Planning for Fast Connectivity Updates

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...until June 2008
Jobs, anyone?

Connectivity in *Changing* Graphs

- insert(edge)
- delete(edge)

} time t_u

- connected(u,v) = are u and v in the same component?

↳ time t_q — well understood: $t_q = \Theta\left(\lg n \big/ \lg \frac{t_u}{\lg n}\right)$

Amortized: randomized $t_u = O(\lg n\,(\lg\lg n)^3)$ [T'05]

deterministic $t_u = O(\lg^2 n)$ [HLT'98]

generation gap

Worst case: deterministic $t_u = O(\sqrt{n})$ [F'83]

What's wrong with amortized?

starting with empty graph

- mathematician:

  "Nothing, but deamortization is a big challenge.
  It's hard therefore it's interesting."

- CS theorist:

  "May spend $O(n)$ per update! Bad for practice!
  And practice is always our main motivation."

- CS practitioner:

  "Does spend $O(n)$ /update at worst possible times...
  But I don't really care anyway."

Emergency Planning

Preprocess graph during good times

… when emergency comes, understand what happened quickly

more???

If one edge goes down, what happens to:

- connectivity: graph bridges
- reachability: [King-Sagert STOC'99]
- shortest path: [Hershberger-Suri FOCS'01, Roditty-Zwick ICALP'05]
- APSP: [Chowdhury-Ramachandran'02, Demetrescu-Thorup SODA'02]

=> Nice way to understand graph structure (algorithmically)

Planning for Connectivity

1. Preprocessing: graph with m edges
   time poly(m) space O(m)
2. Batched updates: d edge deletions, ~~insertions~~
   "understand connectivity" in time $O(d \lg^2 m \lg\lg m)$
   => # connected components
   => size of each connected component
   => oracle ~~~
3. Oracle query: root(v) = ID of connected component
   time $O(\lg\lg m)$ per query

optimal, actually

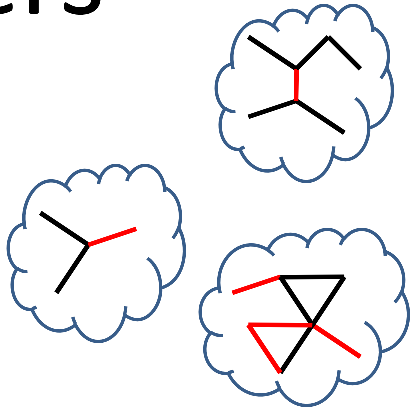
Idea 1: Don't worry, be happy

Any respectable graph is an expander…

let Φ=edge expansion

- preprocessing: I'm feeling lucky
- batched deletions: $O(d/\Phi)$ time
- oracle query: $O(1)$

Exploring Expanders



Grow components around deleted edges:

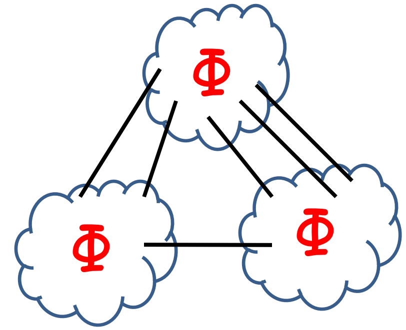
- isolated: no adjacent edges
  => found connected component
- active: #deleted edges > Φ #original edges
  => keep growing component
- passive: otherwise
  not enough deleted edges to destroy expansion
  => eventually, all passive comps will unite into one giant comp
  => no need to explore further

Can only explore $O(d/\Phi)$ edges before everything becomes passive.

Idea 2: Worry later, be happy

Remove cuts sparser than Φ for "later"

=> partition into expanders



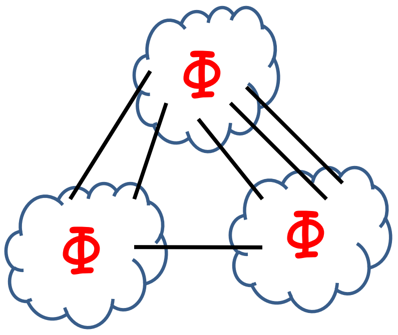
"Later"?

[Henzinger-King STOC'95]

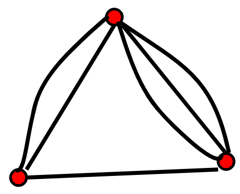
actually $O(d\sqrt{\lg m}/\Phi)$
using $O(\sqrt{\lg m})$ approx for sparsest cut

- set $\Phi = 1/(2 \lg m)$
  => update time $O(d/\Phi)$ still ok
- for every cut, charge Φ to each edge on smaller side
  => # edges cut $\leq$ \$ charged
- each edge charged at most $\lg m$ times
  => total $\$ \leq \Phi \lg m \leq m/2$

Hierarchical Decomposition



- Level 1: original graph
  promote sparse cuts to level 2



- Level 2: at most $m/2$ edges
  contract level-1 expanders
  promote sparse cuts to level 3

- Level 3: at most $m/4$ edges

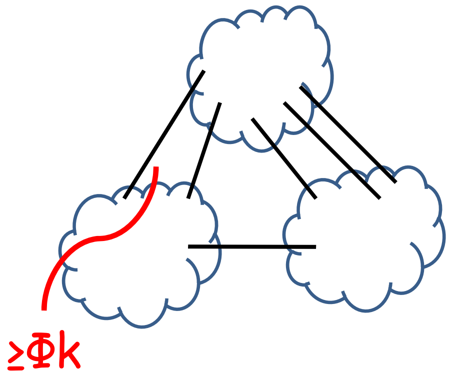
… up to $\leq \lg m$ levels

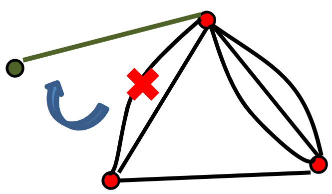
Handling Deletions

In each expander, run expander algorithm.

If an expander is split:

- let k = #edges on smaller side
- => at least Φk edges deleted in expander
- => can afford to inspect edges on smaller side @ next level



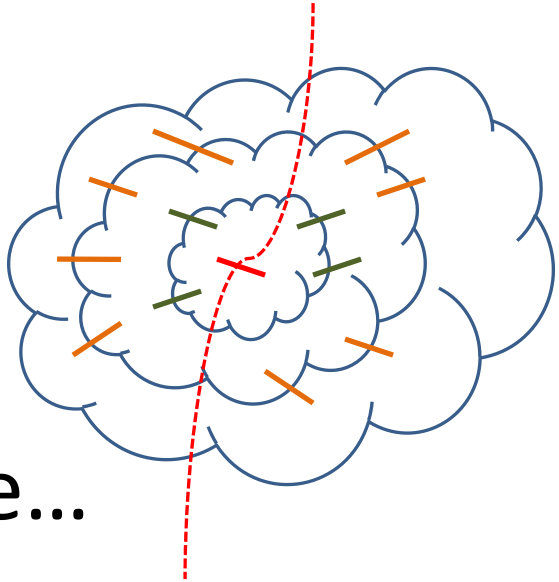


Trouble with Hierarchies in Paradise

Cascading!

- delete 1 edge at level 1
- separates 2 edges at level 2
- separates 4 edges at level 3

…

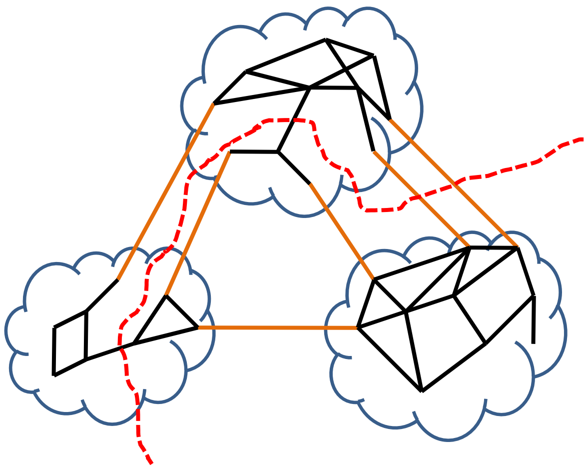


But don't try this at home…

Idea 3: Cut to the Bone

Trouble: cuts that don't look too sparse on level i
but are very sparse viewed from level i' » i

Fix: consider sparsity of cuts that violate levels



Let E_i = edges on level ≥ i

Before: $$\phi_i = \min_S \frac{|E_i \cap (S \times \bar{S})|}{|E_i \cap (S \times S)|}$$

Now: $$\phi_i = \min_S \frac{|E_1 \cap (S \times \bar{S})|}{|E_i \cap (S \times S)|}$$

Thus, we never contract components on higher levels

Levels = reweighting of the graph

Nota Bene

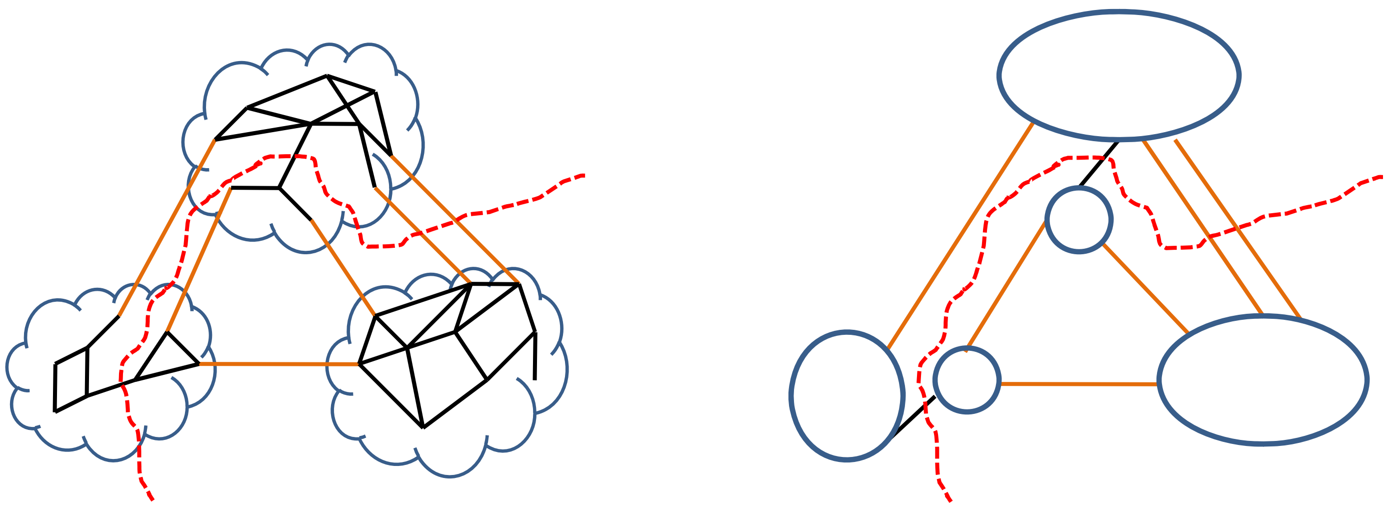
Profile:
e was in a sparse cut on lev 1,2
not on level 3, 4, 5
but again on level 6

Level promotions not monotone

This changes nothing

… but makes ever step of the reasoning a bit trickier

Updating the Analysis of Updates



At level i : Vertices = components united by remaining edges on levels ≤ i

Edges = original edges on level i; deleted edges between components

} expander!

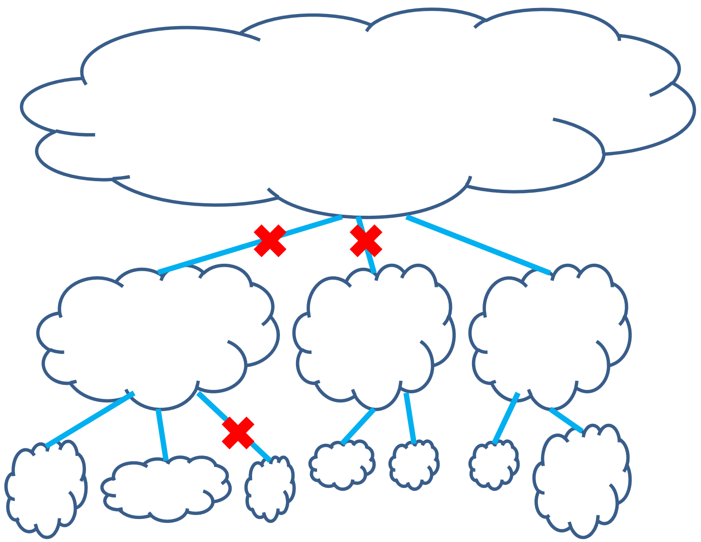
Unfortunately…

- constructing the hierarchy takes poly(m) time
  ☹ need $O(m)$ construction for fully dynamic

- [Spielman-Teng STOC’05]
  construct the original hierarchy in $O(m)$
  local approximation to weighted sparsest cut?

- need better random walks for volume in weighted graphs?

Oracle Queries



Level-i component

= comp induced by edges on levels ≤ i

Hierarchy tree

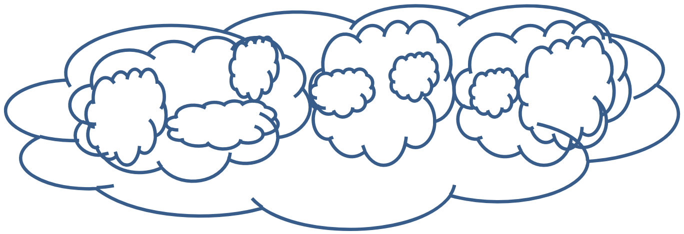
= parent relations between components

Isolated component => break parent pointer

Query = find lowest broken pointer

Binary search on level

=> O(lglg m) time per query



The End

Anarchists question hierarchies