Planar Point Location in Sublogarithmic Time

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Point Location in o(log n) Time, Voronoi Diagrams in o(n log n) Time, and Other Transdichotomous Results in Computational Geometry

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Main Theme

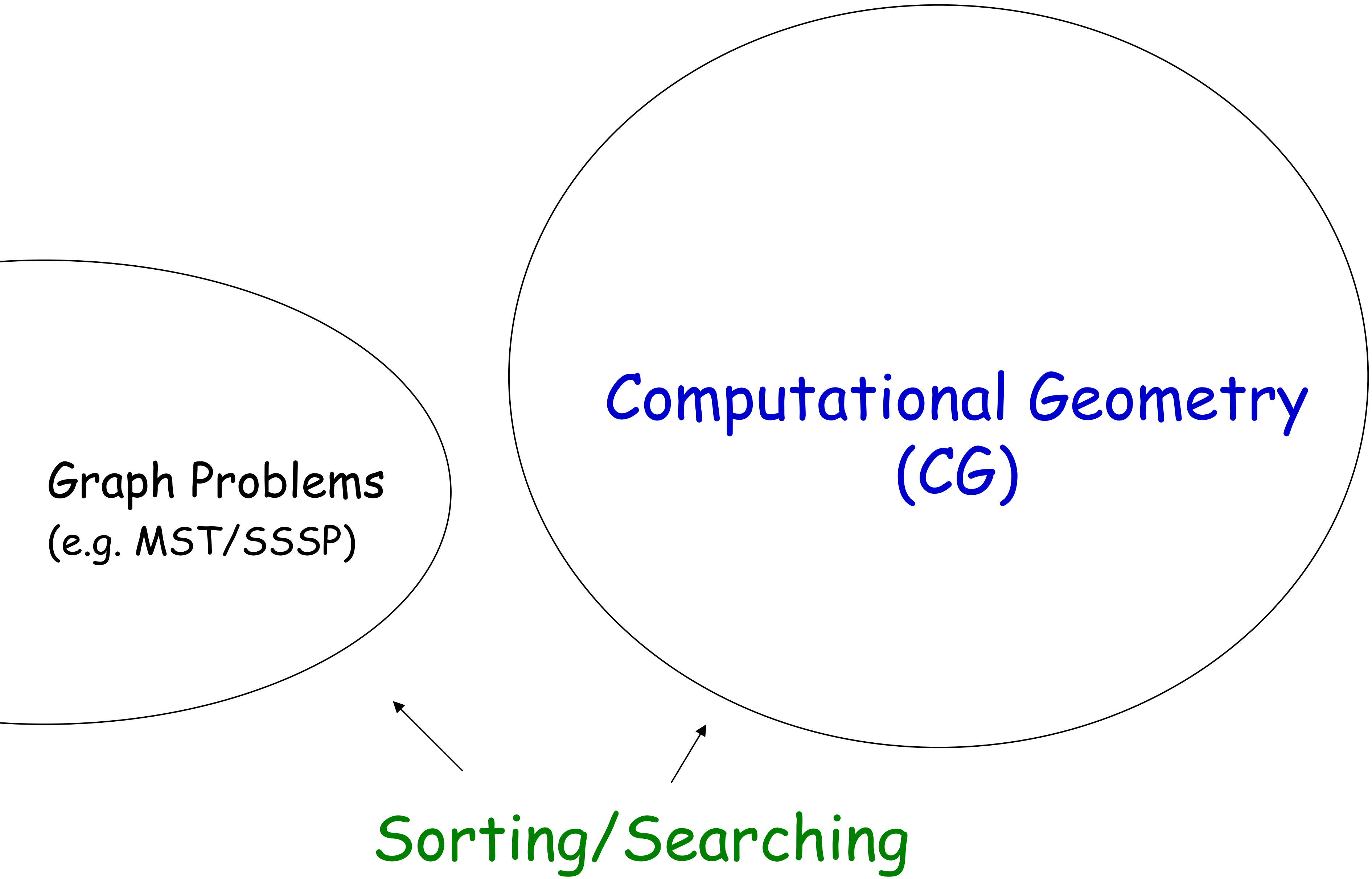
Power of the Word RAM !

Integer Sorting

• Radix-sort <'54	$O(n \log U/\log n)$
• Van Emde Boas '77	$O(n \log\log U)$
• Kirkpatrick, Reisch'84	$O(n \log(\log U/\log n))$
• Fredman, Willard (FOCS'90)	$O(n \log n/\log\log n)$ (det.)
	$O(n\sqrt{\log n})$ (rand.)
• Andersson et al. (STOC'95)	$O(n \log\log n)$ (rand.)
	$O(n)$ for $w \gg \log^2 n$ (rand.)
• Raman (ESA'96)	$O(n\sqrt{\log n \log\log n})$ (det.)
• Andersson (FOCS'96)	$O(n\sqrt{\log n})$ (det.)
• Thorup (SODA'98)	$O(n (\log\log n)^2)$ (det.)
• Han (SODA'01)	$O(n \log\log n \log\log\log n)$ (det.)
• Han (STOC'02)	$O(n \log\log n)$ (det.)
• Han, Thorup (FOCS'02)	$O(n\sqrt{\log\log n})$ (rand.)

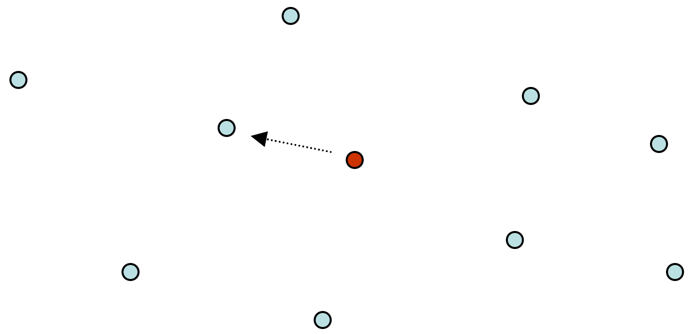
Integer (Predecessor) Searching

- Van Emde Boas '77 ("stratified trees") — $O(\log\log U)$ query time
- Fredman, Willard (FOCS'90) ("fusion trees") — $O(\log n/\log\log n)$, $O(\sqrt{\log n})$
- Beame, Fich (STOC'99) (optimal in cell probe model) — $O(\log\log U/\log\log\log U)$, $O(\sqrt{\log n/\log\log n})$
- Andersson, Thorup (STOC'00) ("exponential search trees") — $O(\sqrt{\log n/\log\log n})$ for query & update
- Pătraşcu, Thorup (STOC'06)
- Etc.


Graph Problems
(e.g. MST/SSSP)
Computational Geometry
(CG)
Sorting/Searching

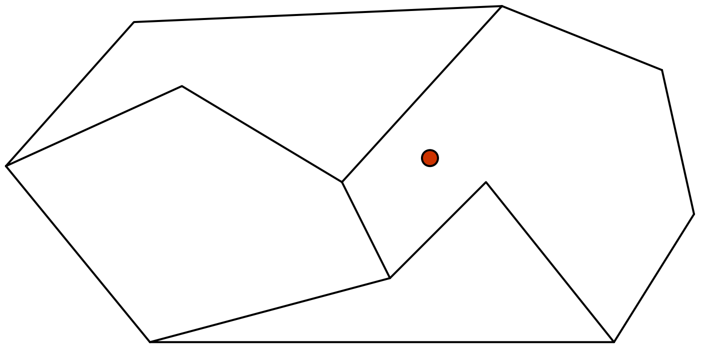
Standard Problems in CG

- 2D nearest neighbor search



O(n log n) preproc.
O(n) space
O(log n) query

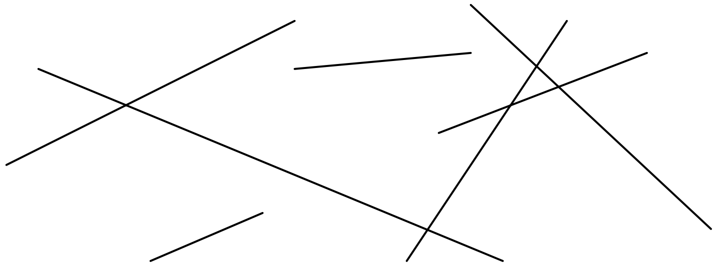
- 2D point location



O(n) preproc.
O(n) space
O(log n) query

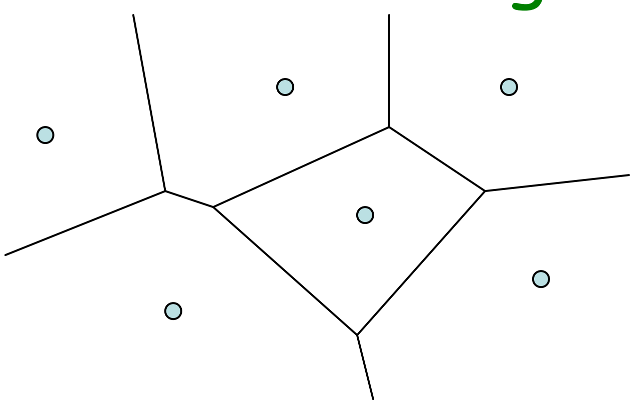
Standard Problems in CG (Cont'd)

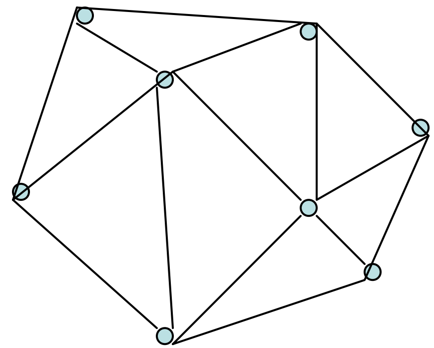
- 2D line segment intersection



O(n log n + K) time

- 2D Voronoi diagrams & 3D convex hulls





O(n log n) time

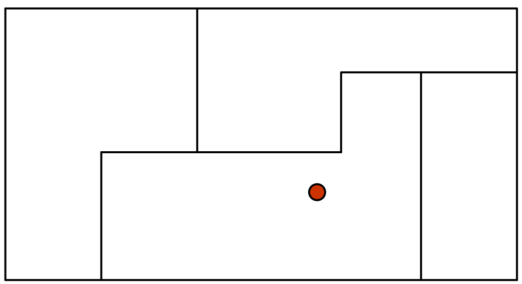
- Etc, etc, etc.

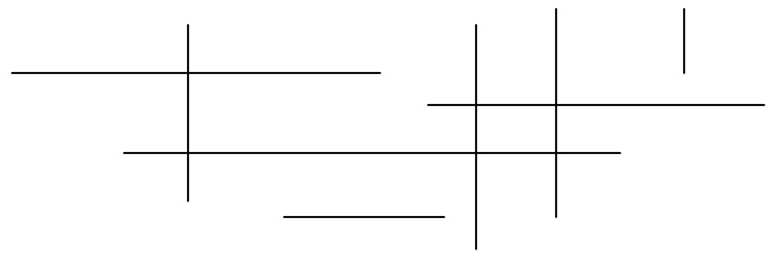
The Model

- Unit-cost RAM with word size w
- Coordinates are integers in {1,…,U}
- $U \le 2^w$, i.e., $w \ge \log U$
- $w \ge \log n$
- Availability of standard ops like <, +, -, *, /, bitwise-&, <<, >>

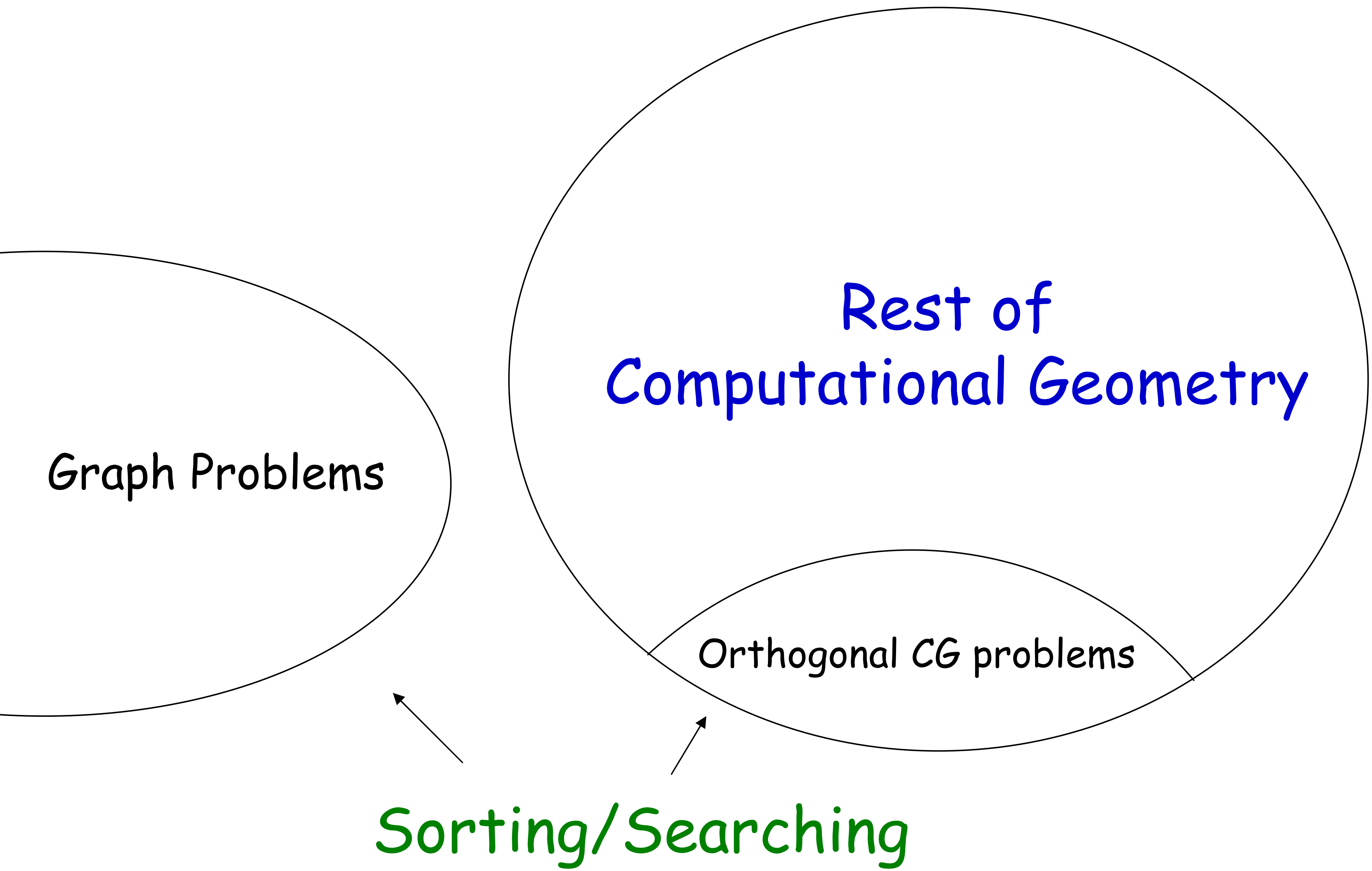
Previous Word RAM Results in CG

- Orthogonal range searching
- Orthogonal cases of 2D point location & segment intersection [e.g. loglog U-type results by Overmars '87]





- L_∞ variants of 2D nearest neighbor/Voronoi diagrams [e.g., loglog U-type results by Karlsson '84]
- NON-orthogonal problems???
  e.g., Willard (SODA'92) asked: standard 2D Voronoi diagrams in o(n log n) time??


Rest of
Computational Geometry
Graph Problems
Orthogonal CG problems
Sorting/Searching

New Results

- 2D Voronoi diagrams
  $O(n \log n / \log \log n)$ time (rand.)
- 3D convex hulls
  $O(n \log n / \log \log n)$ time (rand.)
- 2D line segment intersection
  $O(n \log n / \log \log n + K)$ time (rand.)
- 2D point location & 2D nearest neighbor
  $O(n)$ space, $O(\log n / \log \log n)$ query
- Etc, etc, etc.

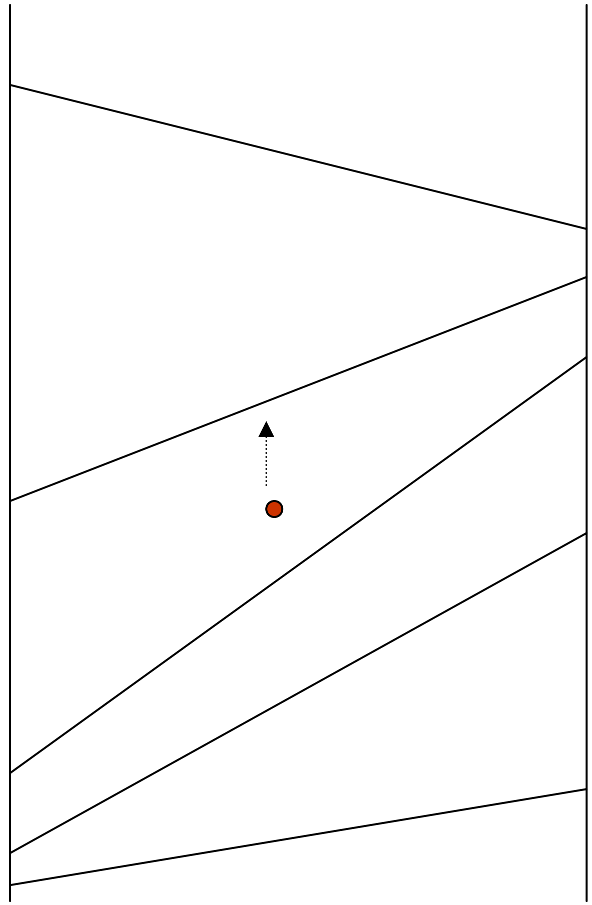
New Results (Cont'd)

- 2D Voronoi diagrams
  $O(n\sqrt{\log U / \log\log U})$ time (rand.)
- 3D convex hulls
  $O(n\sqrt{\log U / \log\log U})$ time (rand.)
- 2D line segment intersection
  $O(n\sqrt{\log U / \log\log U} + K)$ time (rand.)
- 2D point location & 2D nearest neighbor
  $O(n)$ space, $O(\sqrt{\log U / \log\log U})$ query
- Etc, etc, etc.

Key Subproblem: Point Location in a Slab

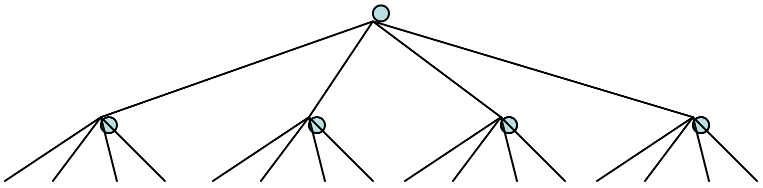
- Given n disjoint line segments spanning vertical slab, how to locate query point in o(log n) time??

  [can't project to 1D, can't build 1D structure at every vertical line, …]



Basic Idea

- Replace binary tree with b-ary tree



- But how?
  [Fredman, Willard's original "fusion tree" does not generalize…]

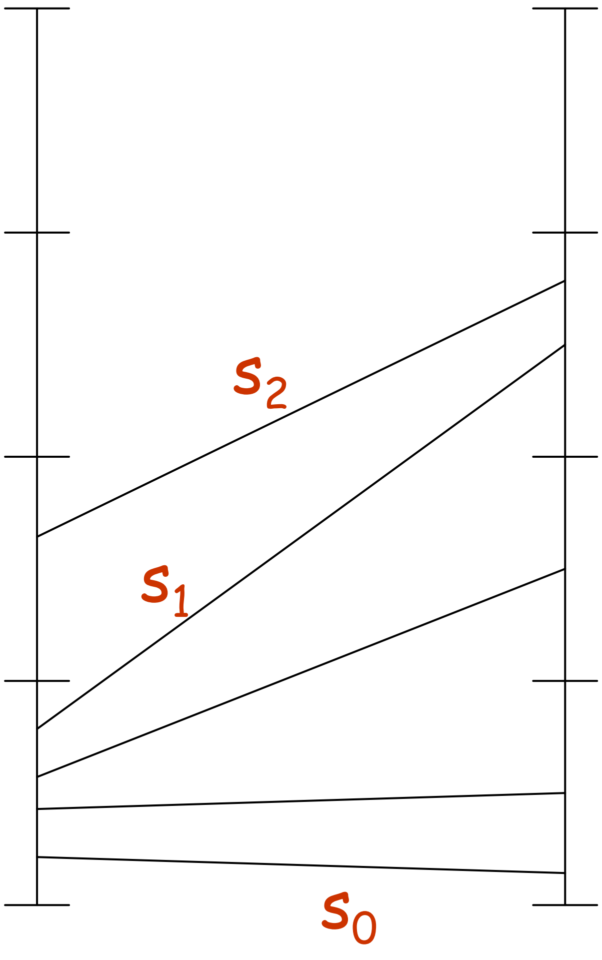
Key Observation

- Fix b, h. Given n segments with left/right endpts in a length-2^ℓ / length-2^m vertical interval, we can find ≤ b segments $s_0, s_1, \ldots$ s.t.
  i. Between s_i & s_{i+1}: there are ≤ n/b segments, OR left endpts are in a length-$2^{\ell-h}$ subinterval, OR right endpts are in a length-2^{m-h} subinterval;
  ii. $\exists$ segments $s_0', s_2', \ldots$, each encodable in O(h) bits, with $s_0 < s_0' < s_2 < s_2' < \ldots$

Proof

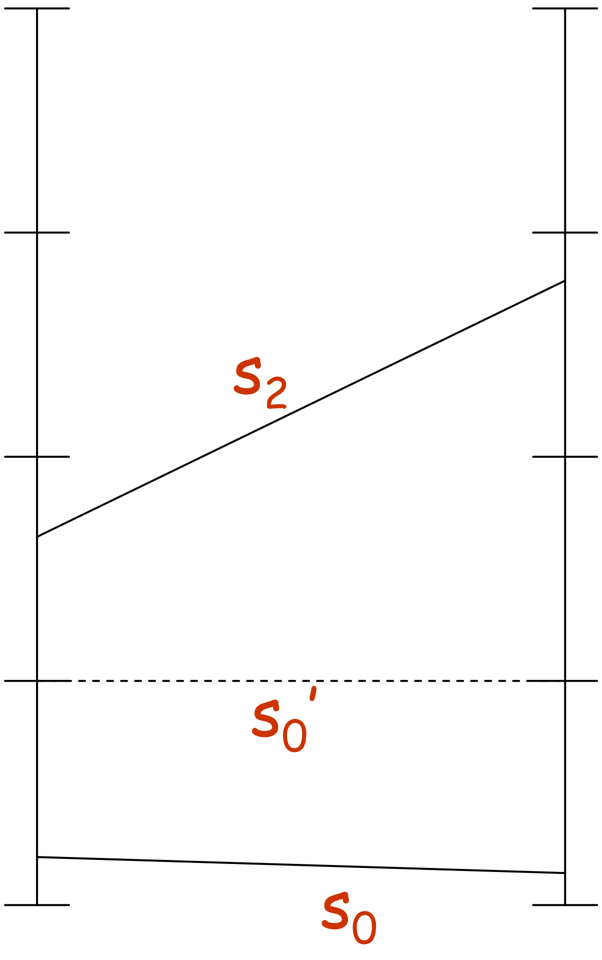
- Divide left/right interval into 2^h grid subintervals
- Draw $(n/b)^{th}$, $(2n/b)^{th}$, $(3n/b)^{th}$,... segment

i. Betwn s_0 & s_1, left endpts are in length-$2^{\ell-h}$ subinterval;
Betwn s_1 & s_2, there are n/b segments



Proof

- $\exists$ s_0' betwn s_0 & s_2 using only grid endpts ($O(h)$ bits)



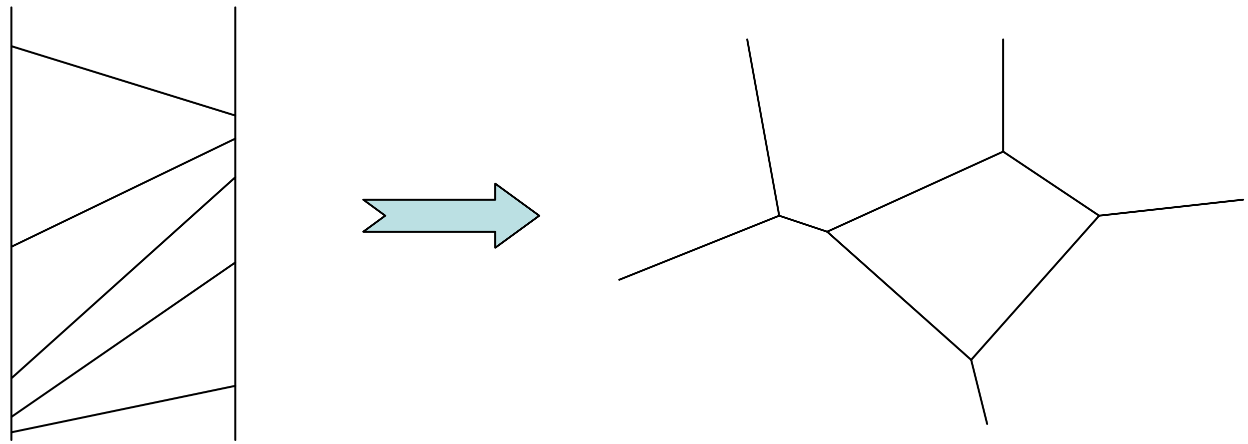
The New "2D Fusion Tree" for the Slab Subproblem

- Just apply Observation recursively!
- Each tree node can be packed in one word if $bh \approx w$ (by ii)
- Query time $=$ tree height (using word ops)

$$\leq \log_b n + 2w/h \quad \text{(by i)}$$
$$= O(\log_b n + b) \quad (\text{set } b = \log^{\varepsilon} n)$$
$$= \boxed{O(\log n / \log\log n)}$$

Switch speakers. Confuse audience

Using the Slab Problem



- talk only about 2D point location
  - a bit more smartness involved for 2D Voronoi, 3D convex hull, line segment intersection…
- before our work:
  - the slab problem is a trivial binary search
  - going to the general case is what’s interesting

Conversion Techniques

Lipton+Tarjan — planar separator

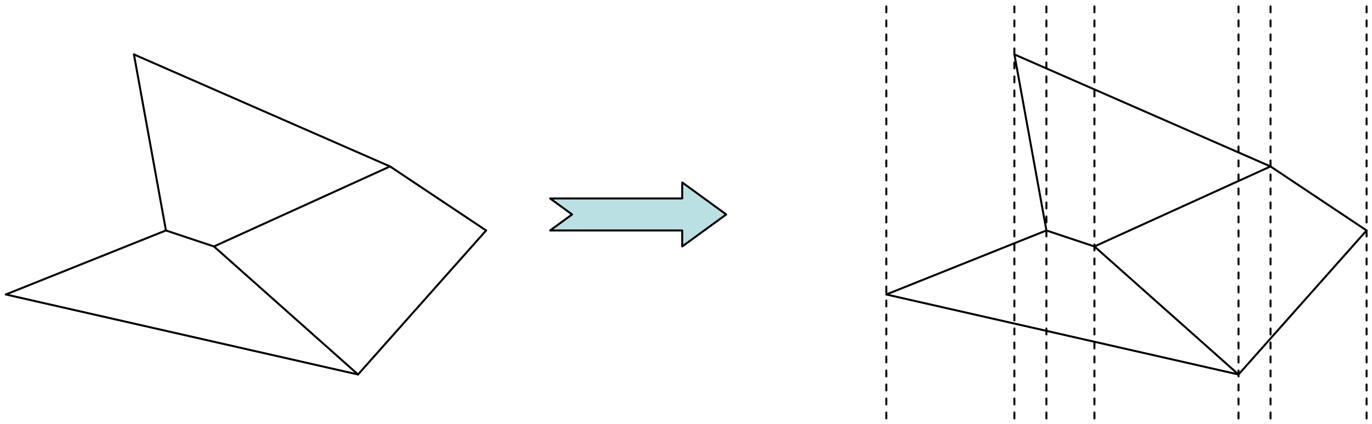
Mulmuley — random sampling

Cole, Sarnak+Tarjan — persistence

- use exponential trees [Andersson] + new ideas

- Kirkpatrick — triangulation refinement
- Edelsbrunner+Guibas+Stolfi — separating chains

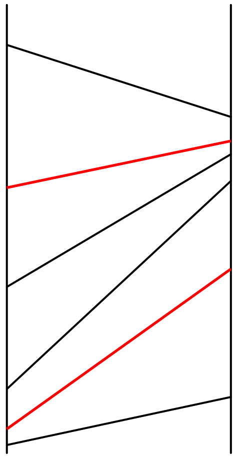
Review: Persistence



- sweep with vertical line
- insert/remove segments into dynamic slab structure

  (next slides)
- keep all past images of the data structure in small space via persistence (can be done)

Review: Exponential Trees



Hope: dynamize slab structure black-box

pick $\sqrt{n}$ separators, $\Theta(\sqrt{n})$ segments apart

construction: static structure on separators; recurse

update: rebuild static structure when needed

separators change infrequently (?)

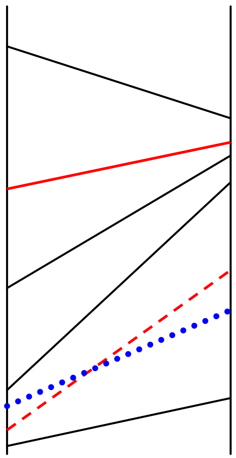
query: $$\frac{\lg \sqrt{n}}{\lg \lg \sqrt{n}} + \frac{\lg \sqrt[4]{n}}{\lg \lg \sqrt[4]{n}} + \ldots \le \frac{\lg n}{\lg \lg n} \cdot (\tfrac{1}{2} + \tfrac{1}{4} + \ldots)$$

2D ≠ 1D

Trouble: segments are not numbers

- remove separator segment
- insert intersecting segment

⇨ separator does not separate



Use offline knowledge:

separators = segments which won't be removed for a long time

Open problems

- <u>count</u> segment intersections in $o(n \log n)$
- derandomize (3D convex hull, 2D Voronoi)
- better algorithms
  - point location problem is offline
- lower bounds for the data structures
  - is exact NN easier than point location?

Thank you!