MIT 6.854/18.415: Advanced Algorithms	Spring 2016
Lecture 10 – March 7, 2016	
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1 Last Time

- min cost flow
- Goldberg-Tarjan

A mandatory homework problem (due April 1st):

Homework 1. *Let F be a family of sets closed under union, i.e.*

$$A, B \in F \implies A \cup B \in F.$$

Prove that there is an element x such that

$$|\{A \in F \text{ s.t. } x \in A\}| \geq \frac{|F|}{2}.$$

2 Introduction to Linear Programs

"Canonical form" of LPs:

- x, y are variables vectors, b, c are constant vectors, and A is a matrix (that represents the constraints in the linear program).
- **Primal (P):**

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax \leq b \\ & x \geq 0 \end{aligned}$$

- **Dual (D):**

$$\begin{aligned} \min\ & y^T b \\ \text{s.t. } & y^T A \geq c^T \\ & y \geq 0 \end{aligned}$$

- We say x is feasible if it meets constraints in (P), and y is feasible if it meets constraints in (D).

3 Weak Duality

Lemma 2 (Weak Duality)**.** *If x, y are feasible, then there's a relationship about "how good they are":*

$$c^T x \leq y^T b$$

Proof. Using $x \geq 0$ and $y^T A \geq c^T$, we find that

$$y^T Ax \geq c^T x.$$

Now, using $Ax \leq b$ and $y^T \geq 0$,

$$y^T Ax \leq y^T b.$$

So, using transitivity, we have shown that

$$c^T x \leq y^T b.$$

□

3.1 Application to Min Cut/Max Flow

Let's apply weak duality to max flow by formulating the problem as a linear program. Let $P_{s,t}$ be all $s - t$ paths in $G = (V, E)$ and let x_P be a weight assigned to each path $P \in P_{s,t}$.

$$\begin{aligned} \max \quad & \sum_{P \in P_{s,t}} x_P \\ \text{s.t.} \quad & \sum_{P \in P_{s,t}, e \in P} x_P \leq u(e) \\ & x_P \geq 0 \end{aligned}$$

Here our constraints can be written in matrix form by letting A be an $|E| \times |P_{s,t}|$ matrix that contains a 1 in position (e, P) if $e \in P$. All other entries are 0.

The dual of the above program is as follows:

$$\begin{aligned} \min \quad & \sum_{e} u(e) y(e) \\ \text{s.t.} \ \forall P \in P_{s,t}, \quad & \sum_{e \in P} y(e) \geq 1 \\ & y(e) \geq 0 \end{aligned}$$

We claim that an $s - t$ cut S corresponds to a feasible solution to the dual. Specifically, y contains an entry for every edge $e = (u, v)$ in G and we can set:

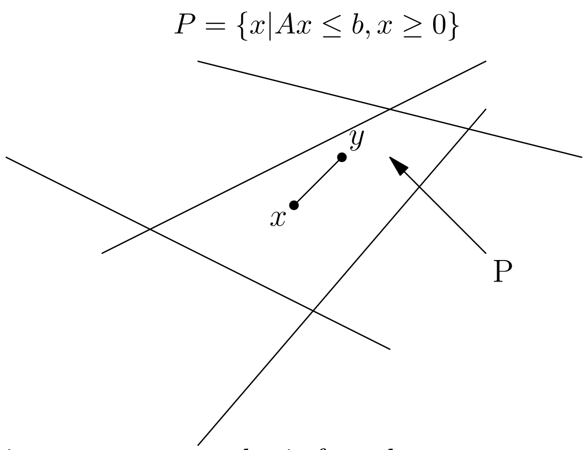
$$y(e) = \begin{cases} 1 & \text{if } u \in S, v \in V \setminus S \\ 0 & \text{else} \end{cases}$$

Then, $\sum_{e\in P} y(e) \geq 1 \; \forall P \in P_{s,t}$. This is because P is a path from s to t and thus at some point must cross the min cut, which means that it contains an edge with $y(e) = 1$.

Moreover, $\sum_e y(e)u(e) = cap(S, V \setminus S)$. Weak duality therefore implies a fact we already know well – any cut can be used to upper bound the maximum obtainable flow.

4 Projection Theorem and Farkas' Lemma

Geometrically, the constraints of a linear program each form a half space that constrains the location of any feasible solution.



This is called a polyhedron; it is convex.
For any $x, y \in P$ and $\lambda \in [0, 1]$:

$$\lambda x + (1-\lambda)y \in P$$

We're going to assume one basic fact about convex regions:

Theorem 3 (Projection theorem)**.** *If P is a nonempty closed, convex set, then:*

(1) For any b, there is a unique minimizer in P to

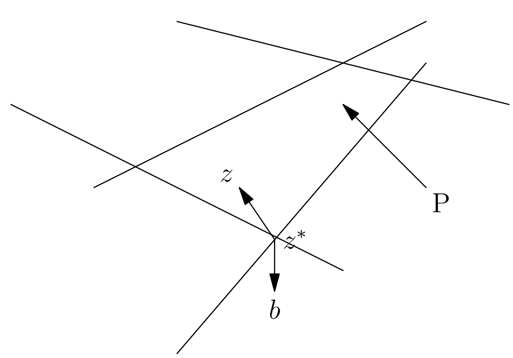
$$f(z) = \|z - b\|^2.$$

Call the optimal point $z^* = proj_P(b)$.

(2) z^* is the projection of b iff

$$(z - z^*)^T(b - z^*) \leq 0 \; \forall z \in P$$

Visually,



The projection theorem can be used to prove another basic fact about convex regions.

Lemma 4 (Farkas' lemma)**.** *Exactly one of the following holds:*

(1) $\exists x$ *s.t.* $Ax = b, x \geq 0$

(2) $\exists y$ *s.t.* $y^T A \geq 0$ *and* $y^T b < 0$

This version of Farkas' lemma corresponds to the standard form of an LP.

Proof. No more than one: if both (1) and (2) hold, set $c = 0$ in the standard form. Then from (2) there must be some y such that $y^T A \geq 0$ and $y^T b < 0$. However, by (1) there is some feasible x for the primal and since $c = 0$, $c^T x = 0$. $y^T b \leq 0 = c^T x$ violates weak duality.

At least one: Assume $\nexists x$ that satisfies (1). Then, we will construct a y that satisfies (2). Let $P = \{Ax \text{ s.t. } x \geq 0\}$. by assumption, $b \notin P$.

Let $\mathfrak{p} = proj_P(b)$. $\mathfrak{p} = Ax$ for some $x \geq 0$. Let $y = \mathfrak{p} - b$. We want to show that y satisfies (2).

Claim 1: $y^T A \geq 0$.

Proof: By the projection theorem,

$$(z - \mathfrak{p})^T (b - \mathfrak{p}) \leq 0 \; \forall z \in P.$$

With $\mathfrak{p} = Ax$, set $z = A(x + e_i)$. Then

$$(A(x + e_i) - Ax)^T (b - \mathfrak{p}) = {e_i}^T A^T (b - p) \leq 0 \implies {e_i}^T A^T y \geq 0.$$

In other words, the i^{th} entry of $A^T y$ is ≥ 0. By setting z for each i, we see that $A^T y \geq 0$.

Claim 2: $y^T b \leq 0$.

Proof: $b^T y = (\mathfrak{p} - y)^T y = \mathfrak{p}^T y - y^T y$. Since b is not in P, $y^T y = \|y\|_2 > 0$, so we just need to show that $\mathfrak{p}^T y \leq 0$. Since $0 \in P$, then by the projection theorem,

$$(0 - \mathfrak{p})^T (b - \mathfrak{p}) \leq 0 \implies \mathfrak{p}^T y \leq 0.$$

Conclusion: So, we've shown that (1) and (2) can't both be true, and that if (1) is true, we just take it; if (1) is not true, then we can construct a y such that (2) is holds.

$\square$