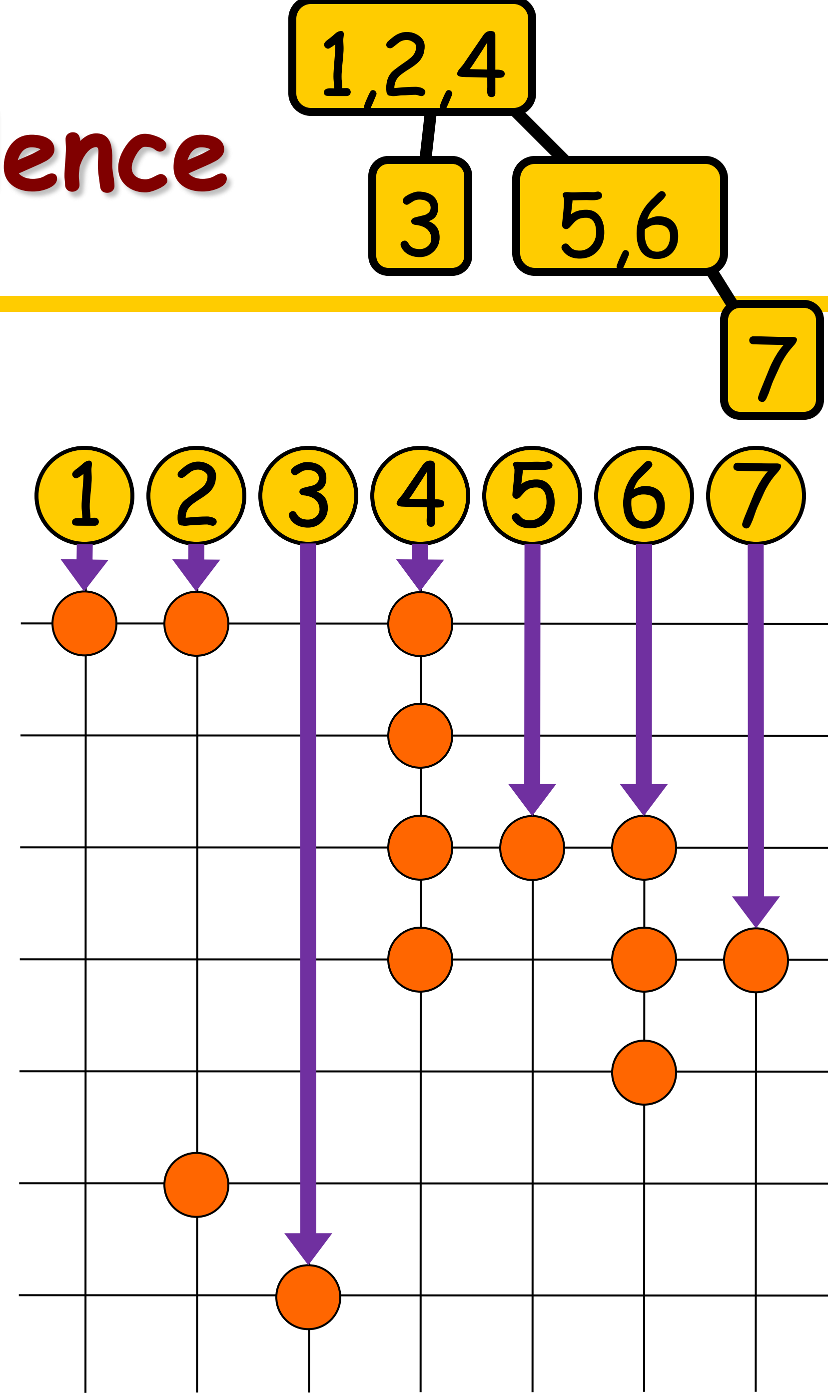


ASS/BST Equivalence

- In fact, any ASS point set is a BST execution!
  - Treap by next access time
- **Corollary:** MinASS(S) = OPT(S)

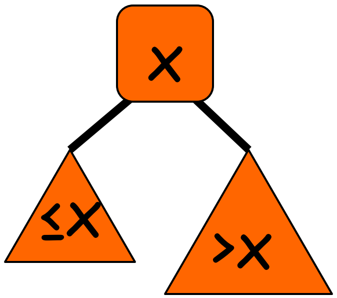


Attacks on OPT(S)

- **Splay trees** [Sleator & Tarjan 1983]
  - **Conjecture:** O(1)-competitive
  - Many nice properties known, but no o(lg n)-competitiveness known
- **Tango trees** [Demaine, Harmon, Iacono, Pătraşcu 2004]
  - O(lg lg n)-competitive
- **GreedyASS** [Lucas 1988; Munro 2000]
  - Proposed as offline BST algorithm ("Order by Next Request")
  - No o(n)-competitiveness known

Online ASS/BST Equivalence

- **Theorem:** If ASS sequence computed online row by row (like GreedyASS), then can convert to an online BST algorithm with constant-factor slowdown
  - Transform to geometry and back
- Main ingredient: **split trees**
  - Support any sequence of n splits in O(n) time
  - Splay trees take $O(n\ \alpha(n))$ time [Lucas 1988]
  - Splay trees take $O(n\ \alpha^*(n))$ time [Pettie 20??]



New Dynamic Optimality Conjecture

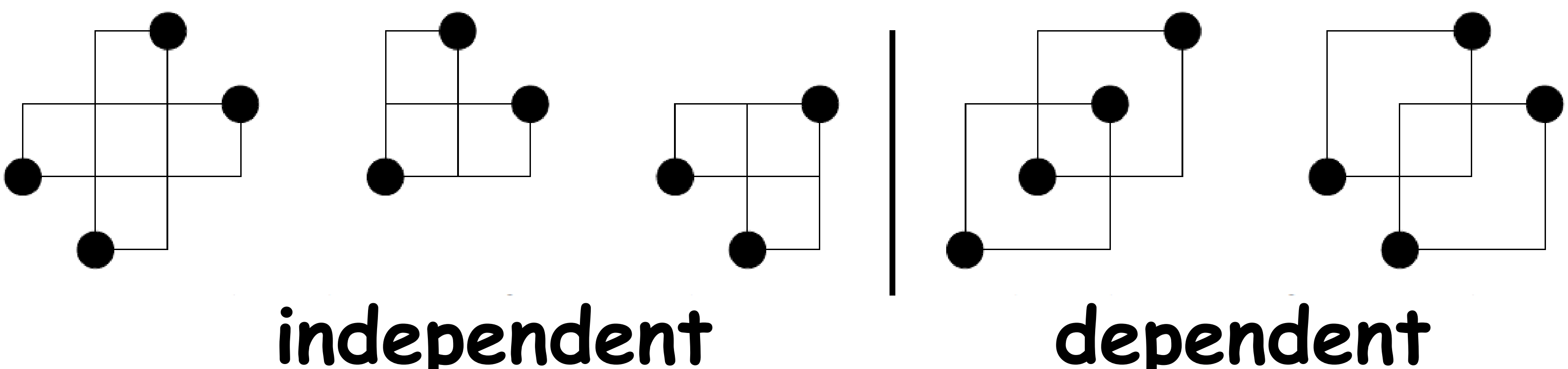
- GreedyASS is now an online algorithm!
- **Conjecture:**
  GreedyASS is O(1)-competitive
- Previously conjectured to be an offline O(1)-approximation to OPT
  [Lucas 1988; Munro 2000]

Lower Bounds

- Need lower bounds on OPT(S) to compare algorithms (like GreedyASS) against
- Wilber [1989] proved two lower bounds:
  - Wilber I used for O(lg lg n)-competitiveness of Tango trees
  - Wilber II used for key-independent optimality [Iacono 2002]
  - **Conjecture:** Wilber II ≥ Wilber I
  - **Conjecture:** OPT(S) = Θ(Wilber II)

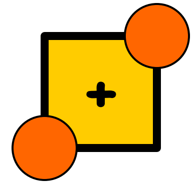
Independent Set Lower Bounds

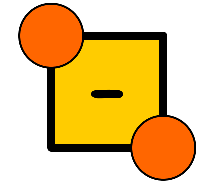
- Wilber I and Wilber II fall in the (new) class of **independent rectangle bounds**



- **Theorem:** MinASS(S) = Ω(largest independent rectangle set)
- What is the best lower bound in this class?

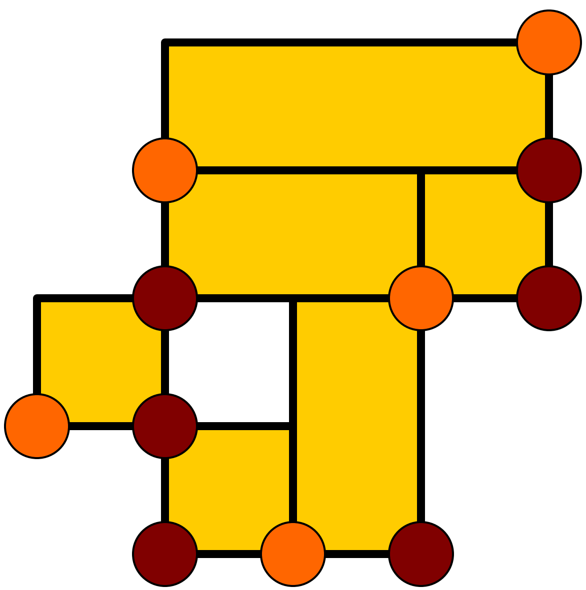
SignedGreedy



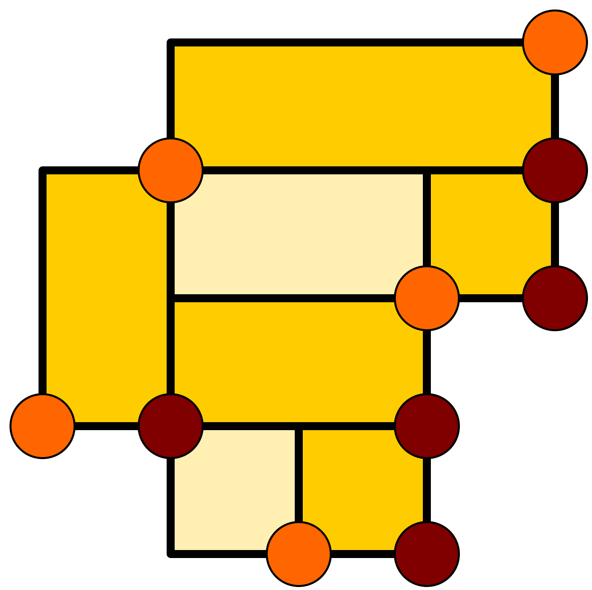


- Max of:

	Just fix empty positive-slope rectangles	Just fix empty negative-slope rectangles



original points
added points



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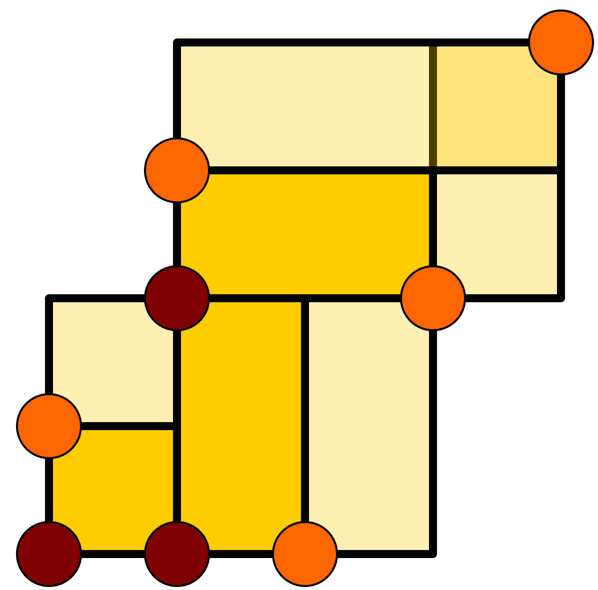


original points
added points

Lower Bounds: SignedGreedy

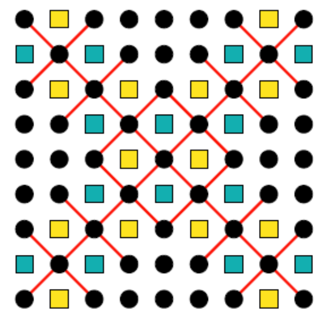
- **Theorem:** SignedGreedy(S) is within a constant factor of the best independent rectangle bound
- **Corollary:**
  - OPT(S) ≥ SignedGreedy(S)
  - SignedGreedy(S) = Ω(Wilber I + WilberII)
- SignedGreedy (***lower bound***) is annoyingly similar to GreedyASS (***upper bound***)

Open Problems

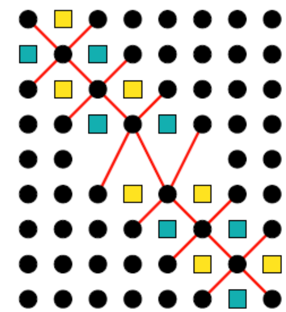
- Does GreedyASS share all the nice properties of splay trees?
  - **Recent result:** [Iacono & Pătraşcu] GreedyASS satisfies access theorem (from splay trees)
    $\Rightarrow$ working-set bound
    $\Rightarrow$ entropy bound
    $\Rightarrow$ static finger bound
    $\Rightarrow$ $O(\lg n)$ amortized per operation
  - Anyone for dynamic finger?

Open Problems

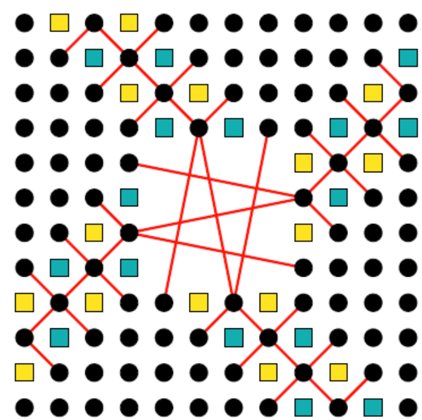
- Hardness of approximability
- NP-hardness of general position



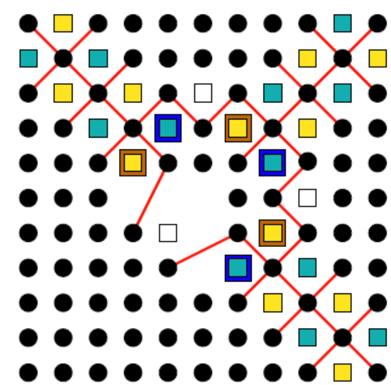
(a) Wire and split gadget.



(b) Shift gadget.



(c) Crossover gadget.



(d) Not-All-Equal clause gadget.

P.S.

- **ASS** = Arborally Satisfied Set
- [**Arboral** = arboreal = related to trees]