

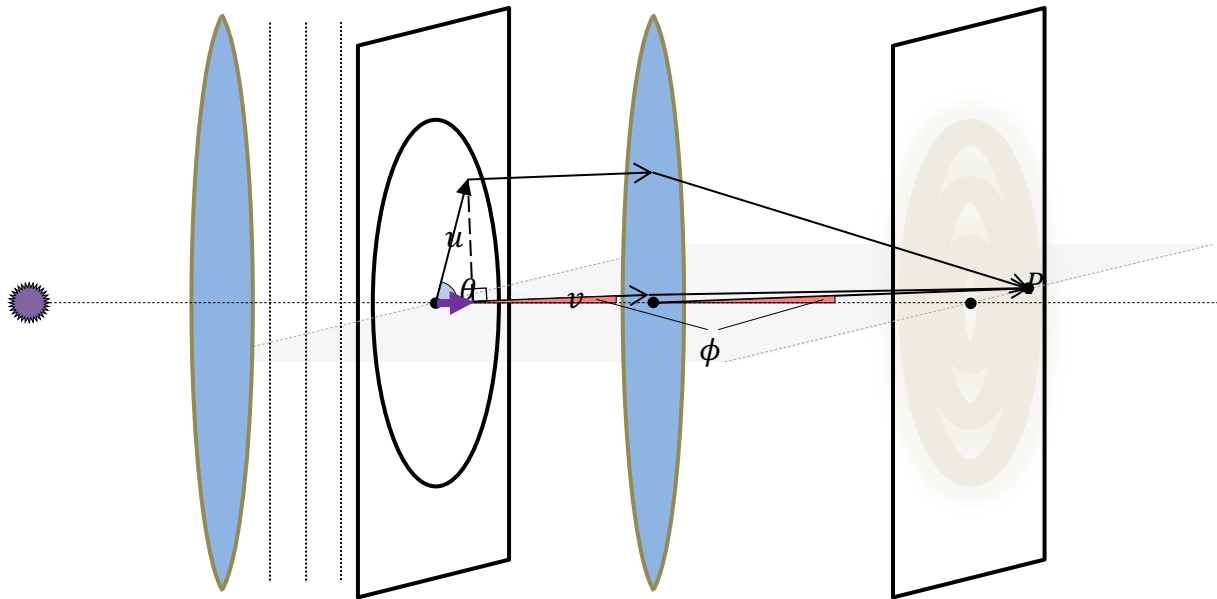
Provable Algorithms for Inverse Problems in Physics?

Ankur Moitra (MIT)

Simons Institute Bootcamp Tutorial, Part 4

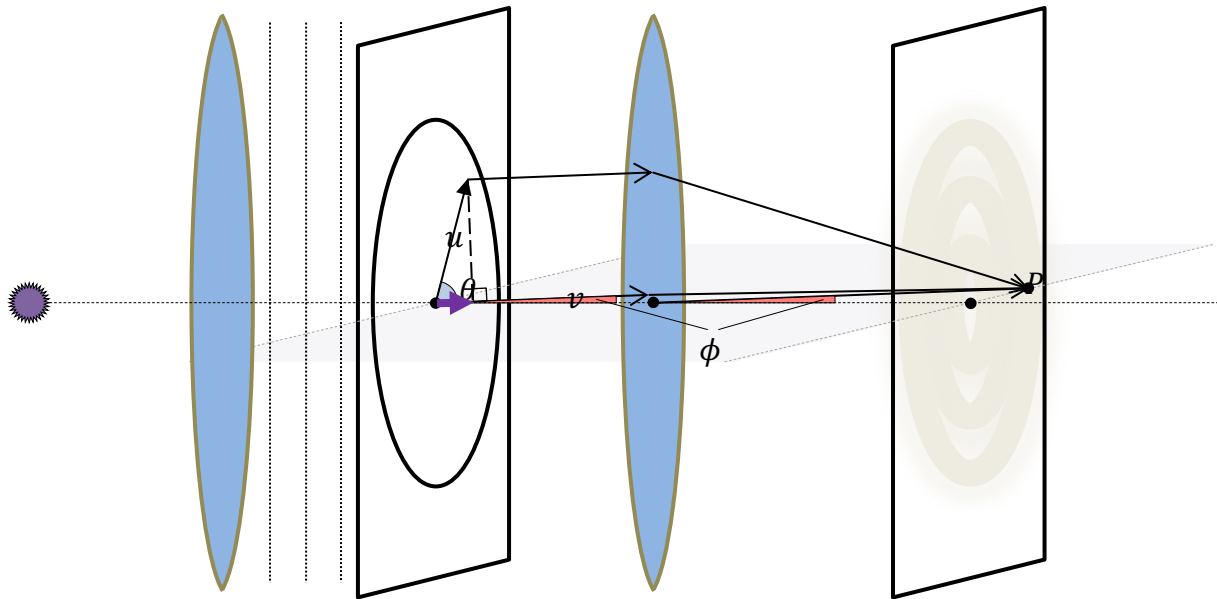
THE PHYSICS OF DIFFRACTION

Observe incoherent illumination from far-away point sources through a circular aperture



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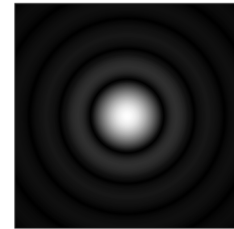


The normalized intensity is called an **Airy disk**

THE PHYSICS OF DIFFRACTION

Definition: The Airy disk is the function

$$I(x) = \frac{1}{\pi\sigma^2} \left(\frac{2J_1(\|x\|_2/\sigma)}{\|x\|_2/\sigma} \right)^2$$

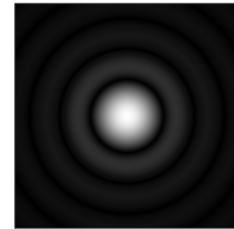


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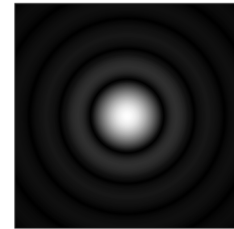
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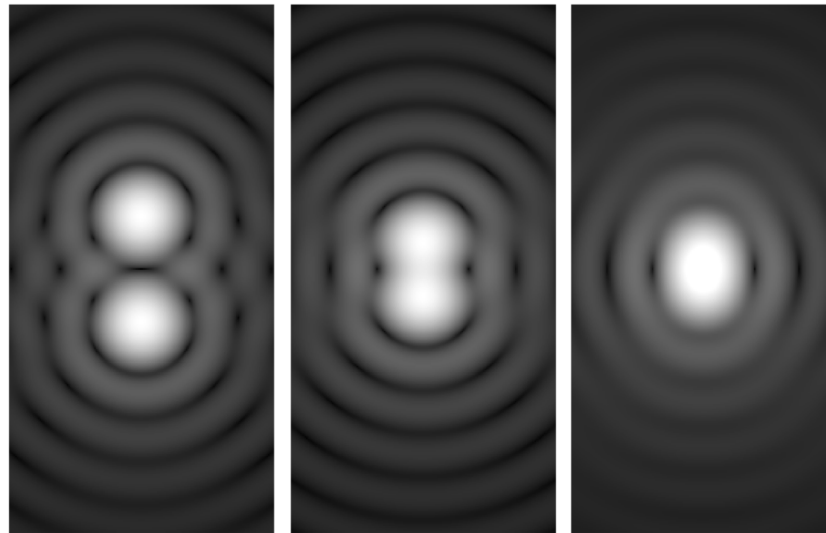
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First explicitly computed by Sir George Biddell Airy in 1835

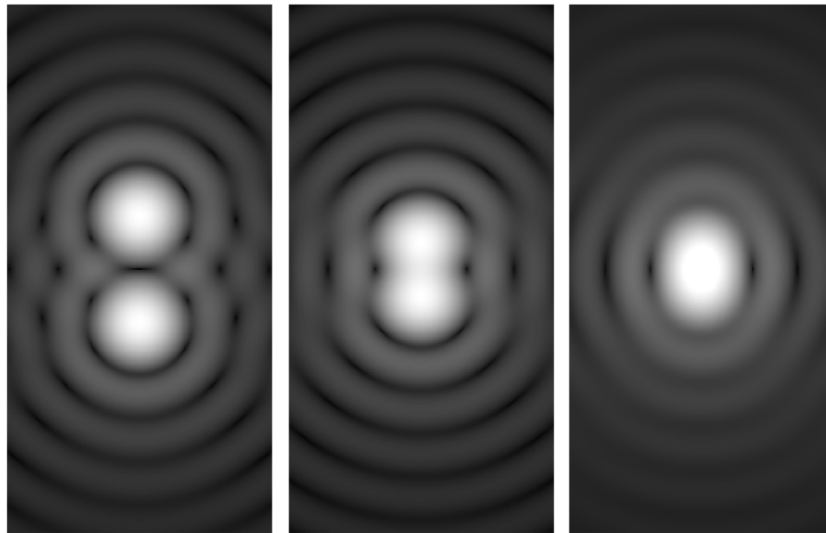
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Main Question: Are there statistical/algorithmic limitations to how accurately we can estimate a mixture of Airy disks?

THE DIFFRACTION LIMIT

In particular, how should the **minimum separation** that you can resolve depend on the parameters of the optical system?

ON SPECTROSCOPIC RESOLVING POWER
By C. H. SPARROW

If a spectroscope is just able to separate two monochromatic lines of equal intensity and wave-lengths λ and $\lambda + \Delta\lambda$, the ratio $\frac{\Delta\lambda}{\lambda}$ is called the resolving power of the instrument for the wave-length λ . This is the definition of resolving power and if we can determine by actual measurement the value of $\Delta\lambda$ for some particular instrument, we can obtain the resolving power of this instrument. If, however, our problem is to calculate the resolving power from the optical theory of the instrument, the definition must be supplemented by a criterion of when two lines will enable us to say when the two lines are to be considered as just resolved. In the case of a prism without absorption, or of a grating with many lines, the criterion proposed by Rayleigh has hitherto been universally adopted. The intensity in a single line being given by:

$$I = I_0 \frac{\sin^2 \alpha}{\alpha^2} \quad (1)$$

and that due to two lines by:

$$I = I_0 \left\{ \frac{\sin^2(\alpha - \beta)}{(\alpha - \beta)^2} + \frac{\sin^2(\alpha + \beta)}{(\alpha + \beta)^2} \right\} \quad (2)$$


Sparrow



Abbe



Daves Houston



Rayleigh

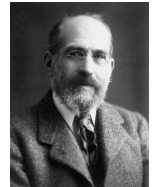
Increase/Reduce Image Size Note on Optical Resolution.
By A. Buxton, M.A. Oxon.*

CONSIDER the reproduction of an infinitely distant self-luminous point-source such as a star, through an optical system such as a telescope. The image in the focal plane consists of a central concentration of light, surrounded by light rings of rapidly diminishing intensity. The intensity at a point near the symmetrical axis of the "spurious-disk" image can be conveniently represented in terms of Bessel Functions, thus:

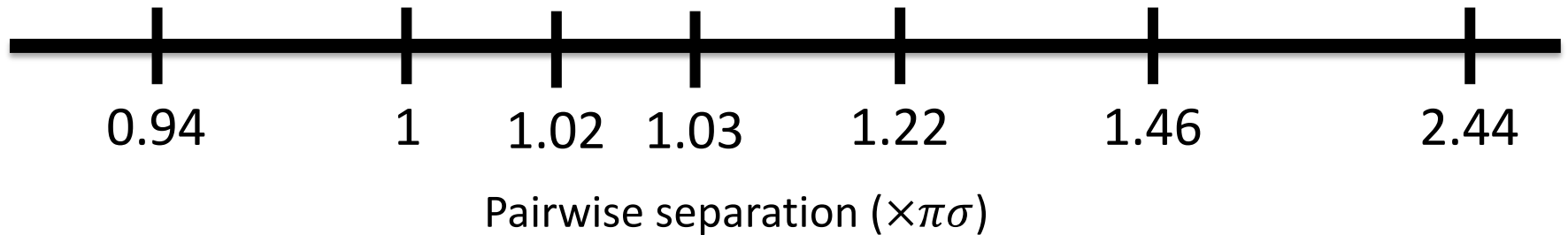
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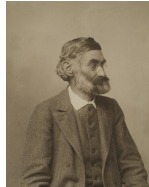
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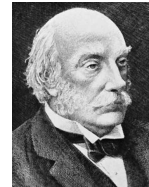
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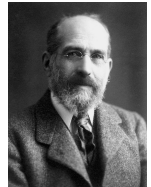
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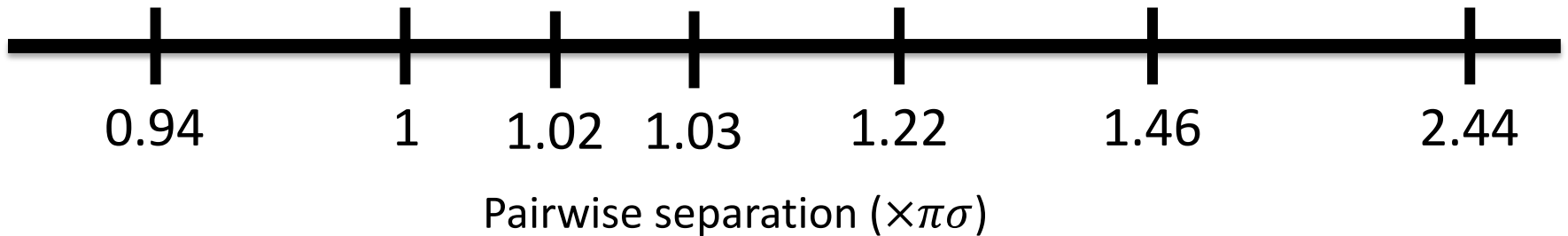
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Which, if any, of these criteria is the right one?

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“ It is obvious that the undulation condition should set an upper limit to the resolving power ... My own observations on this point have been checked by a number of friends and colleagues. ” Carroll Sparrow, 1918

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Can we put the diffraction limit on a rigorous foundation?

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We give the first provable algorithms for learning mixtures of Airy disks that have non-asymptotic guarantees

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Many arguments for the existence of a diffraction limit stem from reasoning about mixtures of two Airy disks --- but there is no fundamental limitation to what can be resolved in this setting!

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Conversely there are $(1 - \epsilon)\gamma_- \pi \sigma$ -separated mixtures of k Airy disks that require exponentially many samples to learn

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With any reasonable physical setup (finite exposure times, finite precision in recording locations of photons) there really is a fundamental limit to resolving many point sources

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2014 Nobel Prize in Chemistry!

Super-resolution through stimulated emission

Eric Betzig, Stefan Hell, William Moerner

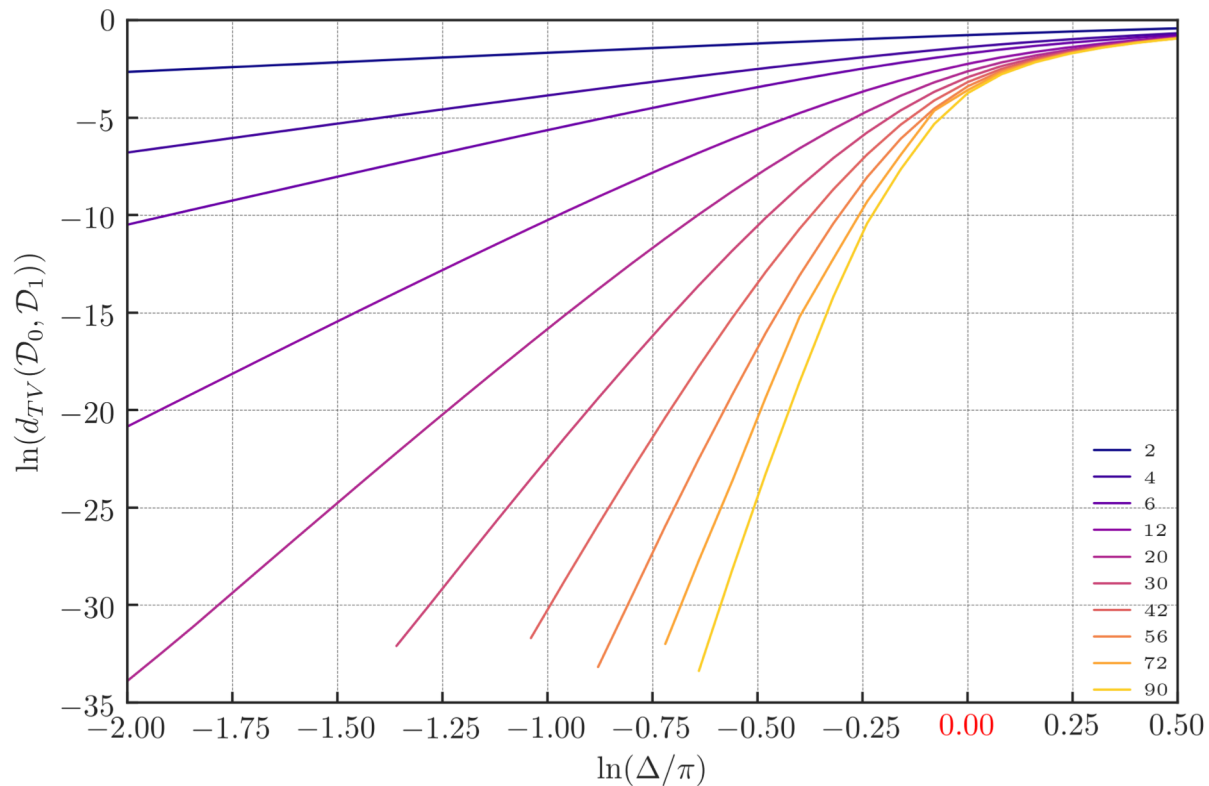


VISUALIZING THE DIFFRACTION LIMIT

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Now can we **remove** the $\hat{I}(\omega)$ term, at least in the region where it is nonzero?

Lemma: For any $\|\omega\| < \frac{1}{\pi\sigma}$ we can simulate noisy access to the exponential sum

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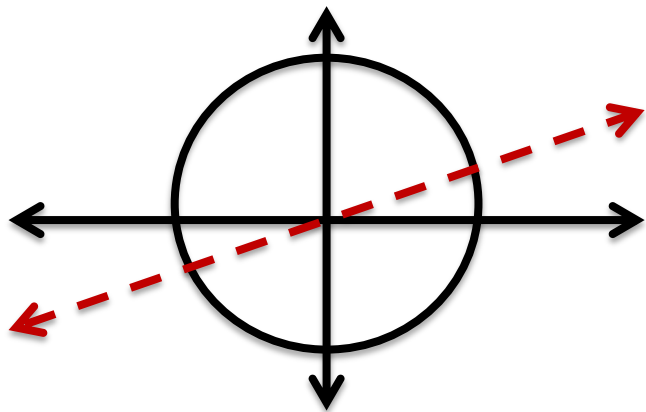
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Now, can we estimate the centers from the exponential sum?

REDUCING TO ONE DIMENSION

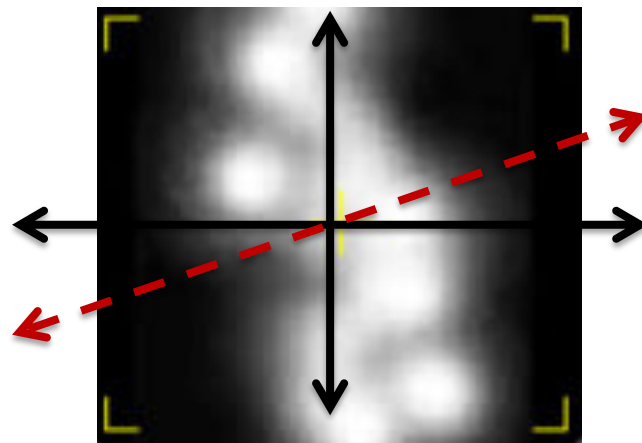
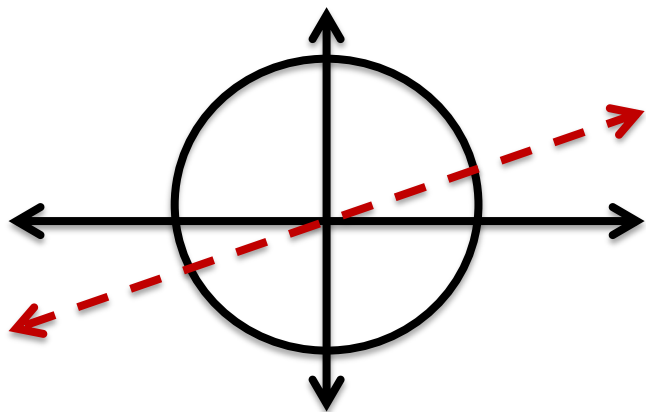
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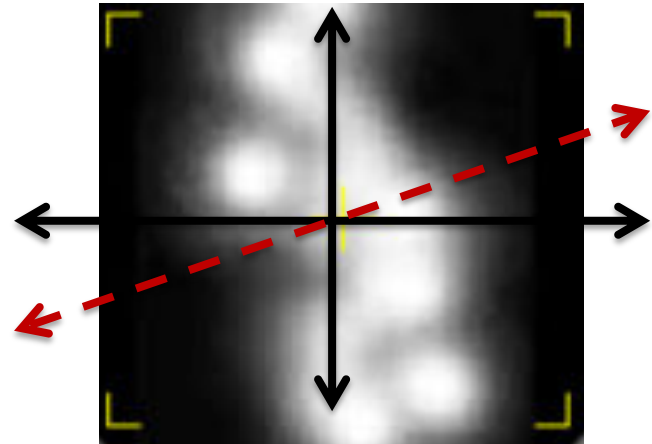
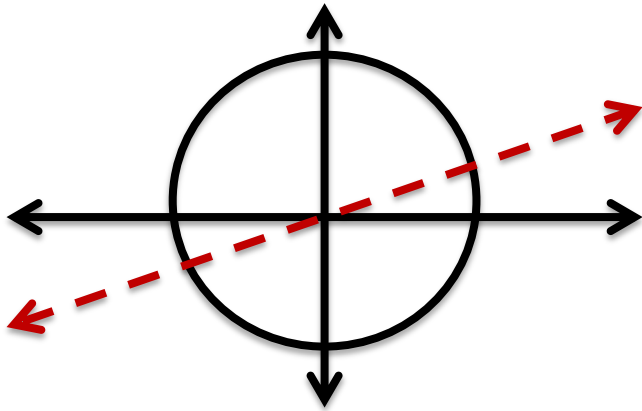
Equivalent to **projecting** the Airy disks onto a line



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Equivalent to **projecting** the Airy disks onto a line



This could decrease the separation and make resolution harder, but let's figure out what our queries look like, mathematically

Fact: Suppose we sample $f(\omega)$ at the sequence of points on a line

$$a, a + b, a + 2b, \dots$$

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Then our vector of measurements can be expressed as

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where $\alpha_j = e^{-2\pi i \langle \mu_j, b \rangle}$ and $c_j = \lambda_j e^{-2\pi i a}$

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Lemma [Moitra '15]: The stability of the Matrix Pencil Method depends on the condition number of the Vandermonde matrix

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The intuition is the Vandermonde matrix is merely exponentially ill-conditioned (i.e. finding coefficients from querying a polynomial)

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The intuition is the Vandermonde matrix is merely exponentially ill-conditioned (i.e. finding coefficients from querying a polynomial)

Thus if we take an exponential in k number of samples, we can accurately determine the projected centers of the disks

Lemma: If we choose a line to restrict to at random and the measurements are finely spaced then

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Thus if we take an exponential in k number of samples, we can accurately determine the projected centers of the disks

Then repeat for new lines and piece together the estimates

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In 1-D the smallest singular value of the Vandermonde matrix is much larger when the projected centers are separated

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The approach was based on a 1-D extremal function from number theory called the **Beurling-Selberg Majorant** and is the best smooth approximation to the sgn function

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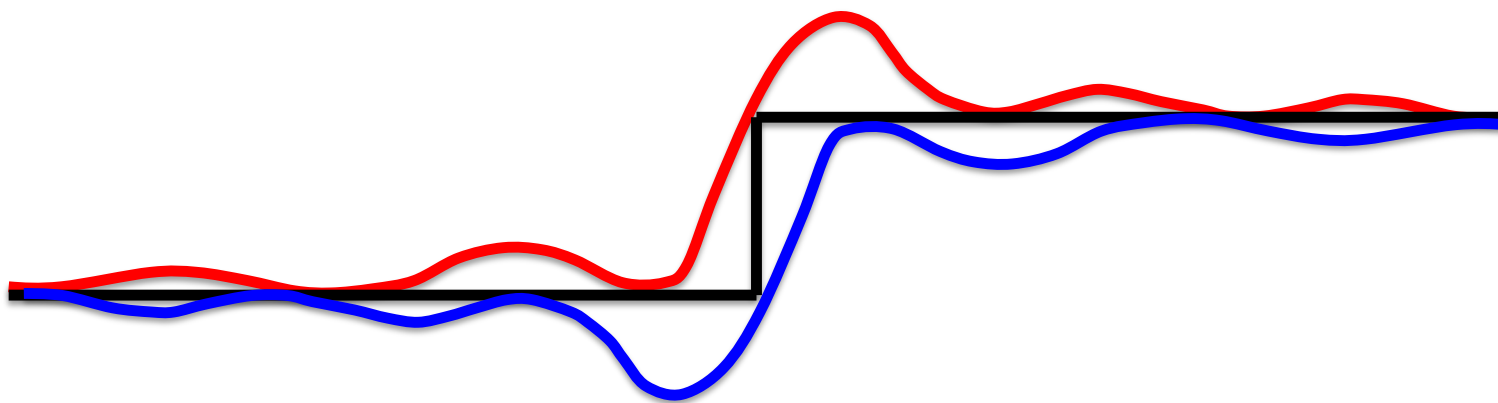
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So can we improve the dependence on k in 2-D when the centers are separated?

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Sometimes projection just doesn't work!

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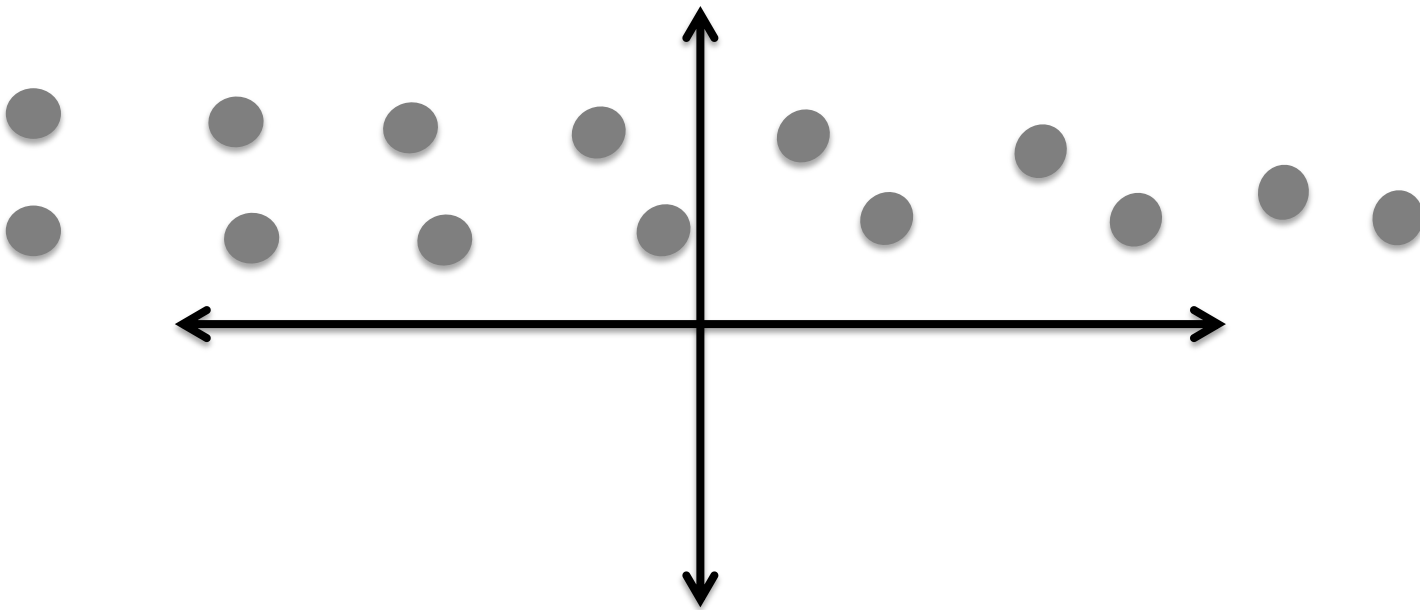
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A NO-GO EXAMPLE

There are 2-D configurations where there is no 1-D projection that even approximately preserves the min separation

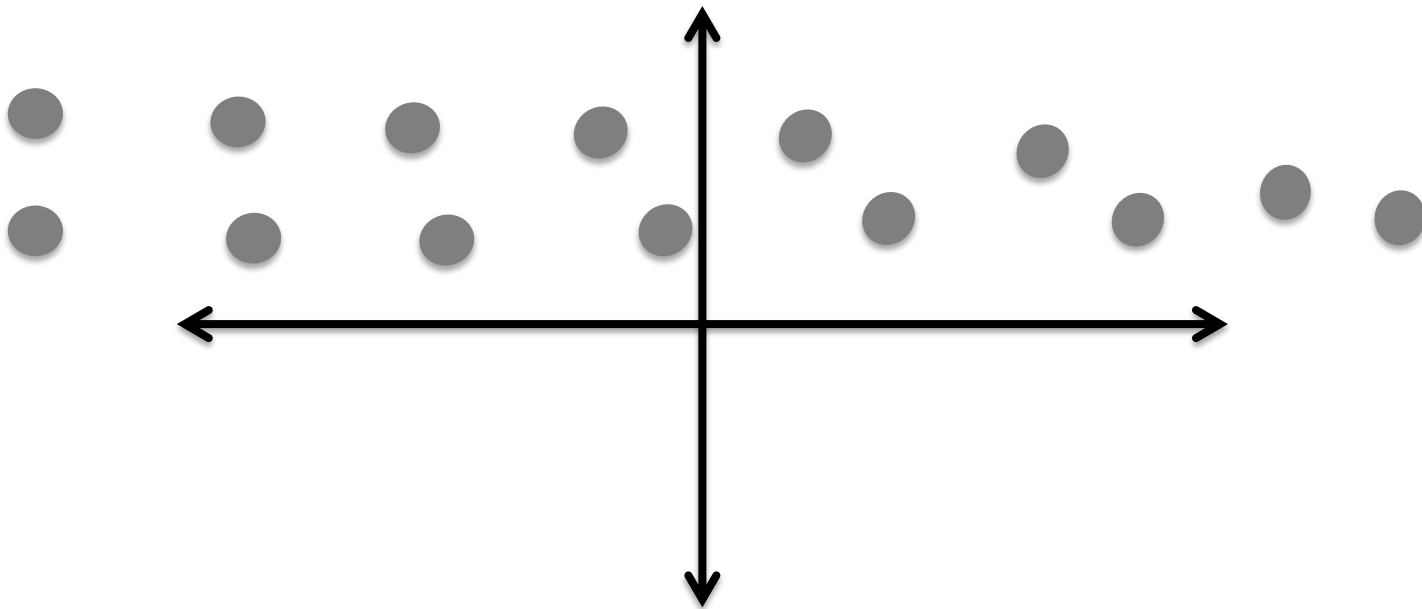
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It's not just a failure of the technique! In fact the true threshold for 2-D problem is bounded away from that of the 1-D problem

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These bounds arise through the study of de Branges spaces of entire functions

[Huang, Kakade '15]: Introduced a tensor method for recovering exponential sums whose analysis depends

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We can use the 2-D extremal functions to show that random ω_j 's from the $B(0, \gamma_+)$ have bounded condition number whp

SUMMARY

With all due apologies to Carroll Sparrow's friends and colleagues:

ON SPECTROSCOPIC RESOLVING POWER
By C. M. SPARROW

If a spectrograph is just able to separate two monochromatic lines of equal intensity and wave-lengths λ and $\lambda + \Delta\lambda$, the ratio $\frac{\Delta\lambda}{\lambda}$ is called the resolving power of the instrument for the wave-length λ . This is the definition of resolving power, and if we can determine by actual measurement the value of $\Delta\lambda$ for some particular instrument, we can obtain the resolving power of that instrument. If, however, our problem is to calculate the resolving power from the optical theory of the instrument, the definition must be supplemented by a criterion of some sort which will enable us to say when the two lines are to be considered as just resolved. In the case of a prism without absorption, or of a grating with many lines, the criterion proposed by Rayleigh has hitherto been universally adopted. The intensity in a single line being given by

$$I = I_0 \frac{\sin^2 \alpha}{\alpha^2} \quad (1)$$

and that due to two lines by

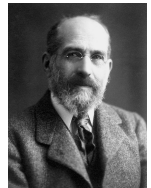
$$I = I_0 \left\{ \frac{\sin^2(\alpha - \beta)}{(\alpha - \beta)^2} + \frac{\sin^2(\alpha + \beta)}{(\alpha + \beta)^2} \right\} \quad (2)$$


increase/Reduce image size
Note on Optical Resolution.
By A. BUXTON, M.A. Oxon.*

CONSIDER the reproduction of an infinitely distant self-luminous point-source such as a star, through an optical system such as a telescope. The image in the focal plane consists of a central concentration of light, surrounded by light rings of rapidly diminishing intensity. The intensity at a point near the symmetrical axis of the "spurious-disk" image can be conveniently represented in terms of Bessel Functions, thus:

$$I = \text{Intensity} = \left\{ \frac{2}{z} J_1(z) \right\}^2 \quad \text{where } z = \frac{2\pi R y}{\lambda f}$$

* Communicated by the Author.



Sparrow

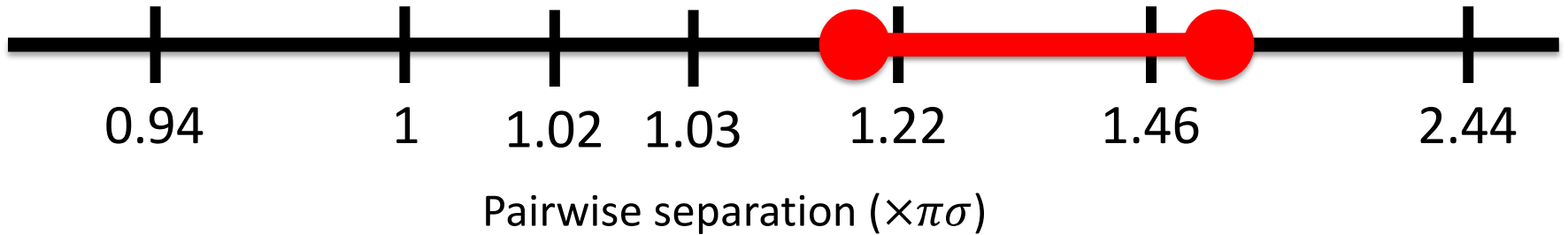
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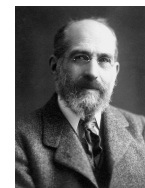
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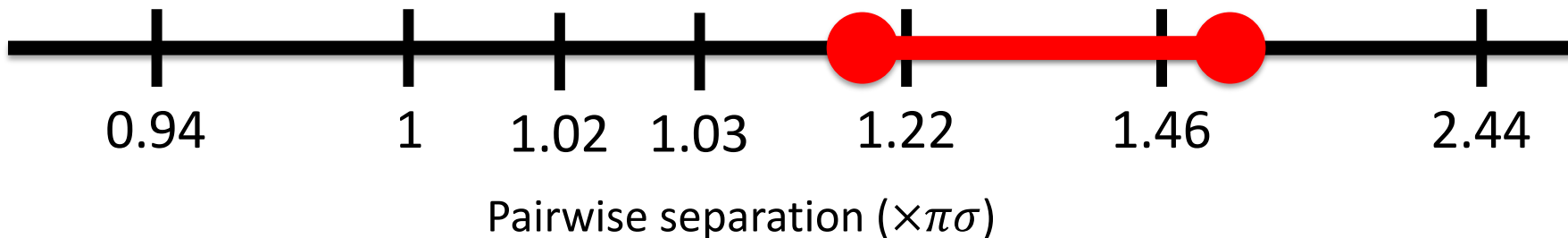
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Finding the sharp diffraction limit remains a challenging problem in harmonic analysis

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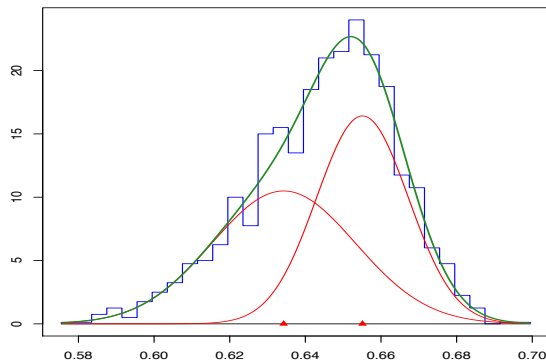
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TAKING A STEP BACK

In theoretical machine learning we have provable algorithms for parameter learning, e.g. mixtures of gaussians

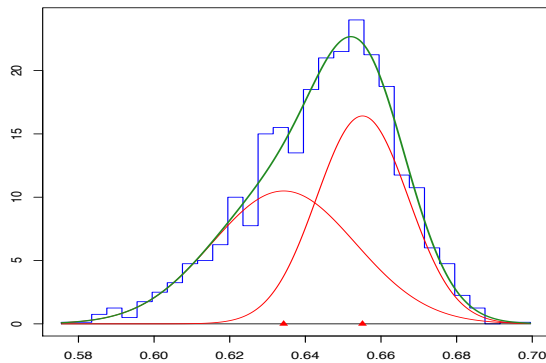
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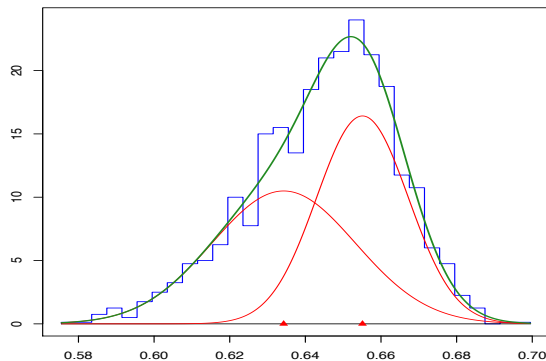
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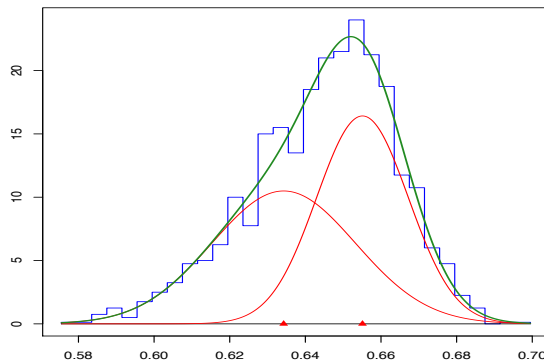


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Do tools from theoretical machine learning have more to say about provable algorithms for inverse problems in science?

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- The diffraction limit is an inverse problem
- Fundamental limits in optics can be understood as a statistical phase transition
- **Are other inverse problems in the Sciences amenable to tools from theoretical ML?**

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Thanks! Any Questions?