On the Possibility of Faster SAT Algorithms

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SAT Problems

\[ \text{SAT} = \{ \Phi = (x_1 \lor x_7 \lor \overline{x_8}) \land (\overline{x_5} \lor x_8) \land \cdots | \Phi \text{ satisfiable} \} \]

\[ k\text{-SAT} = \text{all clauses have} \leq k \text{ literals} \]

Parameters:

\[ n = \text{number of variables} \]
\[ m = \text{number of clauses} \]

Upper bounds:

\[ \text{SAT:} \quad 2^n \left(1 - \frac{1}{O(\log(m/n))}\right) \cdot \text{poly}(m) = 2^{n-o(n)} \]

\[ k\text{-SAT:} \quad 2^n \left(1 - O\left(\frac{1}{k}\right)\right) \cdot \text{poly}(m) = 2^{s_k n} \]
Hardness Assumptions

**ETH:** 3-SAT cannot be solved in $2^{o(n)}$

Assuming ETH, $s_k$ is increasing. [IP’01]

**Hard SAT:** SAT requires $2^{n-o(n)}$

If SAT takes $2^{\delta n}$, $s_k \leq \delta \left(1 - \Omega\left(\frac{1}{k}\right)\right)$. [IP’01]

**Strong ETH:** $s_k \to 1$

**Open problem.**
Say $s_k \to \frac{1}{2}$. Can SAT be solved in $2^{0.99n}$?
Applications: Lower Bounds

\(d\)-SUM
Given \(S = \{n \text{ numbers}\}\), are there \(x_1, \ldots, x_d \in S\) with \(x_1 + \cdots + x_d = 0\) ?

ETH \(\Rightarrow n^{\Omega(d)}\) time.

\(k\)-Dominating Set
Given graph, find \(S \subset V, |S| = k\) such that \(N(S) = V\).

Hard SAT \(\Rightarrow O(n^{k-\varepsilon})\) impossible.

3-Party Set Disjointness
Alice, Bob, Carmen hold \(A, B, C \subset [n]\).
Goal: determine whether \(A \cap B \cap C = \emptyset\).

Number on forehead

Strong ETH \(\Rightarrow\) no \(o(n)\) protocol.
$k$-Dominating Set Requires $n^{k-o(1)}$

$n$ variables $\mapsto k$ blocks of $\frac{n}{k}$ variables

Block $\mapsto 2^{n/k}$ nodes (partial assignments)
... Plus one supernode connected to block’s assignments
  $\Rightarrow$ much select exactly one assignment in each block

$m$ clauses $\mapsto m$ nodes
Edges from close $C_i$ to partial assignment satisfying it
When doing reductions, $m$ matters!

Is sparse SAT still hard? No: $2^{(1-\varepsilon)n}$.

How about sparse $k$-SAT?

**Lemma (Sparsification Lemma)**

Complexity of $k$-SAT with $m = f(k, \varepsilon) \cdot n$

\[ \leq \left[ \text{Complexity of general } k\text{-SAT} \right]/2^{\varepsilon n} \]

ETH $\Rightarrow$ may assume $m = O(n)$. 

Pătrașcu and Williams  Lower Bounds from SAT
### Reduction to $d$-SUM

<table>
<thead>
<tr>
<th>Problem</th>
<th>Variables</th>
<th>Clauses</th>
<th>Why</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$-SAT</td>
<td>n</td>
<td>m=O(n)</td>
<td>sparsification</td>
</tr>
<tr>
<td>3-SAT</td>
<td>$O(nk)$</td>
<td>$O(nk)$</td>
<td>[Cook]</td>
</tr>
<tr>
<td>1-in-3-SAT</td>
<td>$N = O(nk)$</td>
<td>$M = O(N)$</td>
<td>[GJ]</td>
</tr>
</tbody>
</table>

Partition variables $\rightarrow$ $d$ blocks of $\frac{N}{d}$ variables

Block $\rightarrow$ $2^{N/d}$ numbers of $M$ digits

digit $[i] = 1 \iff$ clause $i$ satisfied

Must find numbers to sum to 11...11.
Reduction to Set Disjointness

Partition variables \( \rightarrow X \cup Y \cup Z, \ |X| = |Y| = |Z| = \frac{n}{3}. \)

\( x \) induces \( S(x) = \{ \text{clauses not satisfied by } x \} \subseteq [m] \)
\( \Phi(xyz) = \text{true} \iff S(x) \cap S(y) \cap S(z) = \emptyset \)

Run communication protocol for “\( S(x) \cap S(y) \cap S(z) = \emptyset \)?”

- \( o(m) = o(n) \) bits of communication [Sparsity!]
- so enumerate all transcripts \( \pi \) ending in “Disjoint!”

Tripartite graph \( G: V = X \cup Y \cup Z \)
\( (x, y) \in X \times Y \iff \) Alice follows \( \pi \) on \( S(x), S(y) \)
\( (y, z) \in Y \times Z \iff \) Bob follows \( \pi \) on \( S(y), S(z) \)
\( (x, z) \in X \times Z \iff \) Carmen follows \( \pi \) on \( S(x), S(z) \)
Find triangle in \( O(N^{2.376}) = O((2^{n/3})^{2.376}) = O(1.74^n). \)
THE END