1 Overview

Recall:

- Weak Duality
- Projection onto a convex set
- Farkas’ Lemma

Today:

- Strong Duality
- Zero Sum Games
- Complementary Slackness + relation to strong and weak duality

2 Farkas’ Lemma

Recall standard form of a linear program:

\[
\begin{align*}
\text{(primal)} & \quad \max c^T x \text{ s.t. } Ax = b, \ x \geq 0 \\
\text{(dual)} & \quad \min y^T b \text{ s.t. } y^T A \geq c^T
\end{align*}
\]

And the original form of Farkas’ lemma:

**Lemma 1** (Farkas’). *Exactly 1 of the following holds:*

(1) \( \exists x \text{ s.t. } Ax = b, \ x \geq 0 \)

(2) \( \exists y \text{ s.t. } y^T A \geq 0, \ y^T b < 0 \)

That is, either there is a feasible \( x \), or there is a \( y \) that certifies no such \( x \) exists.

We now prove that an alternate form of Farkas’ lemma holds.

**Lemma 2** (Farkas’, alternate form). *Exactly 1 of the following holds:*

(1’) \( \exists x \text{ s.t. } Ax \leq b \)

(2’) \( \exists y \text{ s.t. } y^T A = 0, \ y \geq 0, \ y^T b < 0 \)
Proof. We map (1') into the form (1). Consider the following linear system.

\[
\begin{pmatrix}
A & -A & I
\end{pmatrix}
\begin{pmatrix}
x^+ \\
x^- \\
s
\end{pmatrix} =
\begin{pmatrix}
b \\
0 \\
0
\end{pmatrix}
\geq 0
\]

(1'"

or expanded out

\[Ax^+ - Ax^- + s = b, x^+ \geq 0, x^- \geq 0, s \geq 0\]

s represents “slack” variables (the amount of room remaining for each constraint), and A is the same as in (1').

Claim 3. 1'" and 1' are equivalent. We can convert any solution of 1' to a solution of 1" and vice versa.

Suppose we have a solution to 1', x with \(Ax \leq b\). Then we can construct a solution to 1" as follows.

\[
\begin{cases}
x^+ = \max(x, 0) \\
x^- = -\min(x, 0) \\
s = b - Ax
\end{cases}
\]

This is a solution to 1".

Conversely, suppose we are given a solution \(x^+, x^-, s\) to the 1" system. Then,

\[x = x^+ - x^-\]

is a valid solution.

Now looking back at the original Farkas’ lemma, we find the corresponding 2" having the form

\[y^T (A - A I) \geq 0, y^T b < 0\]

1" has no solution if and only if 2" holds.

2" is equivalent to 2' because \(y^T A \geq 0\) and \(-y^T A \geq 0\) implies \(y^T A = 0, y \geq 0, y^T b < 0\). \(\square\)

3 Strong Duality

Let \(z^* \in \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}\) be the optimal value for (p). Note that \(-\infty\) corresponds to (p) being infeasible and \(+\infty\) corresponds to an unbounded objective value.

Let \(w^* \in \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}\) be optimal value for (d). Here \(-\infty\) corresponds to an unbounded objective value and \(+\infty\) corresponds to (p) being infeasible.

Theorem 4. If either (p) or (d) is feasible, \(z^* = w^*\).
Proof. Assume without loss of generality that (p) is feasible.

If (p) is feasible and (p) is unbounded, then \( z^* = +\infty \). Then \( w^* = +\infty \) by weak duality.

Otherwise, let \( x^* \) be an optimal solution to (p). Then \( z^* = c^T x^* \).

We’re looking for \( y \) with \( b^T y \leq 0 \) and \( A^T y \geq c \) or equivalently,

\[
\begin{pmatrix} -A^T \\ b^T \end{pmatrix} y \leq \begin{pmatrix} -c \\ z^* \end{pmatrix}
\]

If there is no such \( y \), then Farkas’ lemma tells us that there exists an \( x, \lambda \) such that

\[
(*) \left( \begin{array}{c} x^T \\ -A \end{array} \right) \left( \begin{array}{c} -A^T \\ B^T \end{array} \right) = 0 x \geq 0, \lambda \geq 0, -x^T c + \lambda z^* < 0
\]

Case 1: If \( \lambda > 0 \), then rescale both \( x \) and \( \lambda \) by \( \frac{1}{\lambda} \) to get \( (x, \lambda) = \left( \frac{x}{\lambda}, 1 \right) \). This is still feasible by (*)

The new system satisfies \( Ax = b, x \geq 0 \) and \( x^T x > z^* \). However, since \( z^* \) was optimal we have a contradiction.

Case 2: If \( \lambda = 0 \), then \( Ax = 0, x \geq 0 \) and \( c^T x > 0 \). Now, \( x^* + x \) is feasible for (p) so \( c^T (x^* + x) > c^T x^* = z^* \), but since \( z^* \) is optimal this is a contradiction.

\[
\square
\]

4 Zero Sum Games

A powerful application of strong duality is to zero sum games. A zero sum game associates with every strategy \( a \) for player \( A \) and \( b \) for player \( B \), a known payoff of \( M_{a,b} \) for \( A \) and \(-M_{a,b}\) for \( B \).

The Colonel Blotto Games are examples of zero-sum games. Imagine \( A \) has \( r \) armies and \( B \) has \( s \) armies. Both players divide their armies among 2 mountain passes. \( A \) gets \(-1\) if he is outnumbered on either pass. Otherwise he gets \( 1 \).
Theorem 5. (Von Neumann) There are randomized strategies \((x, y)\) where \(x, y \geq 0\), \(\sum x_i = 1\) and \(\sum y_i = 1\), which represent distributions among the strategies, and a value \(V\) such that
\[
\begin{align*}
x^T M &\geq V 1 \\
M y &\leq V 1
\end{align*}
\]
where \(1\) is the vector of all 1s.

Intuitively, (1) corresponds to the amount \(A\) can guarantee by playing \(x\) and (2) corresponds to the amount \(B\) can guarantee by playing \(y\).

We present a sketch of the proof. We first set up (1) as an LP which maximizes \(V\). Then the dual of this linear program will minimize \(V\) and give (2). Use strong duality to finish. □

The \(V\) from above is called the game value. While the theorem above gives a powerful characterization of \(V\), \(V\) can also be computed.

5 Complementary Slackness

Lemma 6. (Complementary Slackness) Let \(x\) and \(y\) be feasible for the primal and the dual respectively. Then both \(x\) and \(y\) are optimal if and only if \(x_i > 0\) implies \((y^T A)_i = c_i\)

Proof. We follow the proof of weak duality. Because \(y^T A \geq c^T\), \(x \geq 0\) and \(b = Ax\), we have
\[
y^T b = y^T Ax \geq c^T x.
\]
If for any \(i\), we have \(x_i > 0\) and \((y^T A)_i > c_i\), then this inequality becomes strict, i.e. \(y^T Ax > c^T x\), and so \(x\) and \(y\) aren’t optimal.
If for all \(i\), \(x_i > 0\) implies \((y^T A)_i = c_i\), then the inequality becomes equality, i.e. \(y^T Ax = c^T x\), and so \(x\) and \(y\) are both optimal. □

6 Physics Interpretation

We will give a physical interpretation of duality through physics. Let
\[
p = \{y | A^T y \geq c\}.
\]

Pictorally the setup looks like:
with gravity in the $-b$ direction. We can create a dictionary between LP terms and physics terms as follows:

<table>
<thead>
<tr>
<th>LPs</th>
<th>Physics</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-b$</td>
<td>gravity</td>
</tr>
<tr>
<td>rows of $A^T$</td>
<td>normals to walls</td>
</tr>
<tr>
<td>$\exists x \geq 0, x^T A^T = b$</td>
<td>forces balance at equilibrium</td>
</tr>
<tr>
<td>complementary slackness: $x_i &gt; 0 \Rightarrow (A^T y)_i = c_i$</td>
<td>only walls touching, exert force</td>
</tr>
</tbody>
</table>

This can be turned into a proof of strong duality, but the details are subtle.