Lecture #20

Last Time: MAXCUT and Semidefinite Programs

(0.878-approx via powerful new convex programs)

Today: Grothendieck's Inequality and Lovasz Theta Fctn

We will see some powerful applications of SDPs. First consider

$$\max_{x_i, y_j \in \{\pm 1\}} \sum_{ij} A_{ij} x_i y_j = OPT$$

Is this problem easy or hard? MAXCUT is a special case where $A_{ii} = 2|E|$ (force $x_i = y_i$)

$$A_{ij} = \begin{cases} -\frac{1}{4} & \text{if } (i,j) \in E \\ 0 & \text{else} \end{cases}$$

Then

$$\sum_{i \neq j} A_{ij} x_i y_j = \frac{1}{2}(\text{edges cut} - \text{edges not cut})$$

$$+ \frac{|E|}{2} \qquad\qquad + \frac{|E|}{2}$$

$$= \text{edges cut.}$$

So for A as above, optimum is achieved by MAXCUT. Let's try to approximate it via an SDP:

$$(*) \qquad \max \sum_{ij} A_{ij} \overset{m \times n}{Z_{ij}}$$

$$\text{s.t.} \quad \underbrace{\begin{array}{c c} & \begin{array}{cc} m & n \end{array} \\ \begin{array}{c} m \\ n \end{array} & \left[\begin{array}{c|c} 1_{1_{1_1}} & Z \\ \hline Z^T & 1_{1_{1_1}} \end{array}\right] \end{array}}_{B} \succeq 0$$

In particular, B is $(m+n)\times(m+n)$ and has ones along the diagonal. As before:

$$B = \begin{array}{c} m \\ n \end{array}\begin{bmatrix} W \end{bmatrix} \begin{array}{c} \quad m \quad n \\ \begin{bmatrix} W^T \end{bmatrix} \end{array}$$

The first m rows of W are $u_1, u_2 \dots u_m$, the last n rows of W are $v_1, v_2 \dots v_n$, all unit vectors.

Thus (*) is equivalent to:

$$\max_{\substack{\|u_i\|=1 \\ \|v_j\|=1}} \sum_{ij} A_{ij} \underbrace{\langle u_i, v_j \rangle}_{z_{ij}} = OPT'$$

<u>Theorem</u> [Grothendieck, Krivine] (0.56-approx)

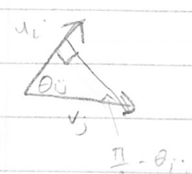
$$OPT \le OPT' \le \frac{\pi}{2\ln(1+\sqrt{2})} OPT$$

Can we just use hyperplane rounding, and set

$$x_i = \text{sgn}(a^T u_i), \quad y_j = \text{sgn}(a^T v_j)$$

for random a? No, consider each term's contribution

<u>to SDP</u>	<u>to value</u>
$A_{ij} \underbrace{\langle u_i, v_j \rangle}_{\cos\theta_{ij}}$	$A_{ij}\left(\underbrace{-\frac{\theta_{ij}}{\pi}}_{\Pr x_i \neq y_j} + \underbrace{\left(1-\frac{\theta_{ij}}{\pi}\right)}_{\Pr x_i = y_j}\right)$



$$= A_{ij}\left(\frac{2}{\pi}\right)\overbrace{\left(\frac{\pi}{2} - \theta_{ij}\right)}^{\arcsin(\langle u_i, v_j \rangle)}$$

Fact: $\mathbb{E}[\text{sgn}(a^T u_i)\,\text{sgn}(a^T v_j)] = \left(\frac{2}{\pi}\right)\left(\arcsin(\langle u_i, v_j \rangle)\right)$

This is a nonlinear relationship – e.g. we could have

SDP	value of rounding
$+7\ -6\ +7\ -6 \ldots$	$+7\alpha\ -6\ +7\alpha\ -6 \ldots$

and we wouldn't get any approx algorithm!

A beautiful idea is to create new vectors:

Lemma: [Main] For any unit vectors $u_1, \ldots u_m$ and $v_1 \ldots v_n$, there is a set of unit vectors $u_1' \ldots u_m'$ and $v_1' \ldots v_n'$ with

$$\mathbb{E}\left[\operatorname{sgn}(a^T u_i') \operatorname{sgn}(a^T v_j')\right] = \frac{2}{\pi} \ln(1+\sqrt{2}) \langle u_i, v_j \rangle$$

Now we can prove the theorem:

Grothendieck-Krivine Hyperplane Rounding:

Z is optimal soln to SDP

Find unit vectors u_i, v_j with $Z_{ij} = \langle u_i, v_j \rangle$

Find unit vectors u_i', v_j' as in Lemma

Choose a uniformly on the sphere, set

$$x_i = \operatorname{sgn}(a^T u_i'), \quad y_j = \operatorname{sgn}(a^T v_j')$$

To analyze the expected value, consider a term

$$A_{ij} \langle u_i, v_j \rangle \longmapsto \frac{2}{\pi} \ln(1+\sqrt{2}) A_{ij} \langle u_i, v_j \rangle$$

Thus summing over i,j we get $OPT \geq \frac{2}{\pi} \ln(1+\sqrt{2})\ OPT'$

Now let's prove the main lemma:

Let $c = \sinh^{-1}(1) = \ln(1+\sqrt{2})$, then by Taylor's Theorem

$$\sin(c\langle u,v\rangle) = \sum_{k=0}^{\infty} (-1)^k \frac{c^{2k+1}}{(2k+1)!} \langle u,v\rangle^{2k+1}$$

Now if $u^{\otimes 2} = [(u_1)^2, u_1 u_2, u_1 u_3 \dots]$, then

$$\langle u,v\rangle^{2k+1} = \langle u^{\otimes 2k+1}, v^{\otimes 2k+1}\rangle$$

Now consider the infinite dimensional vector

$$u' = \left[\sqrt{c}\, u, (-1)\sqrt{\frac{c^3}{3!}}\, u^{\otimes 3}, \dots (-1)^k \sqrt{\frac{c^{2k+1}}{(2k+1)!}}\, u^{\otimes 2k+1} \dots\right]$$

$$v' = \left[\sqrt{c}\, v, \sqrt{\frac{c^3}{3!}}\, v^{\otimes 3}, \dots \sqrt{\frac{c^{2k+1}}{(2k+1)!}}\, v^{\otimes 2k+1} \dots\right]$$

Then $\sin(c\langle u,v\rangle) = \langle u',v'\rangle$ by construction.

Moreover $\left.\begin{array}{l} \|u'\|^2 = \sinh(c\|u\|^2) = 1 \\ \|v'\|^2 = \sinh(c\|v\|^2) = 1 \end{array}\right\}$ by Taylor's Thm and our choice of c

Finally $\mathbb{E}[\text{sgn}(a^T u')\,\text{sgn}(a^T v')] = \frac{2}{\pi}\arcsin(\langle u',v'\rangle)$

$$= \frac{2}{\pi} c \langle u,v\rangle \quad \blacksquare$$

Amazingly, there are many strong connections between functional analysis (e.g. OPT is related within a constant factor to $\|A\|_{\infty \to 1}$) and rounding algorithms for SDPs.

Let's give one more quick application, to see SDPs are important for more than just approx algorithms.

Let $\alpha(G)$ = size of largest independent set

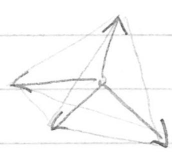
$\bar{\chi}(G)$ = chromatic number of $\bar{G}$ (complement of G)

Then $\alpha(G) \le \bar{\chi}(G)$. Moreover we can write the SDP:

$$\left.\begin{aligned} \Theta(G) = \min\ & k \\ \text{s.t. } & \langle v_i, v_j \rangle = \frac{-1}{k-1} \quad \forall (i,j) \notin E \\ & \langle v_i, v_i \rangle = 1 \end{aligned}\right\} \text{Lovasz theta fctn of } G$$

<u>Theorem:</u> $\alpha(G) \le \Theta(G) \le \bar{\chi}(G)$

<u>Proof:</u> It is easy to see there are k unit vectors $u_1 \dots u_k$ where $\langle u_i, u_j \rangle = \frac{-1}{k-1}$ $\forall i \ne j$



Thus any k coloring of $\bar{G}$ yields a feasible solution.

Conversely let $v_1 \dots v_s$ be vectors in a feasible solution corresponding to a size-s independent set. Then

$2\binom{s}{2}$ terms

$$0 \le \left(\sum_{i=1}^{s} v_i\right)^T \left(\sum_{i=1}^{s} v_i\right) = s + \sum_{i \ne j} \langle v_i, v_j \rangle$$

By averaging $\exists\, i \ne j$ with $\underbrace{\langle v_i, v_j \rangle}_{= \frac{-1}{\Theta(G)-1}} \ge \frac{-s}{2\binom{s}{2}} = -\frac{1}{s-1}$

Rearranging completes proof ☐

Now for an important class of graphs, called <u>perfect graphs</u> $\alpha(G) = \bar{\chi}(G)$.

Thus the Lovasz theta Fctn gives an algorithm to compute the largest independent set (also, largest clique, optimal coloring) for perfect graphs.

The only known combinatorial algorithms are very involved.