On Dynamic Range Reporting in One Dimension

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Range Reporting in 1D

Maintain a set S, $|S| = n$, under:

INSERT(x): $S \leftarrow S \cup \{x\}$

DELETE(x): $S \leftarrow S \setminus \{x\}$

REPORT(a, b): return $S \cap [a, b]$

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Model: Word RAM, w-bit words

$S \subset \{0, \ldots, 2^w - 1\}$

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Alternative query

FINDANY(a, b): return any $y \in S \cap [a, b]$, or EMPTY

Updates maintain S in sorted order.
Then, just scan left or right starting with y.

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Search Problems

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$\text{MEMBER}(x)$: is $x \in S$?

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Search Problems

Range Reporting in 2D

$\text{EMPTY}([a,b]\times[c,d])$: is $S\cap([a,b]\times[c,d])=\emptyset$?

$\Downarrow$ "colored predecessor problem"

Predecessor Search

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- ... and $O(n)$ space [ABR – STOC'01]

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Dynamize these solutions $\Rightarrow$ tradeoff:

- $O(w^{\varepsilon})$ per update, $O(1)$ per query

$\updownarrow$

- $O(\lg w)$ per update, $O(\lg w)$ per query

Not so magical: converges to van Emde Boas.

We achieve:

- $O(\lg w)$ updates
- $O(\lg \lg w)$ queries
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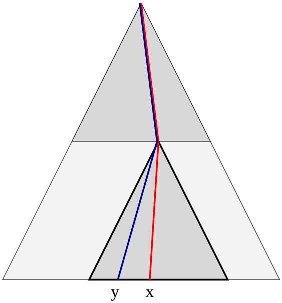
- new, subtle recursion idea
- dynamic perfect hashing in sublinear space

Van Emde Boas Recursion



Binary search for longest common prefix of x and $\textsc{pred}(x)$.

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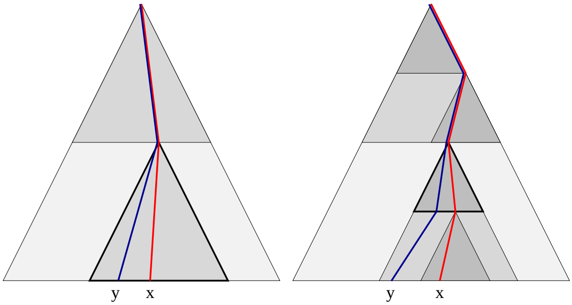

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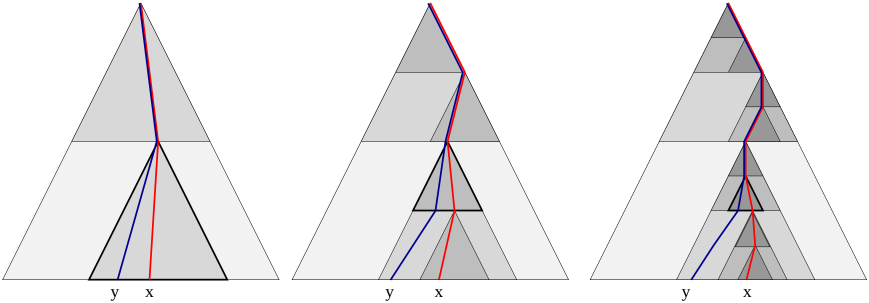

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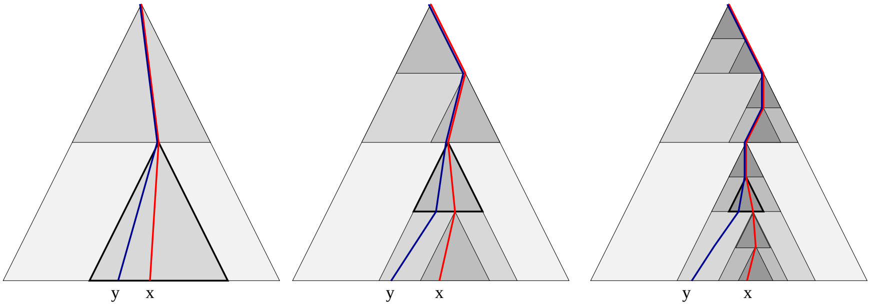

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Interpret integers at different levels of detail.

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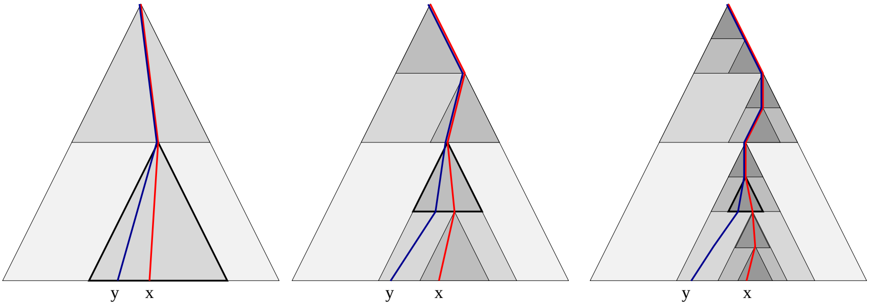

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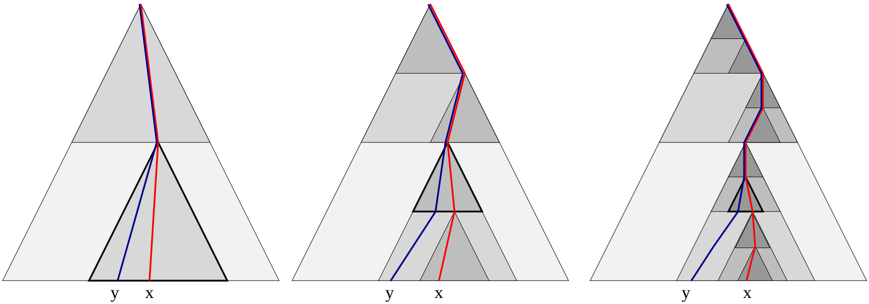

Binary search for longest common prefix of x and $\text{PRED}(x)$.

Interpret integers at different levels of detail.

- van Emde Boas examines levels of detail sequentially
- we do a binary search on the levels of detail

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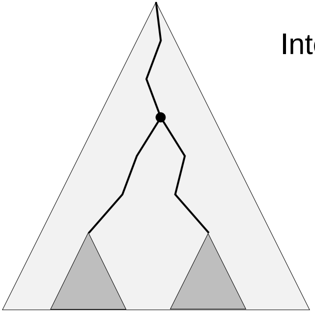

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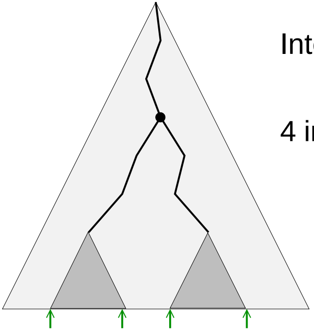

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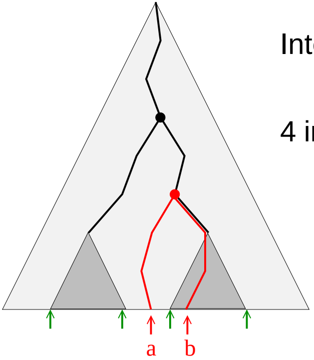

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FINDANY(a, b)

- compute LCA(a, b)
- find lowest branching ancestor of the LCA
- check if any extreme point is in $[a, b]$

Lowest branching ancestor?

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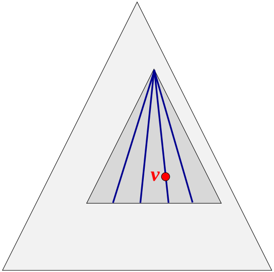
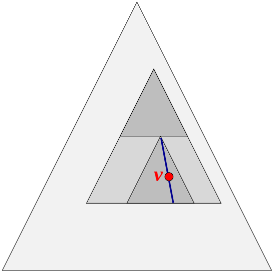
Happens only when $S \cap [a, b] = \emptyset$
$\Rightarrow$ witness verification catches the error

Binary search for level L such that:



Level $L - 1$: branching



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Binary search on levels of detail

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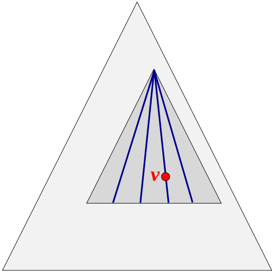

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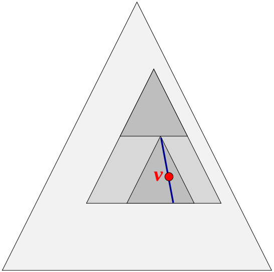

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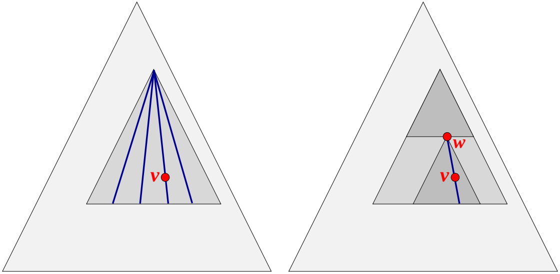

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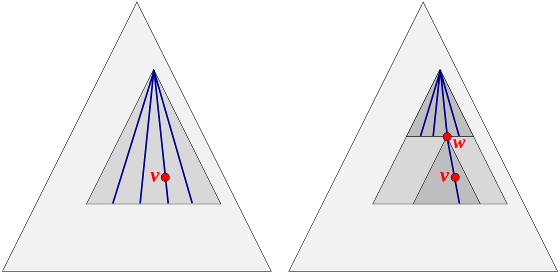

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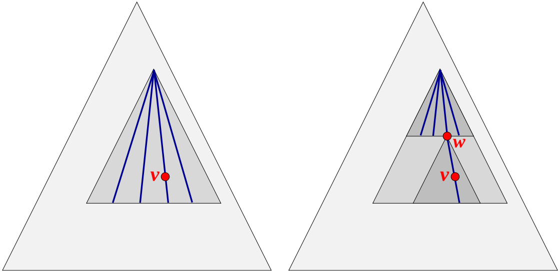

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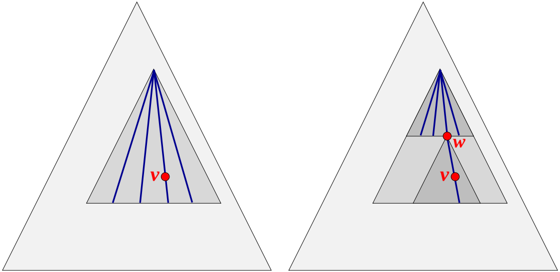

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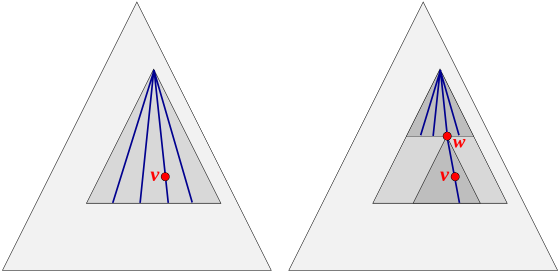

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Allow one-sided error:

- if $A[i] \neq 0$, answer must be correct
- if $A[i] = 0$, answer can be wrong

Low-Space Structures

First sublinear-space solution to dynamic Bloomier filters:
$O(n(r + \lg \lg u))$ bits

Via first sublinear-space solution to dynamic perfect hashing

We prove matching lower bounds
improves [CKRT – SODA'04]

Story Time

Almost done. . .

Story Time

$\mathcal{THE\ END}$