Towards Polynomial Lower Bounds for Dynamic Problems

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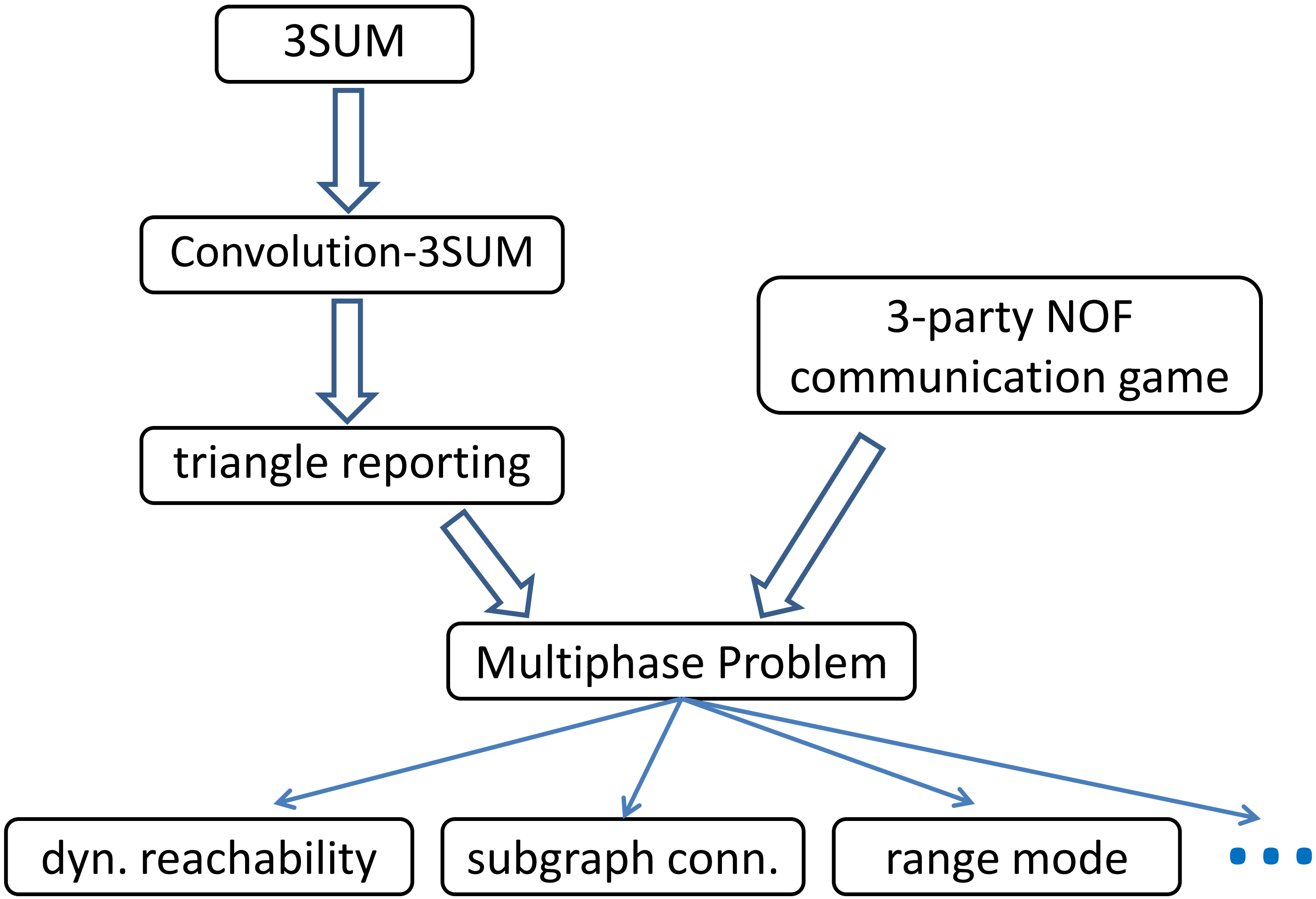
at&t

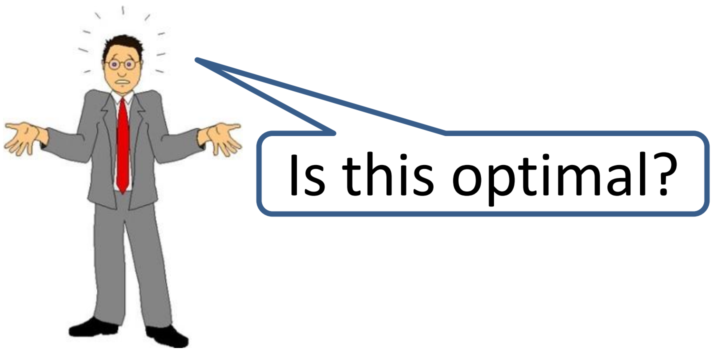
STOC 2010

Reduction Roadmap



Complexity inside P

MaxFlow: $O(m^{1.5})$ time



3SUM: $O(n^2)$ time

"S = {n numbers}, $(\exists) x,y,z \in S$ with $x+y+z=0$?"

Wouldn't it be nice…

"If 3SUM requires $\Omega^*(n^2)$ ➔ MaxFlow requires $\Omega^*(m^{1.5})$"

3SUM

- $O(n^2)$ [Gajentaan, Overmars'95]
- FFT ➔ $O(U \lg U)$ if $S \subseteq [U]$
- smart hashing ➔ roughly $O(n^2/\lg^2 n)$ [Baran, Demaine, P.'06]

Hardness:

- $\Omega(n^2)$ for low-degree decision tree [Erickson'95,'99] [Ailon-Chazelle'04]
- $n^{o(d)}$ for d-SUM => $2^{o(n)}$ for k-SAT, $\forall k$=O(1) [P.-Williams'10]

3SUM-hardness

- ∃ 3 collinear points?
- ∃ line separating n segments in two?
- minimum area triangle
- Do n triangles cover given triangle?
- Does this polygon fit into this polygon?
- Motion planning: robot, obstacles = segments

} "require" $\Omega^*(n^2)$ time

Algebraic reductions… E.g. $a \mapsto (a,a^3)$

$(a,a^3) - (b,b^3) - (c,c^3)$ collinear $\Leftrightarrow$ $a+b+c=0$

A "Fancier" Reduction

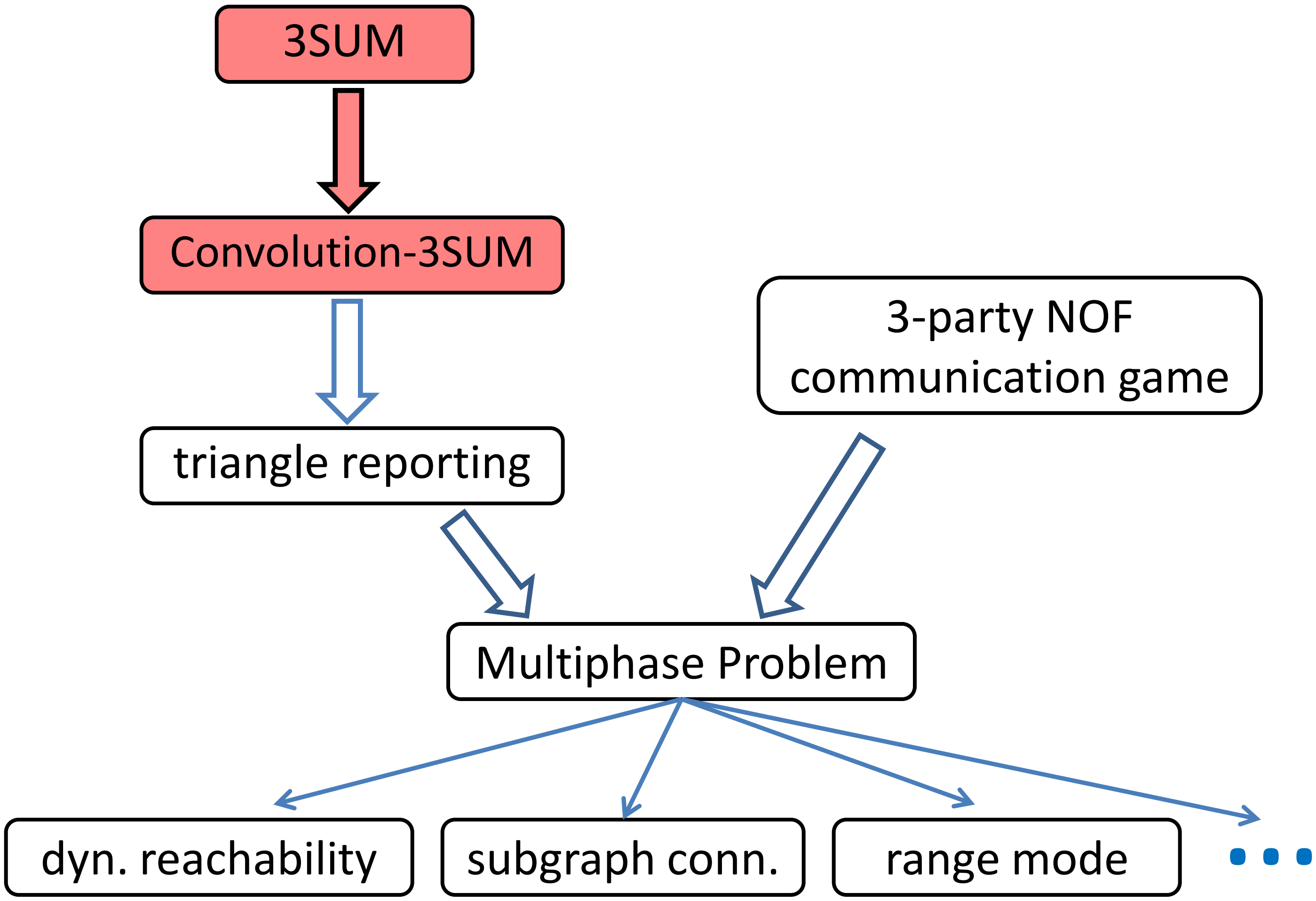
Theorem: If 3SUM requires $\Omega^*(n^2)$
➔ reporting *m* triangles in a graph requires $\Omega^*(m^{4/3})$

Assuming FMM takes $O(n^2)$:

- Triangle detection: $O(m^{4/3})$
- Reporting *m* triangles: $O(m^{1.4})$ [Pagh]

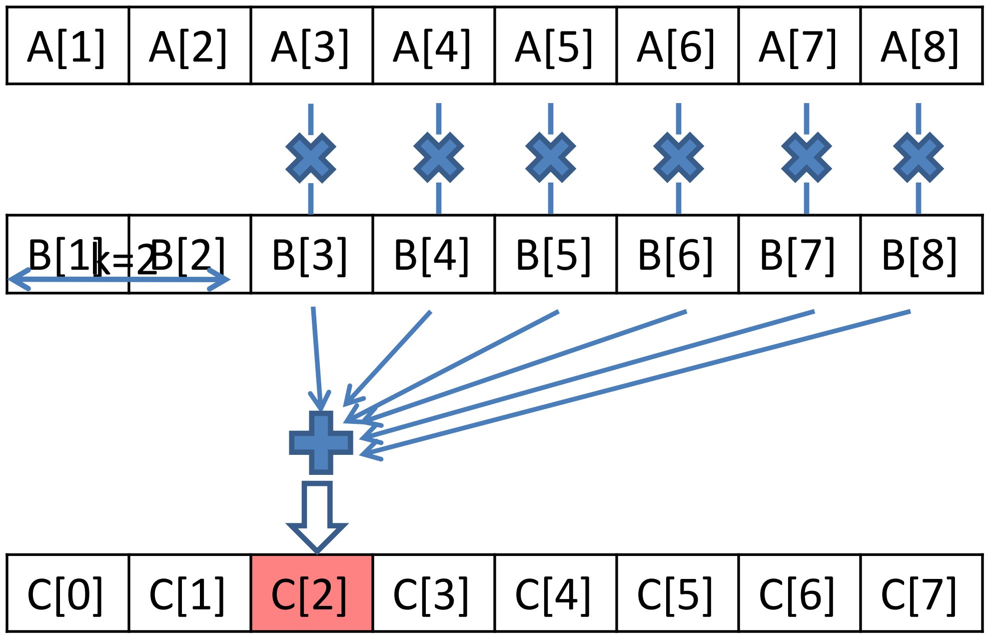
Recently, more reductions by [Vassilevska, Williams]
inc. reporting triangles ⇔ triangle detection

Reduction Roadmap



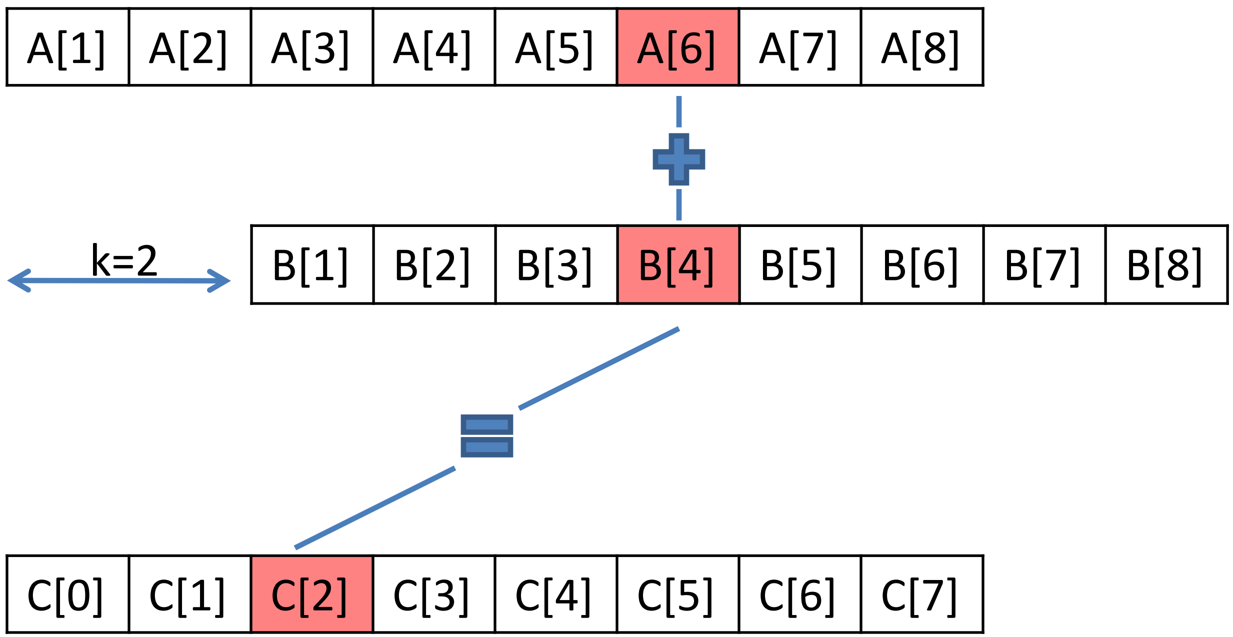
Convolution-3SUM

Convolution: A[1..n], B[1..n] ➔ $C[k] = \sum A[i+k] \cdot B[i]$



Convolution-3SUM

Convolution-3SUM: (∃) i, k ? C[k] = A[i+k] + B[i]



Convolution-3SUM

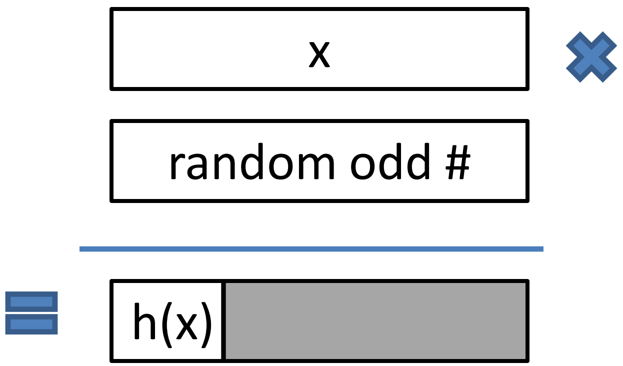
Convolution-3SUM: $(\exists)$ i, k ? $C[k] = A[i+k] + B[i]$

3SUM: $(\exists)$ i, j, k ? $C[i] = A[i] + B[j]$

Theorem: 3SUM requires $\Omega^*(n^2)$ time
iff Convolution-3SUM requires $\Omega^*(n^2)$ time

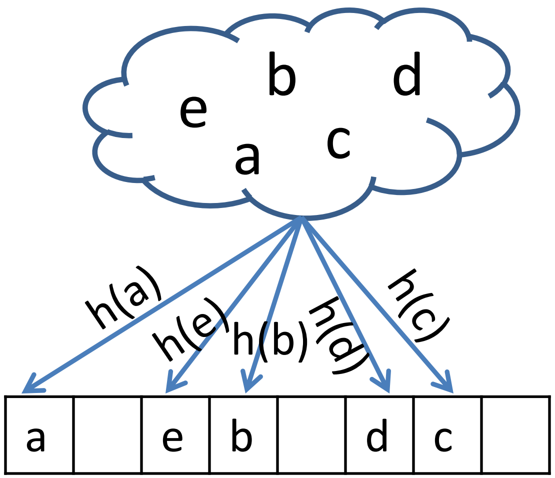
Linear Hashing

Want: x ↦ h(x) linear & few collisions



- almost linear (±1)
- surprisingly good load balancing

3SUM → Conv-3SUM



$x + y = z$

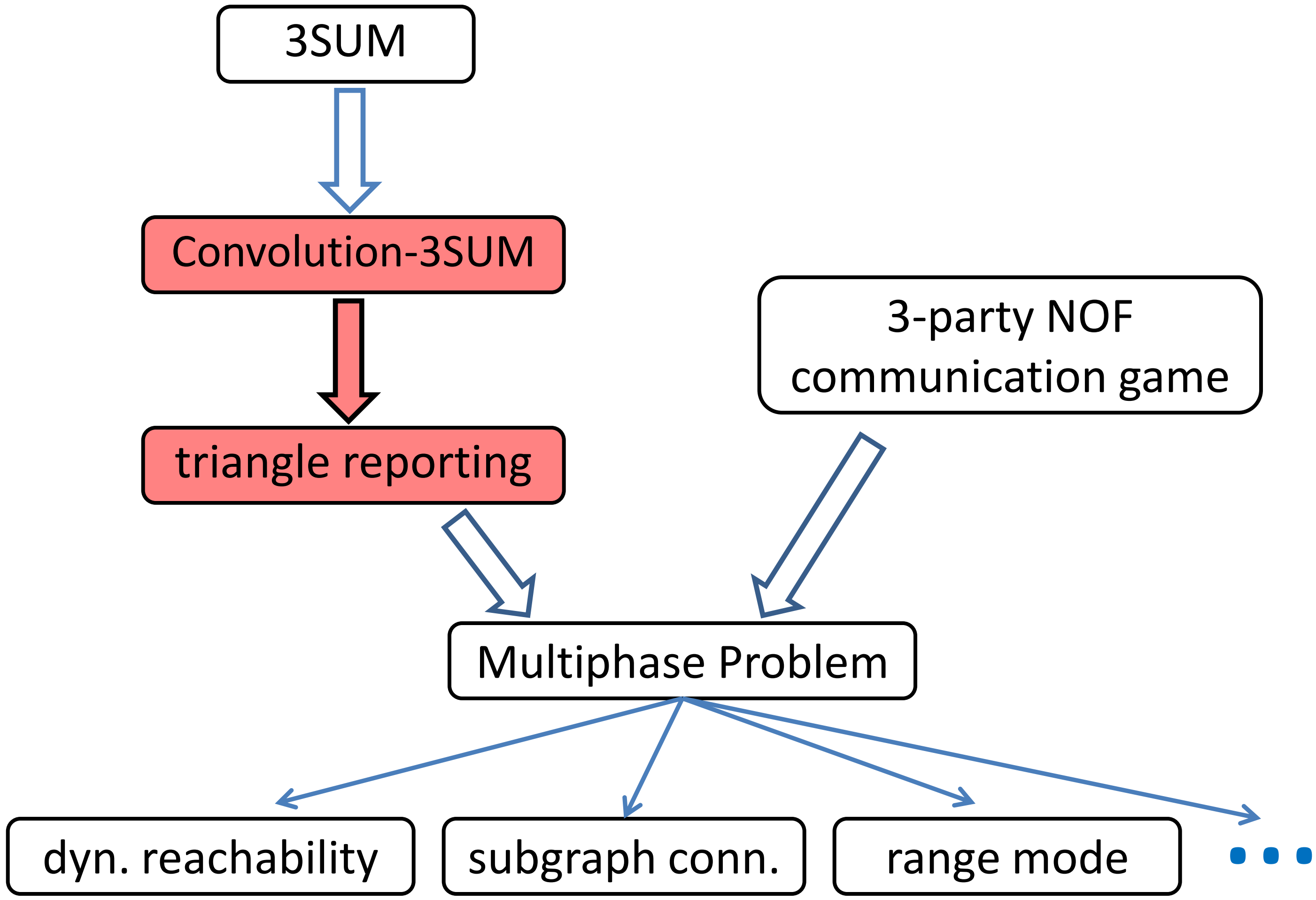
➔

$h(x)+h(y) = h(z)$

➔

$A[h(x)]+B[h(y)]=C[h(z)]$

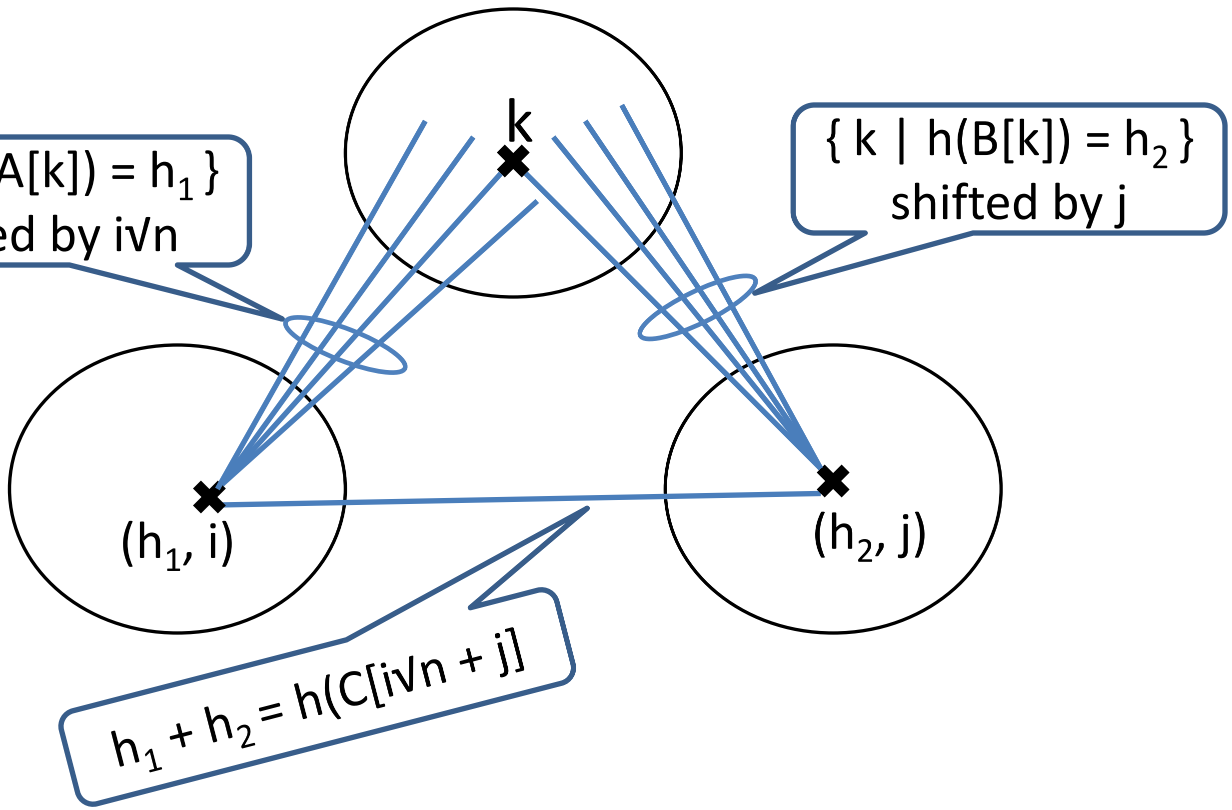
Reduction Roadmap



Conv-3SUM → Triangle Reporting

Hash A, B, C ↦ range [√n]

- $O(n^{1.5})$ false positives ➔ can check all if reported fast



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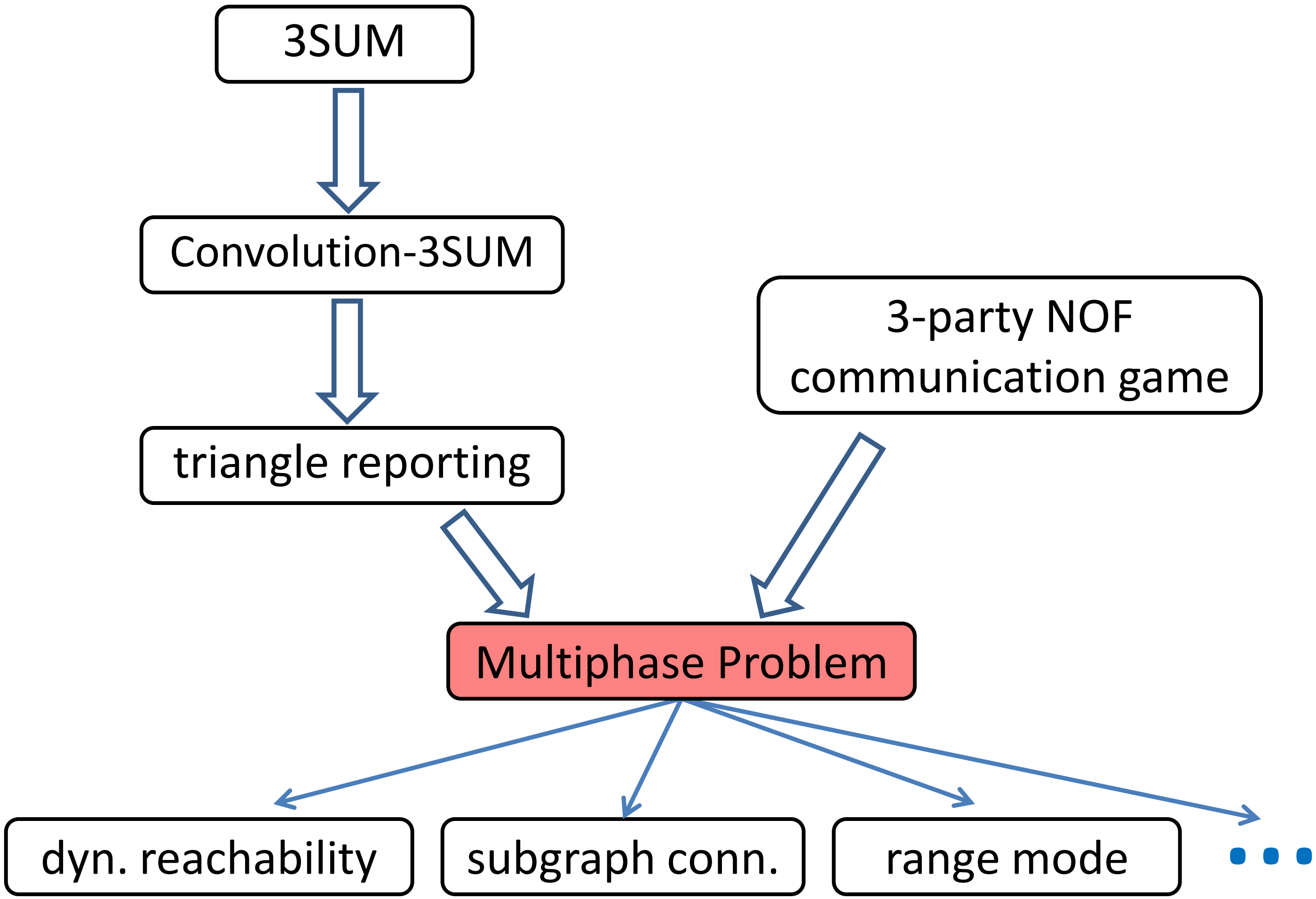
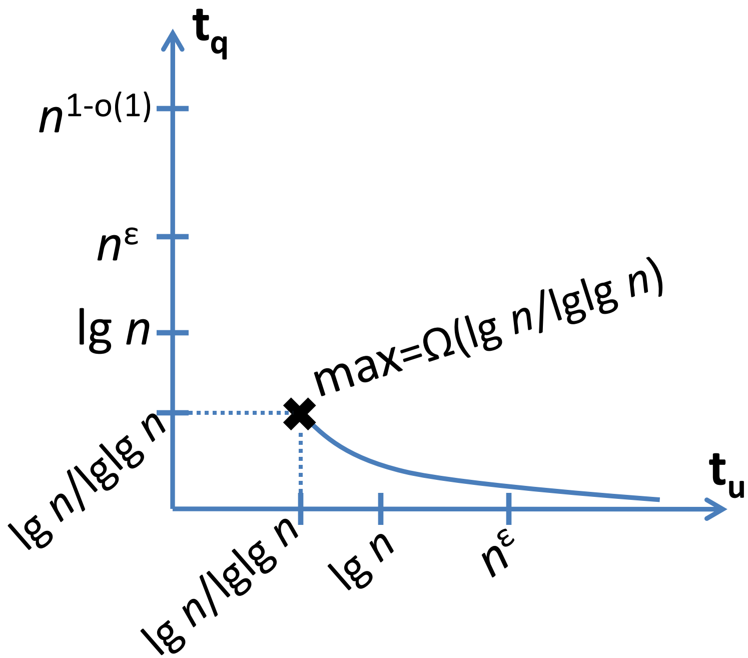


Dynamic Lower Bounds

[Fredman, Saks STOC'89]

$t_q = \Omega(\lg n / \lg (t_u \lg n))$



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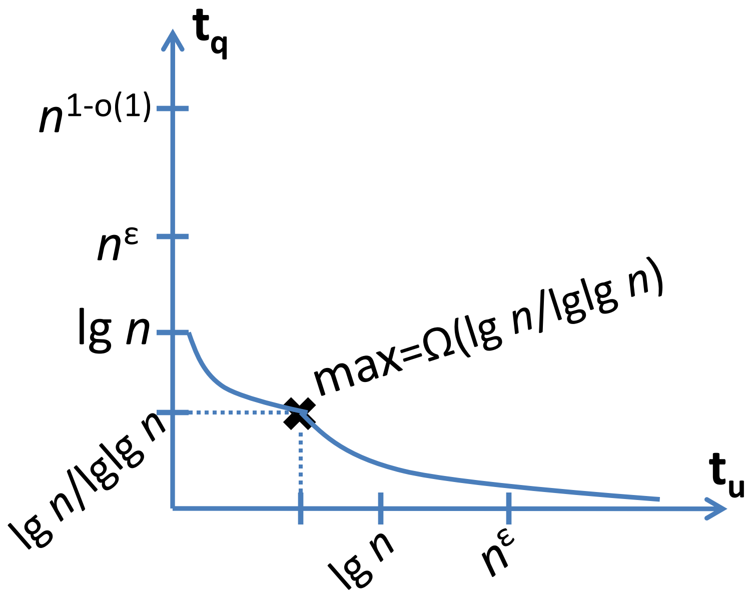

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[Alstrup, Husfeldt, Rauhe FOCS'98]

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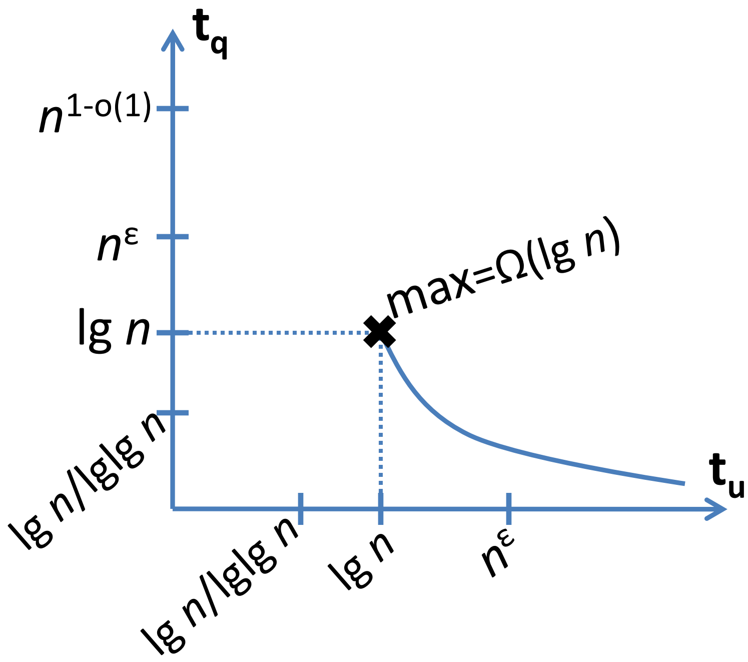

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[P., Demaine STOC'04]

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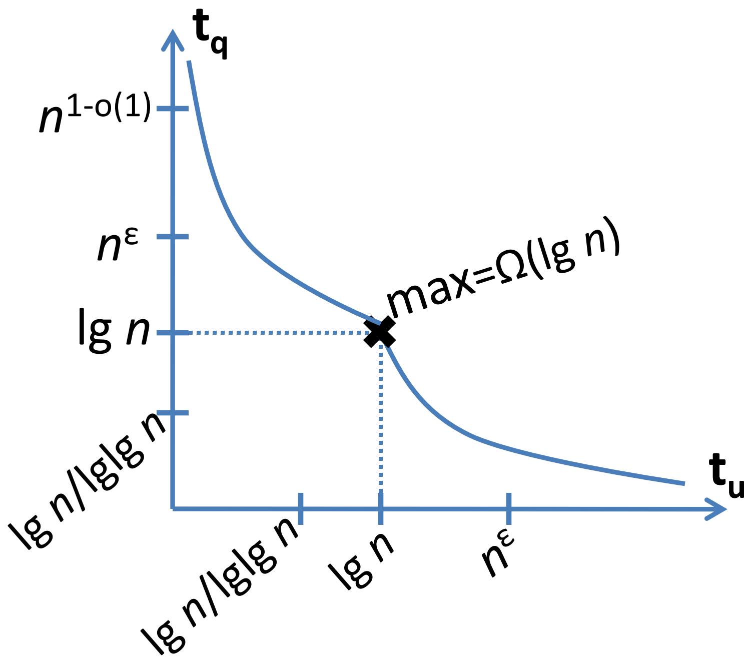

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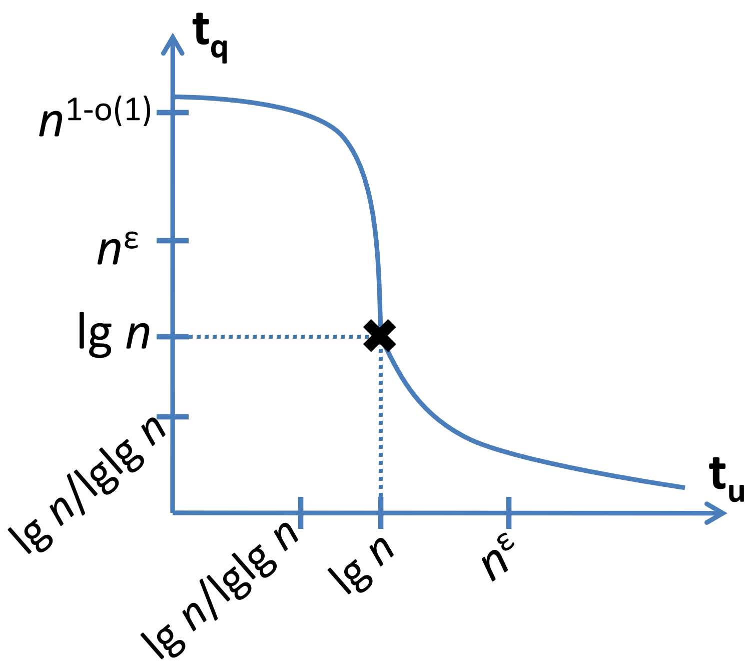
Dynamic Lower Bounds



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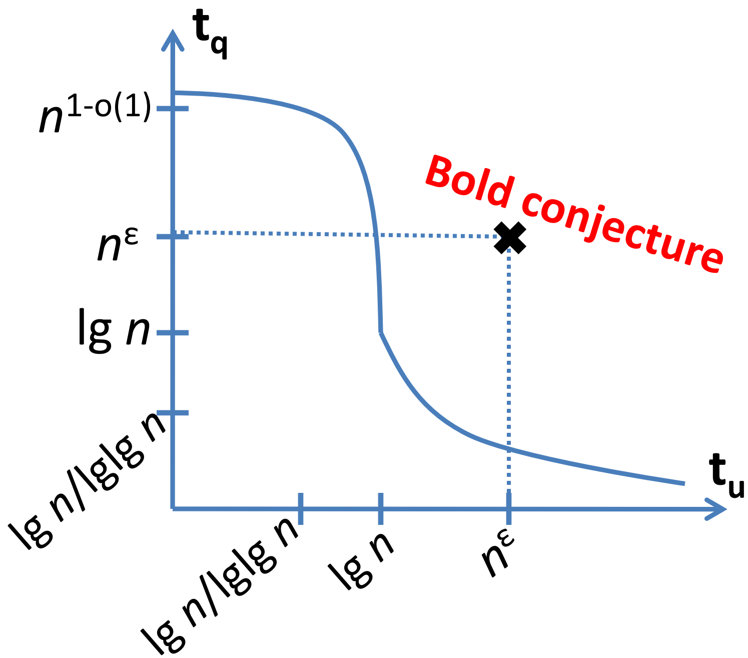
$t_u = \Omega(\lg n / \lg (t_q / \lg n))$

[P., Thorup '10]

$t_u = o(\lg n) \Rightarrow t_q \geq n^{1-o(1)}$

Dynamic Lower Bounds



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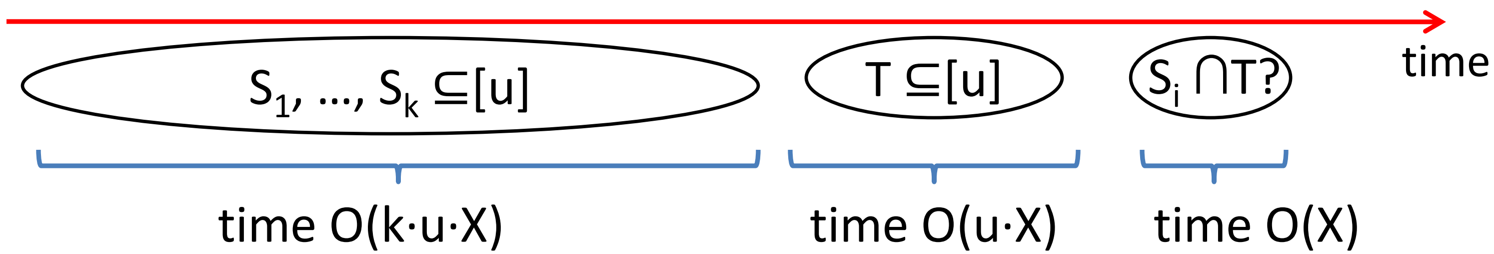
The Multiphase Problem



Conjecture: if $u \cdot X \ll k$, must have $X = \Omega(u^{\varepsilon})$

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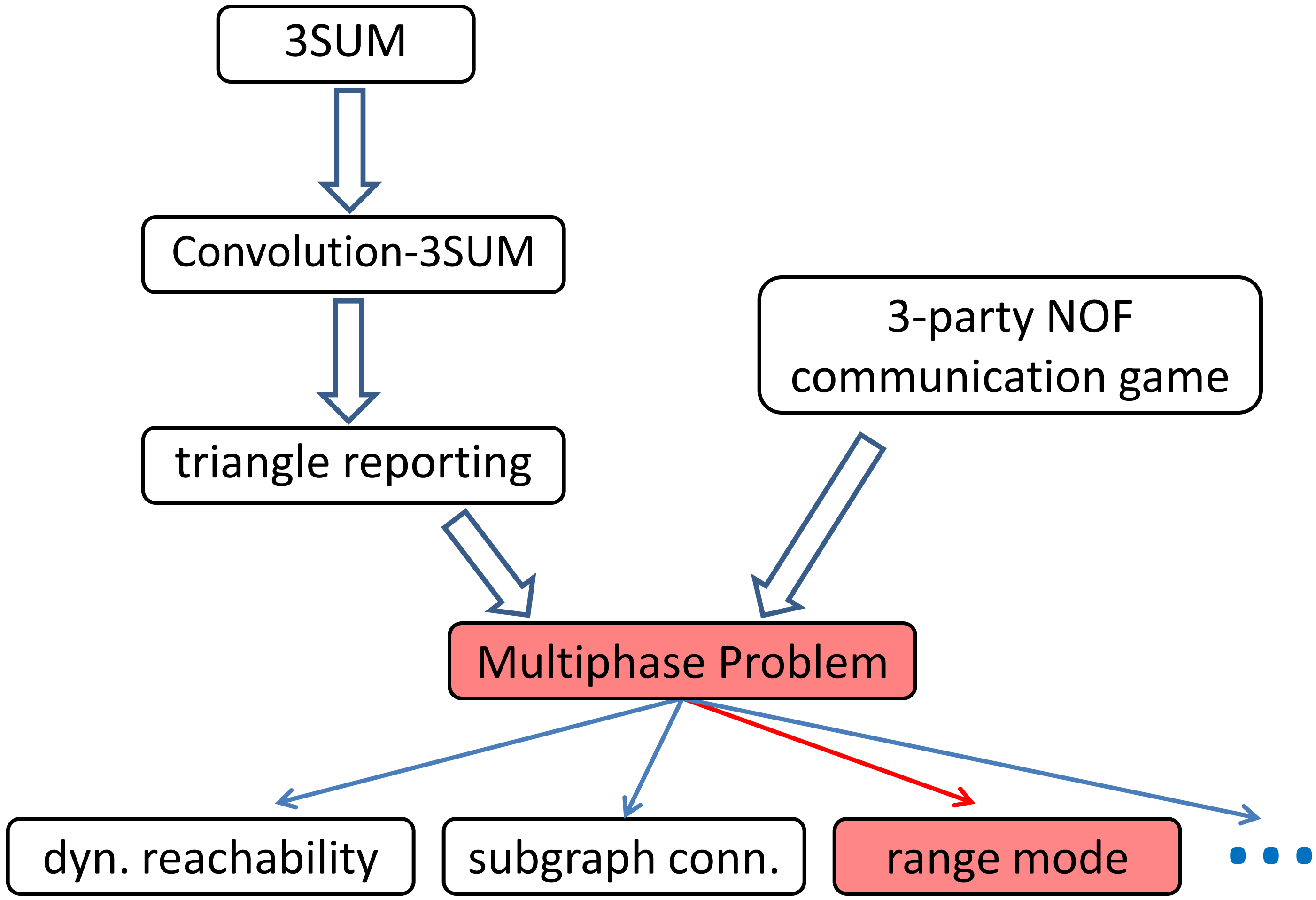

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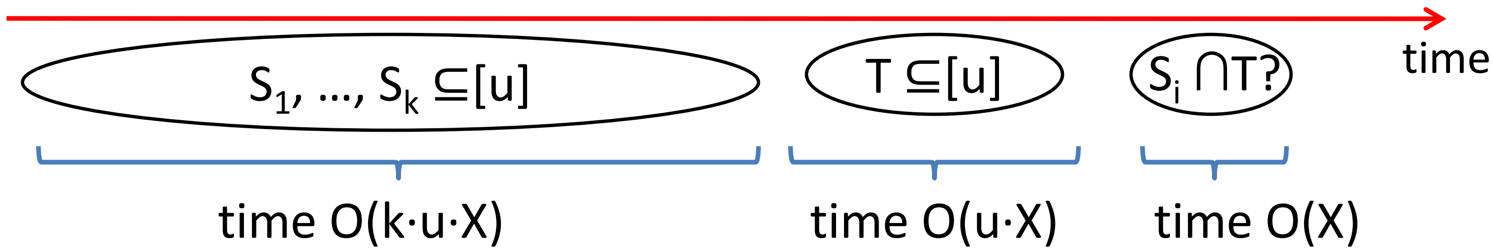
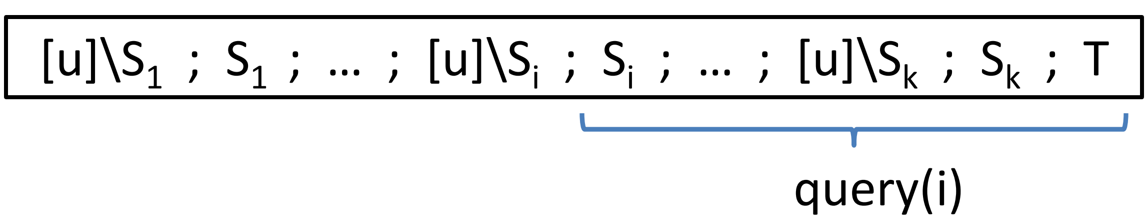


Conjecture: if $u \cdot X \ll k$, must have $X = \Omega(u^{\varepsilon})$

Sample application: maintain array A[1..n] under updates
Query: what's the must frequent element in A[i..j]?

Conjecture ➔ $\max\{t_u, t_q\} = \Omega(n^{\varepsilon})$



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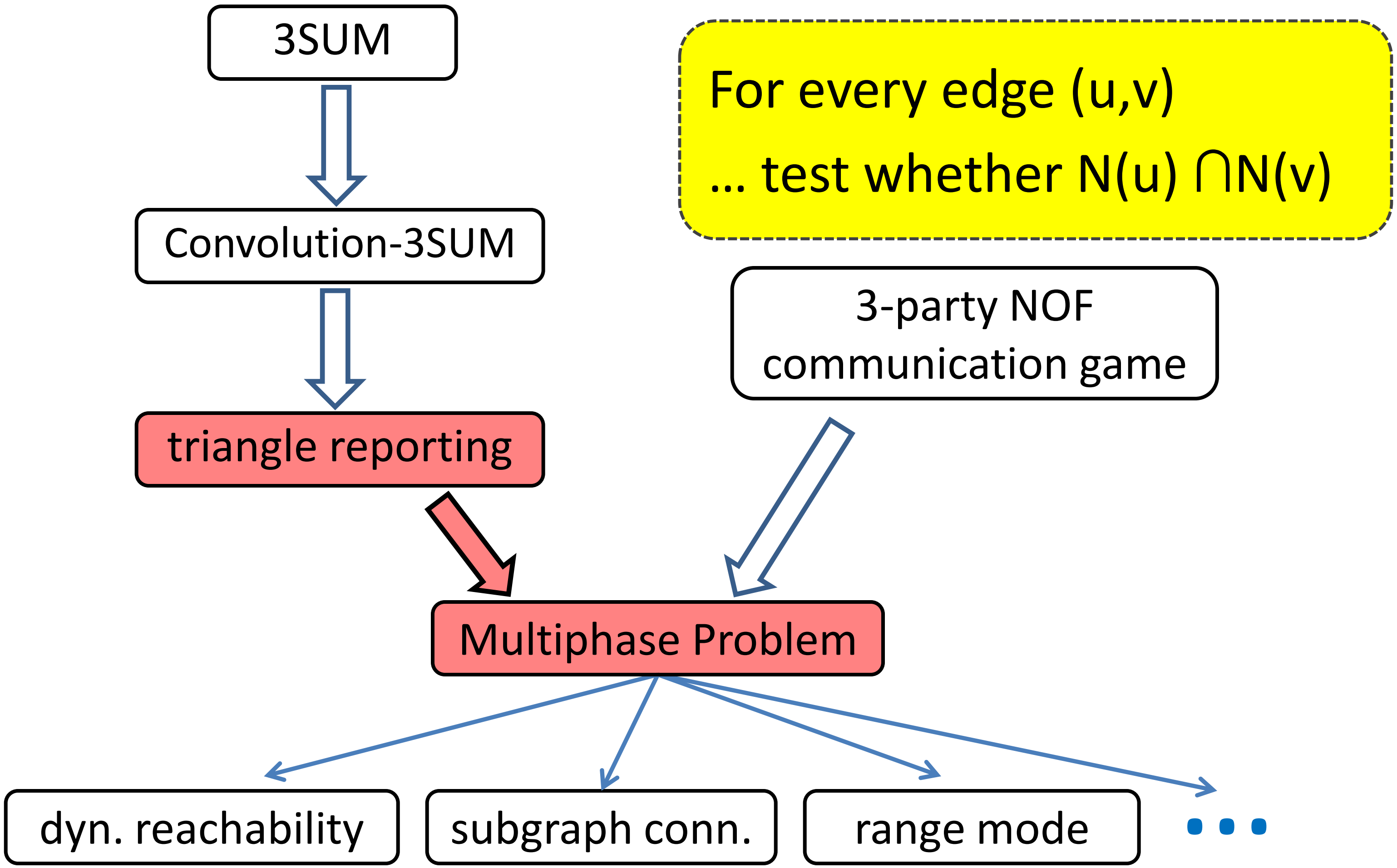

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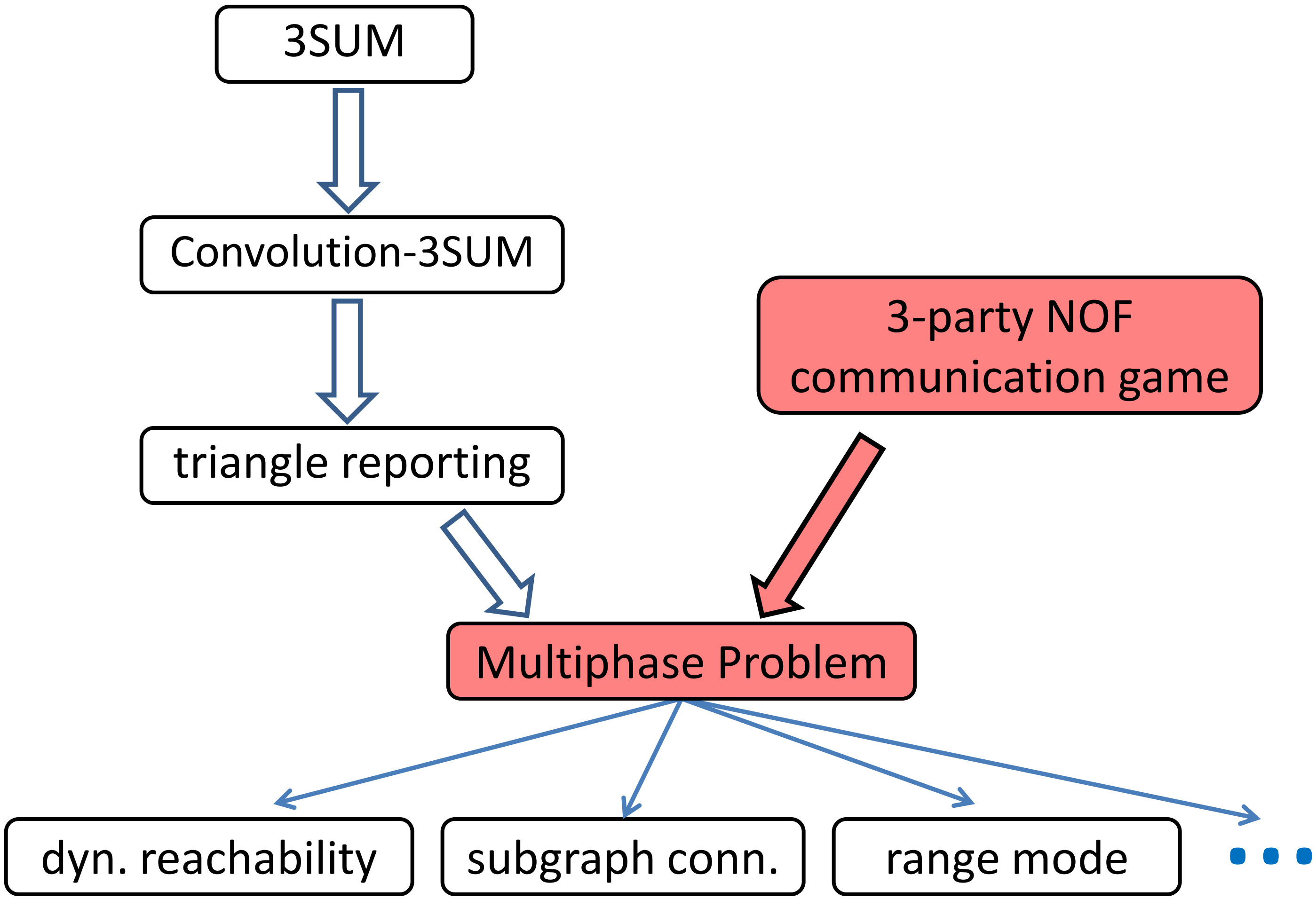
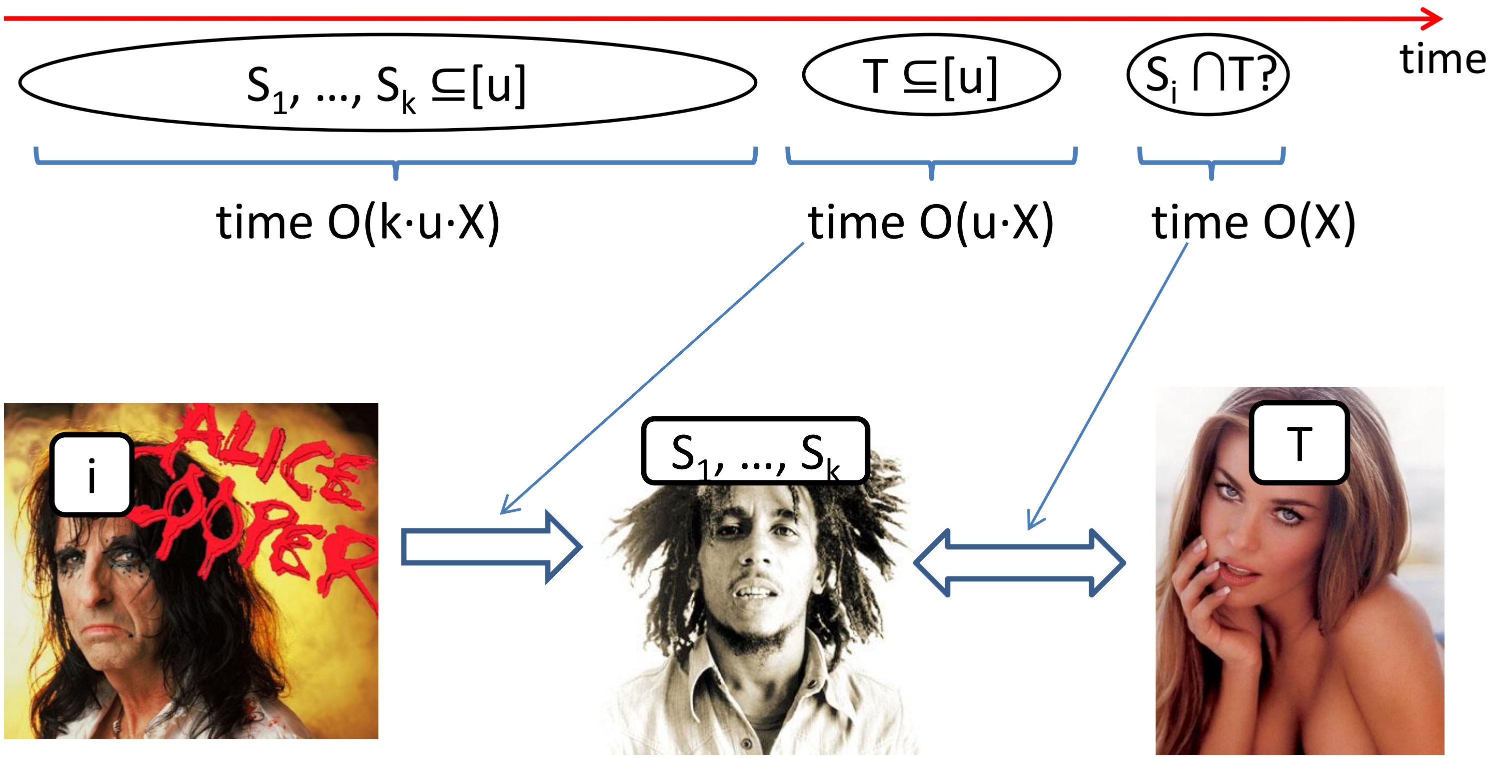


3-Party, Number-on-Forehead



The End