Lecture #10

<u>Last Time</u>: Min Cost Flow and Goldberg-Tarjan

augment along cycle of min mean cost

reminder: fixed proof of Lemma 3

<u>Today</u>: Introduction to Linear Programs

Many examples so far of pattern

optimization problem	$\longleftrightarrow$	certificate of optimality
max flow	$\longleftrightarrow$	min cut
min cost flow	$\longleftrightarrow$	potentials with nonnegative reduced costs

Many more examples in combinatorial optimization, most derived from <u>linear</u> <u>programming</u>:

Canonical form

(P) $\max\ c^T x$ s.t. $Ax \le b$, $x \ge 0$

(D) $\min\ y^T b$ s.t. $y^T A \ge c^T$, $y \ge 0$

Here "$\le$" is a component wise constraint

We say that x is feasible if it meets constraints in (P), similarly for y.

Lemma: (weak duality) If x and y are feasible then $c^Tx \le y^Tb$

Proof: Using that $x \ge 0$ and $y^TA \ge c^T$ we have

$$y^TAx \ge c^Tx$$

Now using that $Ax \le b$ and $y \ge 0$ we have

$$y^TAx \le y^Tb$$

Combining inequalities completes proof $\boxtimes$

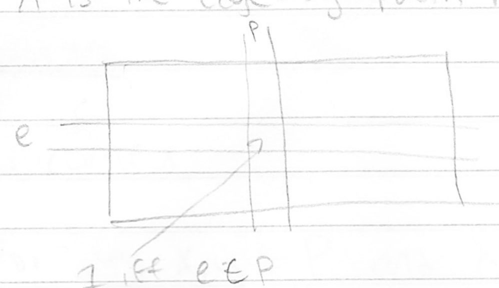
Let's apply this to max flow. Let $\mathcal{P}_{s,t}$ be all s-t paths in G.

$$\max \sum_{p \in \mathcal{P}_{s,t}} x_p \qquad\qquad \min \sum_e u(e)\,y(e)$$

$$\text{s.t.} \sum_{\substack{p \in \mathcal{P}_{s,t} \\ p \ni e}} x_p \le u(e) \qquad\qquad \text{s.t.} \sum_{e \in p} y(e) \ge 1 \quad \forall p \in \mathcal{P}_{s,t}$$

$$x_p \ge 0 \qquad\qquad y(e) \ge 0$$

Here A is the edge-by-path matrix



How is an s-t cut a feasible solution to the dual? Let $e = (u,v)$

$$\text{Let } y(e) = \begin{cases} 1 & \text{if } u \in S,\ v \in V \setminus S \\ 0 & \text{else} \end{cases}$$

Then $\sum_{e \in p} y(e) \geq 1 \quad \forall p \in P_{s,t}$, because the s-t path must cross at least one $y(e)=1$ edge

Moreover $\sum_{e} y(e) u(e) = cap(S, V \setminus S)$

In fact, duality gives us a more general way to prove c.b.s. via fractional cuts, but we do not need them here

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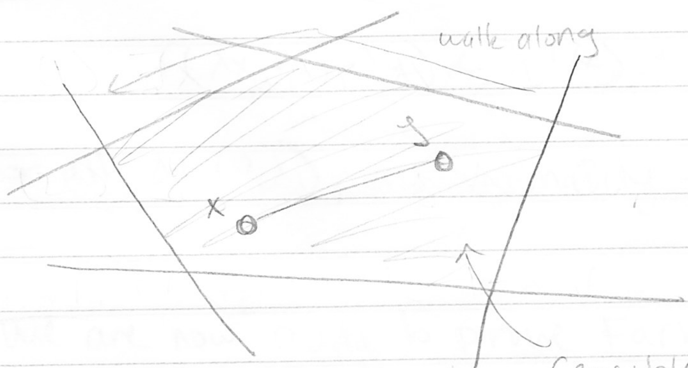
Let's draw a picture:



$P = \{x \mid Ax \leq b, x \geq 0\}$

We call a finite intersection of halfspaces a polyhedron

It is convex:

for any $x, y \in P$ and $\lambda \in [0,1]$ we have

$$\lambda x + (1-\lambda) y \in P$$

basic feasible soln $\equiv$ vertex $\equiv$ extreme point

intersection of n tight constraints   not convex combination of x ≠ y   unique optimal solution in some direction projection thm

Thm. If P is a nonempty, closed convex set then

(1) For any b, there is a unique minimizer to

$$f(z) = \|z-b\|^2$$

over P. Call the optimal point p - the projection $p = \text{proj}_P(b)$

(2) p is the projection of b iff

go back to picture

$$(z-p)^T(b-p) \leq 0 \qquad \forall z \in P$$

Idea: Follows from fact that $f(z)$ is strictly convex:

$$f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda) f(y)$$

for all $\lambda \in (0,1)$ and $x \neq y$, and optimality condition $\nabla f(p)^T (z-p) \geq 0$

We are now ready to prove Farkas' Lemma:

Lemma: Exactly one of the following holds

(1) $\exists x$ s.t. $Ax = b$, $x \geq 0$

(2) $\exists y$ s.t. $y^T A \geq 0$ and $y^T b < 0$

another duality pair "standard form"

Proof: If both (1) and (2) hold set $c = 0$.

Then $y^T b < c^T x$, but since x and y are feasible, this violates weak duality for "standard form"

Now assume $\nexists\, x$ that satisfies (1). We will construct y that satisfies (2)

Let $P = \{Ax \text{ s.t. } x \geq 0\}$, by assumption $b \notin P$

Let $p = \text{Proj}_P(b)$, $p = Ax$ for $x \geq 0$ (since $p \in P$) and $y = p - b$

<u>Claim #1</u> $y^T A \geq 0$

<u>Proof:</u> Projection thm $\Rightarrow (z-p)^T(b-p) \leq 0 \quad \forall z \in P$

Let $z = A(x + \underset{\uparrow}{e_i})$ (i^{th} standard basis vector), then

$$(z-p)^T(b-p) = (A(x+e_i) - Ax)^T(b-p)$$

$$= e_i^T A^T (b-p) \leq 0$$

This is equivalent to $e_i^T A^T y \geq 0$, and this holds for all i. ☒

<u>Claim #2</u> $b^T y < 0$

<u>Proof:</u> $b^T y = (p-y)^T y = p^T y - y^T y$

Now $0 \in P$, then again by projection thm

$$(z-p)^T\underbrace{(b-p)}_{-y} = p^T y \leq 0$$

And since $y^T y > 0$, we're done ☒

This establishes both constraints in (2) ☒

There are many equivalent formulations

(1') $[A, -A, I] \begin{bmatrix} x^+ \\ x^- \\ s \end{bmatrix} = b, \quad \begin{bmatrix} x^+ \\ x^- \\ s \end{bmatrix} \geq 0$

This is a system of inequalities as in (1), equivalent to

$$A\overset{\text{unconstrained}}{x} \leq b$$

Thus Farkas' lemma yields that exactly one of (1') and

(2') $y^T [A \ -A \ I] \geq 0, \quad y^T b < 0$

holds. Moreover (2') is equivalent to

$y^T A = 0, \quad y \geq 0$ and $y^T b < 0$

Lemma: (alternate Farkas') Exactly one of the following conditions holds

(1') $\exists x$ s.t. $Ax \leq b$

(2') $\exists y$ s.t. $y^T A = 0$, $y \geq 0$ and $y^T b < 0$