Lecture #9

Last Time: Capacity Scaling and Min Cost Matching
how to reduce dependence from U to $\log U$

Today: Min Cost Flow and Goldberg-Tarjan

Recall:

Klein's Cycle Cancelling

Find maximum s-t flow

while $\exists$ negative cost cycle $\mathcal{C}$ in residual

Augment along $\mathcal{C}$ by $\Delta_H(\mathcal{C})$

update residual

Last time we talked about optimality conditions for min cost P.M., but analogue

Lemma: A maximum s-t flow f has minimum cost (among all? max flows) iff there is no negative cost cycle in residual

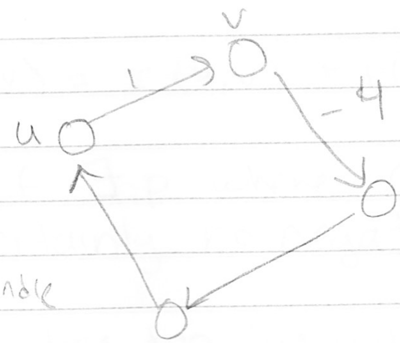
How do we implement the algorithm, and find a negative cost cycle?

Vote: (a) Djikstra's algorithm

* (b) Bellman-Ford *

Recall: Bellman-Ford computes cheapest path from $s \to v$, for all v but gets stuck if negative cost cycle

technicality: might not be reachable from s, but easy to handle



e.g. cheapest path from $s \to v$ is $-\infty$ cost

running time: $O(mn)$

How many iterations does Klein's algorithm take?

All capacities, costs integers at most U and C

bd on magnitude of costs

Initial cost (in residual)	Final cost
0	$\geq -mCU$

$\Rightarrow$ at most mCU iterations

overall running time: $O(m^2 n C U)$

what is this called? quasipolynomial

Goldbeg-Tarjan: cycle of minimum mean cost in residual

choose C that minimizes $\dfrac{c_H(C)}{|C|}$

number of arcs in C

Let's work towards analyzing this...

We need notion of <u>reduced costs</u>

For any $p: V \to \mathbb{R}$, we define

$$c_p(u,v) = c(u,v) + p(u) - p(v)$$

<u>Intuition</u>: If $\exists\, p$ where $c_p(u,v) \geq 0$ then there is certainly no negative cost cycle

special case

Converse is true too (we will prove something stronger)

Let $\mu = \min \frac{c_H(\mathcal{C})}{|\mathcal{C}|}$ and let ε be defined as

largest value δ s.t. $\exists\, p$ with $c_p(u,v) \geq \delta \;\; \forall (u,v)$

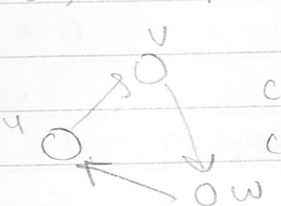
<u>Lemma 1</u>: $\mu = \varepsilon$

claim 2 & claim 2

<u>Proof</u>: <u>First we show $\mu \geq \varepsilon$</u>. This follows because

$$c(\mathcal{C}) = c_p(\mathcal{C}) \text{ for any cycle (telescoping)}$$

e.g.



$$c(\mathcal{C}) = c(u,v) + c(v,w) + c(w,v)$$

$$\begin{aligned} c_p(\mathcal{C}) = {} & c(u,v) + p(u) - p(v) \\ & + c(v,w) + p(v) - p(w) \\ & + c(w,u) + p(w) - p(u) \end{aligned}$$

and $c_p(u,v) \geq \delta \;\; \forall (u,v) \Rightarrow \frac{c_p(\mathcal{C})}{|\mathcal{C}|} \geq \delta$

<u>Next we show $\mu \leq \varepsilon$</u>. Define a helper cost

$$c'(u,v) = c(u,v) - \mu$$

By definition of μ, c' has no negative cost cycles

Set $p(u) =$ cost of cheapest path according to c' from $s \to u$

Then $p(v) \le p(u) + c'(u,v)$ by dfn of cheapest path

Rearranging we have

$$0 \le c'(u,v) + p(u) - p(v) = \underbrace{c(u,v) + p(u) - p(v)}_{c_p(u,v)} - \mu$$

This completes the proof. ∎

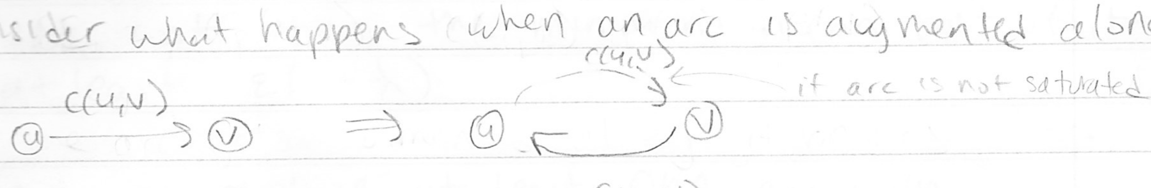
Lemma 2: In the Goldberg-Tarjan algorithm, ε is nondecreasing

i.e. let ε_1 be current value for residual, and ε_2 be new value after augmenting then $\varepsilon_2 \ge \varepsilon_1$

Proof: Let p be such that $c_p(u,v) \ge \varepsilon \;\; \forall (u,v)$

Recall $\varepsilon < 0$ (since we only augment when negative cycle).

Consider what happens when an arc is augmented along:



If we use same potential p we get new reduced cost

$$-c(u,v) + p(v) - p(u) = -c_p(u,v)$$

When we augment along a cycle with minimum mean cost, we only use arcs whose reduced

why? otherwise wouldn't be minimum mean cost.

cost is exactly ε. Thus some reduced cost(s) of ε

get replaced with $-\varepsilon$, and same potential p gives reduced costs of at least ε after augmentation $\boxtimes$

Now we can analyze Goldberg-Tarjan

Theorem: The number of iterations in the Goldberg-Tarjan algorithm is at most

$$O(mn \log nC)$$

Let $\varepsilon(f) = \varepsilon$ for residual graph for flow f

Lemma 3: Let f be a maximum s-t flow, and f' be obtained by m iterations of augmenting via Goldberg-Tarjan rule, then

$$\varepsilon(f') \geq \left(1 - \frac{1}{n}\right) \varepsilon(f)$$

fixed on 3/12

Proof: Let p be optimal potential for f.

If at any point we use a cycle with a nonnegative reduced cost, then the minimum mean cost of it is at least $\varepsilon(1 - \frac{1}{n})$

If not, we replace at least one arc with negative reduced cost, by one with positive.

After at most m iterations, we run out of arcs with negative reduced cost. $\boxtimes$

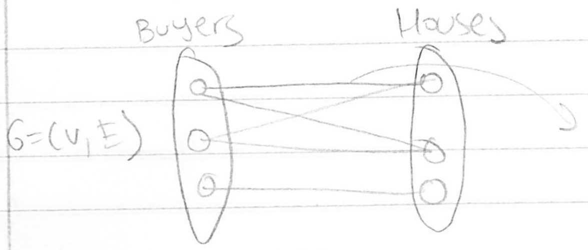
Now finishing proof

$$\mathcal{E}(f_{last}) \geq \left(1-\frac{1}{n}\right)^{2n \ln nC} \mathcal{E}(f_{first})$$

$$\geq \underset{\substack{\text{not reduced, original!}}}{(-C)}\, e^{-2\ln nC} = \frac{-C}{n^2C^2} > \frac{-1}{n}$$

Since all costs are integer this implies there is no negative cost cycle. $\boxtimes$

An Application



$v(a,b)$ = value of house b to buyer a

Let $c(a,b) = -v(a,b)$, recall can find minimum cost perfect matching M via Goldberg-Tarjan
(suppose M is unique min cost p.m)

$\exists\ p: A \cup B \to \mathbb{R}$ be potentials so that

$$c_p(a,b) = c(a,b) + p(a) - p(b) = 0 \quad \forall (a,b) \in M$$

$$c_p(a,b) > 0 \quad \forall (a,b) \in E \setminus M$$

If we choose $-p(b)$ = price of house b, then a will "buy" house b if

$$v(a,b') + p(b') < v(a,b) + p(b) \qquad \forall b' \neq b$$

Rearranging

$$c(a,b) - p(b) < c(a,b') - p(b')$$

$$+ p(a) \qquad\qquad + p(a)$$

$$0 = c_p(a,b) < c_p(a,b')$$

thus for these prices, everyone will agree on maximum utility soln.