On the Possibility of Faster SAT Algorithms

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SAT Problems

$$\text{SAT} = \{\Phi = (x_1 \vee x_7 \vee \overline{x_8}) \wedge (\overline{x_5} \vee x_8) \wedge \cdots \mid \Phi \text{ satisfiable}\}$$

$$k\text{-SAT} = \text{all clauses have} \leq k \text{ literals}$$

Parameters:

$$n = \text{number of variables}$$

$$m = \text{number of clauses}$$

Upper bounds:

SAT:

$$2^{n\left(1-\frac{1}{O(\log(m/n))}\right)} \cdot \text{poly}(m) = 2^{n-o(n)}$$

k-SAT:

$$2^{n\left(1-O(\frac{1}{k})\right)} \cdot \text{poly}(m) = 2^{s_k n}$$

Hardness Assumptions

ETH: 3-SAT cannot be solved in $2^{o(n)}$

Assuming ETH, s_k is increasing. [IP'01]

Hard SAT: SAT requires $2^{n-o(n)}$

If SAT takes $2^{\delta n}$, $s_k \leq \delta\big(1 - \Omega(\frac{1}{k})\big)$. [IP'01]

Strong ETH: $s_k \to 1$

Open problem.
Say $s_k \to \frac{1}{2}$. Can SAT be solved in $2^{0.99n}$?

Applications: Lower Bounds

d-SUM

Given $S = \{n \text{ numbers}\}$,
are there $x_1, \ldots, x_d \in S$ with $x_1 + \cdots + x_d = 0$?

ETH $\Rightarrow n^{\Omega(d)}$ time.

k-Dominating Set

Given graph, find $S \subset V$, $|S| = k$ such that $N(S) = V$.

Hard SAT $\Rightarrow O(n^{k-\varepsilon})$ impossible.

3-Party Set Disjointness

Alice, Bob, Carmen hold $A, B, C \subset [n]$.
Goal: determine whether $A \cap B \cap C = \emptyset$.
Number on forehead

Strong ETH $\Rightarrow$ no $o(n)$ protocol.

k-Dominating Set Requires $n^{k-o(1)}$

n variables $\mapsto$ k blocks of $\frac{n}{k}$ variables

Block $\mapsto 2^{n/k}$ nodes (partial assignments)
…Plus one supernode connected to block's assignments
$\Rightarrow$ much select exactly one assignment in each block

m clauses $\mapsto$ m nodes
Edges from close C_i to partial assignment satisfying it

Harder Reductions: Sparsity Matters

When doing reductions, m matters!

Is sparse SAT still hard? No: $2^{(1-\varepsilon)n}$.

How about sparse k-SAT?

Lemma (Sparsification Lemma)

Complexity of k-SAT with $m = f(k, \varepsilon) \cdot n$

$$\leq \left[\textit{Complexity of general } k\textit{-SAT} \right] / 2^{\varepsilon n}$$

ETH $\Rightarrow$ may assume $m = O(n)$.

Reduction to d-SUM

Problem	Variables	Clauses	Why
k-SAT			
k-SAT	n	m=O(n)	sparsification
3-SAT	$O(nk)$	$O(nk)$	[Cook]
1-in-3-SAT	$N = O(nk)$	$M = O(N)$	[GJ]

Partition variables $\rightarrow$ d blocks of $\frac{N}{d}$ variables

Block $\rightarrow$ $2^{N/d}$ numbers of M digits

digit $[i] = 1 \iff$ clause i satisfied

Must find numbers to sum to $11\ldots11$.

Reduction to Set Disjointness

Partition variables $\rightarrow X \cup Y \cup Z$, $|X| = |Y| = |Z| = \frac{n}{3}$.

x induces $S(x) = \{\text{clauses not satisfied by x}\} \subseteq [m]$

$$\Phi(xyz) = \text{true} \iff S(x) \cap S(y) \cap S(z) = \emptyset$$

Run communication protocol for "$S(x) \cap S(y) \cap S(z) = \emptyset$?"

- $o(m) = o(n)$ bits of communication [Sparsity!]
- so enumerate all transcripts π ending in "Disjoint!"

Tripartite graph G: $V = X \cup Y \cup Z$

$(x, y) \in X \times Y \iff$ Alice follows π on $S(x), S(y)$

$(y, z) \in Y \times Z \iff$ Bob follows π on $S(y), S(z)$

$(x, z) \in X \times Z \iff$ Carmen follows π on $S(x), S(z)$

Find triangle in $O(N^{2.376}) = O((2^{n/3})^{2.376}) = O(1.74^n)$.

Thank you!

THE END