Lecture #11

<u>Last Time</u>: Introduction to Linear Programs
weak duality, projection onto convex set, Farkas' Lemma

<u>Today</u>: Strong Duality and Complementary Slackness

Recall "standard" form

(P) $\max\ c^T x$ s.t. $Ax = b$, $x \geq 0$

(D) $\min\ y^T b$ s.t. $y^T A \geq c^T$

And Farkas' Lemma (proved last time)

<u>Lemma</u>: Exactly one of the following holds

(1) $\exists x$ s.t. $Ax = b$, $x \geq 0$ "(P) is feasible"

(2) $\exists y$ s.t. $y^T A \geq 0$, $y^T b < 0$

Many forms of LPs, weak duality, Farkas' lemma, etc

Once you have one, can obtain others by reduction

<u>Lemma</u>: Exactly one of the following holds

(1') $\exists x$ s.t. $Ax \leq b$

(2') $\exists y$ s.t. $y^T A = 0$, $y \geq 0$, $y^T b < 0$

Proof: Let's map (1') into the form of (1)

Consider the linear system:

$$(1) \quad [A, -A, I] \begin{bmatrix} x^+ \\ x^- \\ s \end{bmatrix} = b, \quad \begin{bmatrix} x^+ \\ x^- \\ s \end{bmatrix} \geq 0$$

slack vars

given a soln x with $Ax \leq b$, we can take

$$x^+ = \max(x, 0), \quad x^- = -\min(x, 0), \quad s = b - Ax$$

to get a soln to (1). Conversely given a soln to (1), set

$$x = x^+ - x^-$$

and it satisfies $Ax \leq b$.

Now appealing to our first version of Farkas' lemma, if (1) has no soln then

$$y^T [A, -A, I] \geq 0, \quad y^T b < 0$$

does. This is equivalent to

$$y^T A = 0, \quad y \geq 0, \quad y^T b < 0 \quad \blacksquare$$

Pictorially, the proof technique is:

$$\begin{tikzcd} (1) \arrow[r, leftrightarrow, "\text{Farkas}"] & (2) \arrow[d, "\text{unwrapping reduction}"] \\ (1') \arrow[u, "\text{reduction}"] & (2') \end{tikzcd}$$

This is a general pattern how to obtain alternate forms for facts about LPs.

Now let's prove strong duality.

Let $z^* \in \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ be optimal value of (P)

$\uparrow$ infeasible $\quad$ $\uparrow$ unbounded objective value

Let $w^* \in \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ " of (D)

$\uparrow$ unbounded $\quad$ $\uparrow$ infeasible

<u>Theorem</u> If either (P) or (D) is feasible

$$z^* = w^*$$

assuming (P) is feasible, similar argument works if (D) is feasible

<u>Proof:</u> If (P) is unbounded and $z^* = +\infty$, then by weak duality $w^* = +\infty$.

Otherwise let x^* be optimal solution to (P), $z^* = c^T x$
we are looking for y with

$$\left\{ \begin{array}{l} b^T y \le z^* \\ A^T y \ge c \end{array} \right\} = \left\{ \begin{bmatrix} -A^T \\ b^T \end{bmatrix} y \le \begin{bmatrix} -c \\ z^* \end{bmatrix} \right\}$$

If there is no such y then by Farkas' Lemma (alt version)

$$\exists x, \lambda \text{ s.t. } [x^T, \lambda] \begin{bmatrix} -A^T \\ b^T \end{bmatrix} = 0 \qquad (*)$$

$$x^T \ge 0,\ \lambda \ge 0 \text{ and } -x^T c + \lambda z^* < 0$$

Case #1: If $\lambda > 0$ then rescale

$$x \leftarrow \frac{x}{\lambda}$$
$$\lambda \leftarrow 1$$

still feasible for (*) and $Ax = b$, $x \geq 0$, $c^T x > z^*$, thus y^* was not optimal $\Rightarrow\Leftarrow$

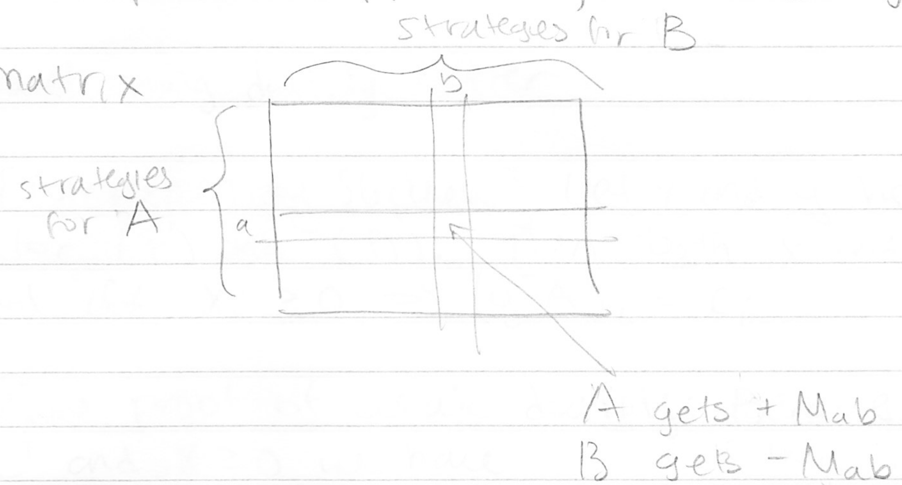
case #2: If $\lambda = 0$ then $Ax = 0$, $x \geq 0$, $c^T x > 0$

Now $x^* + x$ is feasible for (P), but strictly larger objective value $\Rightarrow\Leftarrow$

Thus there is a y that certifies x^* is optimal. $\square$

Let's see a powerful application, to zero sum games

Payoff matrix



Example: Colonel Blotto e.g.

(1) A has r armies, B has s armies, both divide their armies among two mountains

(2) A loses if he is outnumbered in either, otherwise wins

$x \geq 0$ and $\sum x_i = 1$
$y \geq 0$ and $\sum y_i = 1$

Theorem [von Neumann] There are randomized strategies x, y and a value v s.t.

(1) $x^T M \geq v \cdot \vec{1}$ "amount A can guarantee"

(2) $M y \leq v \vec{1}$ "amount B can guarantee"

Proof [sketch]: Set up (1) as an LP, then (2) is the dual $\boxtimes$
$\uparrow$ max v ... $\uparrow$ min v

v is called the game value, and not only does the theorem above give a powerful characterization, it can even be computed.

Lots of things go wrong when not zero sum, or with more than two players.

Let's unravel strong duality further

Lemma [Complementary Slackness] Let x and y be feasible for (P) and (D). Then both x and y are optimal iff $x_i > 0 \Rightarrow (y^T A)_i = c_i$

Proof: Follow proof of weak duality: Because $y^T A \geq c^T$ and $x \geq 0$ we have

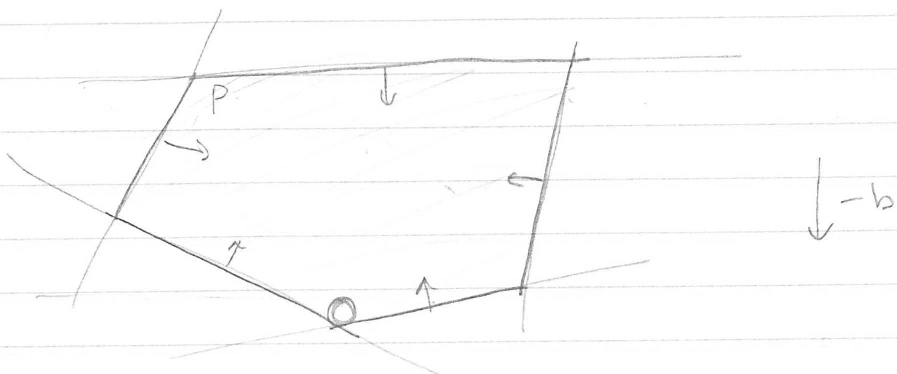
$$y^T b = y^T \underbrace{Ax}_{=b} \geq c^T x$$

If for any i, $x_i > 0$ and $(y^T A)_i > c_i$ then "$\geq$" is actually "$>$", thus x and y cannot both be optimal.

If for all i, either $y_i = 0$ or $(y^T A)_i = c_i$ or both then "$\geq$" is actually "$=$" which implies both x and y are optimal. $\boxtimes$

Now let's give an interpretation of duality thru physics

Let $P = \{ y \text{ s.t. } A^T y \geq c \}$



Dictionary between LPs and physics

LPs	Physics
$-b$	"gravity"
rows of A^T	normals to walls
$\exists x \geq 0,\ x^T A^T = b$	forces balance at equlibrium
$x_i > 0 \Rightarrow (A^T y)_i = c_i$	only walls touching, exert force

complementary slackness

Note: If x and y satisfy complementary slackness, there is no duality gap. This can be turned into proof of strong duality, but details are subtle