1 Semidefinite Programming

Earlier we saw a framework for approximating NP-hard problems by relaxing integer linear programming (ILP) to general linear programming (LP). Here we see a problem for which this method does not work and introduce a more powerful technique, semidefinite programming, to solve it.

2 MAXCUT

Given $G = (V, E)$ we want to choose a subset $U \subseteq V$ so as to maximize $|E(U, V \setminus U)|$. I.e. we want to maximize the number of edges connecting a node in $U$ with a node outside of $U$.

Observe that the maximum cut equals $|E|$ iff $G$ is bipartite.

Let’s give a first attempt at solving this problem via integer linear programming:

**Integer Linear Program:**

$$\text{max } \sum_{(u,v) \in E} Z_{(u,v)}$$

subject to $Z_{(u,v)} \leq X_u + X_v$

$$Z_{(u,v)} \leq (1 - X_u) + (1 - X_v)$$

$Z_{(u,v)}, X_v \in \{0, 1\}$

$Z_{(u,v)}$ can only be set to 1 when $X_u \neq X_v$. Setting $X_u = 1$ corresponds to placing $u$ in $U$ and we can increase our objective function by 1 for every $v$ where $X_v = 0$ (i.e. where $v$ was not also placed in $U$).

**ILP relaxation:** If we relax the integer constraint $Z_{u,v}, X_v \in \{0, 1\}$ to $Z_{u,v}, X_v \in [0, 1]$ we can satisfy the LP constraints by setting every $Z_{(u,v)} = 1$ and every $X_v = \frac{1}{2}$. This achieves a value of $|E|$. However, for the complete graph, MAXCUT $\approx \frac{|E|}{2}$. Accordingly, we can’t expect any rounding strategy to achieve better than a $1/2$-approximation.

And in fact, we can trivially obtain a $1/2$ approximation in expectation: assign each node independently with probability $\frac{1}{2}$ to $U$. 


3 Positive Semidefinite Matrices

Let $X \in \mathbb{R}^{n \times n}$ be symmetric. We say $X$ is positive semidefinite (PSD or $X \succeq 0$) if the following equivalent statements are true:

1. $\forall a \in \mathbb{R}^n \ a^T X a \geq 0$
2. $X = B^T B$ for some $B$
3. All of $X$’s eigenvalues are non-negative

4 Semidefinite Programs (SDP)

The standard form of a semidefinite program is analogous to the standard form of a linear program. In the following equations, let $C$, $X$, and $A_i$ be $n \times n$ matrices.

$$
\begin{align*}
\min_{X} & \quad \langle C, X \rangle = \sum_{i,j} c_{i,j} x_{i,j} \text{ (the Frobenius product of $C$ and $X$)} \\
\text{s.t.} & \quad \langle A_i, X \rangle = b_i \quad \forall i \in (1, ..., m) \\
& \quad X \succeq 0
\end{align*}
$$

Note that this program corresponds exactly to linear programming when all matrices are diagonal. The feasible region for the SDP is $\{ X \mid \langle A_i, X \rangle = b_i \quad \forall i \in (1, ..., m), \ a^T X a \geq 0 \quad \forall a \}$. So, the requirement that $X$ be positive semidefinite effectively creates an infinite number of linear constraints on $X$.

We can solve semidefinite programs using either the ellipsoid method or interior-point methods. However, unlike linear programs we can only obtain solutions to within arbitrary accuracy, not exact solutions. This is because the bit-complexity of solutions to linear programs are bounded by a function of the size of the original problem, meaning that if we converge “close enough” for a linear program we can obtain an exact answer. This property does not hold for semidefinite programs.

4.1 Duality

The dual of the program above can be written as

$$
\begin{align*}
\max_{y} & \quad b^T y \\
\text{s.t.} & \quad \sum_{i=1}^{m} y_i A_i + S = C \\
& \quad S \succeq 0
\end{align*}
$$

where $y$ is a length-$m$ vector and $S$ is an $n \times n$ matrix.
4.2 Basic Facts

**Fact 1** $\langle A, X \rangle = \text{Tr}(A^T X)$

This fact follows from the definition of matrix multiplication.

**Fact 2** $\text{Tr}(AB) = \text{Tr}(BA)$

More generally, cyclic permutations of the order in which matrices are multiplied do not affect the trace. E.g. $\text{Tr}(ABC) = \text{Tr}(CAB) = \text{Tr}(BCA)$.

4.3 Weak Duality

**Lemma 1.** Let $X$ be symmetric, then $X \succeq 0 \iff \langle A, X \rangle \geq 0 \forall A \succeq 0$

**Proof** Suppose $X \not\succeq 0$. Then $a^T X a < 0$ for some vector $a$.

Let $A = aa^T$. Clearly, $A \succeq 0$.

Then $\langle A, X \rangle = \text{Tr}(A^T X) = \text{Tr}(aa^T X) = \text{Tr}(a^T X a) < 0$

Now suppose $X \succeq 0$

Let $A \succeq 0$. Then $A = BB^T = \sum_i b_i b_i^T$ for some matrix $B$ with columns $b_1, b_2, ... b_n$.

Then $\langle A, X \rangle = \sum_i b_i^T X b_i \geq 0$ by the same logic as above.

**Lemma 2** (Weak Duality). If $x/y$ are feasible for $(P)/(D)$ then $b^T y \leq \langle C, X \rangle$.

**Proof** By the feasibility of $y$, we have

$$\langle C, X \rangle = \left\langle \sum y_i A_i, X \right\rangle + \langle S, X \rangle.$$  

By the feasibility of $x$, we have  

$$\left\langle \sum y_i A_i, X \right\rangle = b^T y.$$  

And by Lemma 1, we have  

$$\langle S, X \rangle \geq 0.$$  

Thus  

$$b^T y = \langle C, X \rangle - \langle S, X \rangle \leq \langle C, X \rangle.$$  

So any solution for the dual lower bounds the minimum of the primal.

4.4 Strong Duality

Warning: We won’t cover details, but strong duality “usually holds” for semidefinite programs. Specifically, it holds under the following condition:

**Proposition 3** (Slater’s condition). Strong duality holds if the feasible region has an interior point.
5 Goemans-Williamson

Semidefinite programming provides a generalization of linear programming that is often much more powerful for solving hard approximation problems. Here we will see a famous relax and round procedure for MAXCUT based on SDPs. Specifically, consider the following program:

\[
\begin{align*}
\max & \quad \sum_{(u,v) \in E} \frac{1}{2} - \frac{1}{2} X_{uv} \\
\text{s.t.} & \quad X_{uu} = 1, \forall u \\
& \quad X \succeq 0
\end{align*}
\]

Why is this a relaxation to MAXCUT?

We can construct a feasible solution to this SDP from a solution to MAXCUT.

Let

\[
x_u = \begin{cases} 
1, & \text{if } u \in U, \\
-1, & \text{otherwise.} 
\end{cases}
\]

Set \(X = xx^T\). The objective value is the number of edges across the cut because \(X_{uv}\) is \(-1\) if \(u\) and \(v\) are on opposite sides of the cut and \(X_{uv}\) equals \(1\) otherwise.

6 Hyperplane Rounding

Goemans and Williamson show how to round this semidefinite program to obtain the following approximation guarantee:

**Theorem 4 (Goemans-Williamson [1]).** Let

\[
\alpha_{gw} = \min_{0 \leq \theta \leq \pi} \frac{2}{\pi} \frac{\theta}{1 - \cos \theta} \approx 0.87856 \ldots
\]

There is an algorithm with obtains an \(\alpha_{gw}\)-approximation for MAXCUT in expectation.

We take \(X\) which is the optimal solution to the SDP. Since \(X\) is positive semidefinite and has diagonal entries equal to 1, it can be written as \(X = YY^T\) for some \(Y\). Accordingly, there are vectors \(\{y_u\}\) so that \(X_{uv} = \langle y_u, y_v \rangle\).

Choose a vector \(a\) uniformly on the sphere.

Set \(x_u = \text{sgn}(\langle a, y_u \rangle)\).

We want to analyze the expected contribution of each edge \((u, v)\) to our rounded solution.

The contribution to the SDP is \(\frac{1}{2} - \frac{1}{2} \langle y_u, y_v \rangle = \frac{1}{2} \cos \theta\) where \(\theta\) is the angle between vectors \(y_u\) and \(y_v\).

For a contribution to the cut, we have that \(a\) cuts edge \((u, v)\) if and only if its orthogonal hyperplane lies between the vectors \(y_u\) and \(y_v\). If we assume, without loss of generality, that \(0 \leq \theta \leq \pi\), then
this occurs with probability \( \frac{\theta}{\pi} \). Accordingly, the expected contribution of \((u,v)\) to our cut value is \( \frac{\theta}{\pi} \).

Thus, the worst case contribution for the edge as a fraction of its contribution to the SDP is:

\[
\min_{0 \leq \theta \leq \pi} \frac{\theta}{\frac{1 - \cos \theta}{2}} = \min_{0 \leq \theta \leq \pi} \frac{2\theta}{\pi(1 - \cos \theta)}
\]

as desired.

The Goemans-Williamson rounding scheme gives the best known approximation to MAXCUT and it may be the best approximation possible via any efficient algorithm.

**Theorem 5** (Khot, Kindler, Mossel, O’Donnell [2])(Mossel, O’Donnell, Oleszkiewicz [3]). *Assuming the “Unique Games Conjecture” it is NP-hard to approximate MAXCUT better than \( \alpha_{gw} \).*

The UGC, or Unique Games Conjecture, is a controversial, far-reaching conjecture in complexity theory. It states that there exist constant limits for the best approximation algorithms for certain NP-hard problems, and makes some statements about what those bounds are. Many believe it to be true in some form, many believe it to be false in some form.

**References**

