Lower Bounds for Asymmetric Communication Channels and Distributed Source Coding

Micah Adler\textsuperscript{1}  Erik D. Demaine\textsuperscript{2}  Nicholas J. A. Harvey\textsuperscript{2}  Mihai Pătraşcu\textsuperscript{2}

\textsuperscript{1}University of Massachusetts, Amherst
\textsuperscript{2}MIT

SODA 2006
Data Transmission

Client $\rightarrow$ Server

Send $s \in \{0, 1\}^n$

$s \leftarrow D$, $H(D) < n$

Client sends $\sim H(D)$ bits

$k$ clients $\rightarrow$ 1 server

Send $s_1, \ldots, s_k \in \{0, 1\}^n$

$(s_1, \ldots, s_k) \leftarrow D$ (correlated!), $H(D) < nk$

Clients send $\sim H(D)$ bits in total
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Cost of client not knowing $D$:

- 1 communication by server – optimal
- 2 rounds – quasioptimal

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*Adler, Demaine, Harvey, Pătrașcu*  
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The class of hard distributions $D$
Intuition for hardness

Let $h = \text{height of one layer}$
Let $p = \text{Pr[vestigial child]}$

$\implies H(D) = ph + (1 - p)ph + (1 - p)^2 ph + \ldots$

$H(D)$ is small
$\implies$ one client message cannot talk about many layers \textit{for many samples}

Random choice of vestigial child (left / right)
$\implies$ don’t know which samples need many layers
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Communication Complexity Tools

Message switching

Alice sends a message of $\leq a$ bits
⇒ eliminate, increasing Bob’s message by a factor of $2^a$

Round elimination lemma

Alice gets $x_1, \ldots, x_k$
Bob gets $y, i \in [k]$
\[ \{ \text{they compute } f(x_i, y) \} \]

Alice sends a message of $a \ll k$ bits
⇒ message irrelevant for average $i$; eliminate
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Formal strategy

1. switch client’s message
   NB: need hard upper bound on message size (Markov)

2. round elimination of server’s message
   subproblems: what is below each $T$ leaf
   prefix of client’s sample chooses subproblem

3. repeat, in the smaller probability space where the sample
   is not vestigial at this level

Contradiction
Eliminated $i$ rounds by introducing “small” error
With no rounds, cannot solve better than random guessing
Sample is at level $> i \Rightarrow$ nontrivial problem
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Trouble in paradise

- many complications and subtleties
- innovative communication complexity analysis

Example

Obtaining a hard bound for the client’s messages:

- \( \Pr[\text{sample is from level } \geq i] = (1 - p)^i \)
- error introduced must be small in this space
- hard bound (by Markov) must be huge: \( H(D)/(1 - p)^i \)
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Technical insight: Unilateral error

Regular error

Unilateral error

Application

- Markov on client’s message introduces unilateral error
- Conditioning the sample being from level $\geq i$ does not change the marginal distribution on the client’s input

$\Rightarrow$ much better Markov bound
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