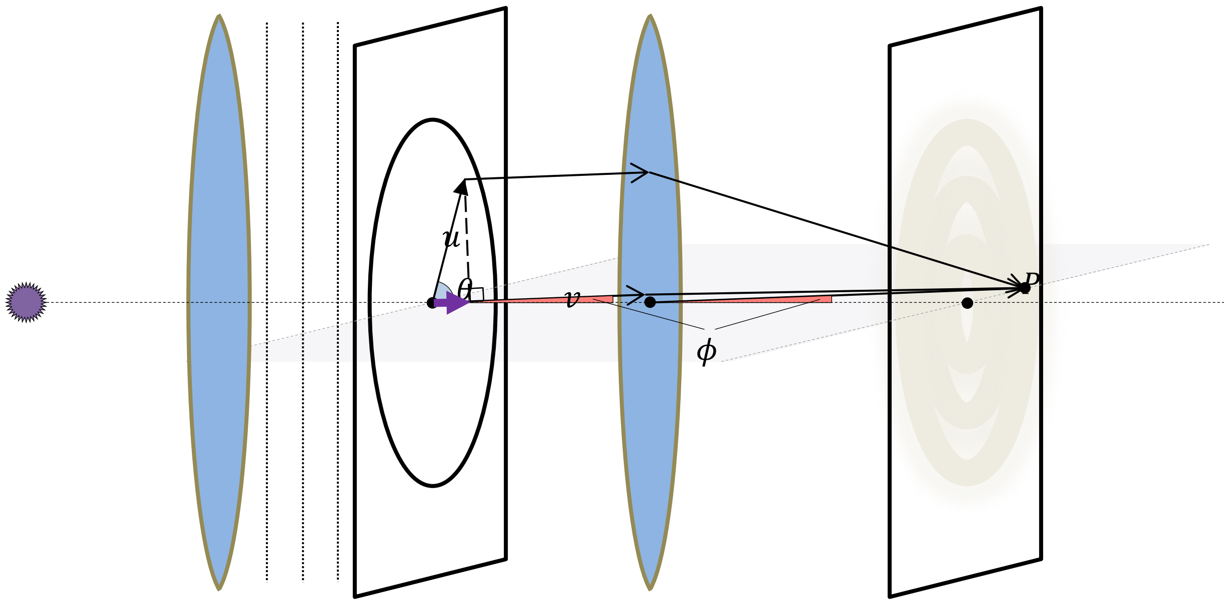
Algorithmic Foundations for the Diffraction Limit

Ankur Moitra (MIT)

UT Austin Machine Learning Lab

THE PHYSICS OF DIFFRACTION

Observe incoherent illumination from far-way point sources through a circular aperture



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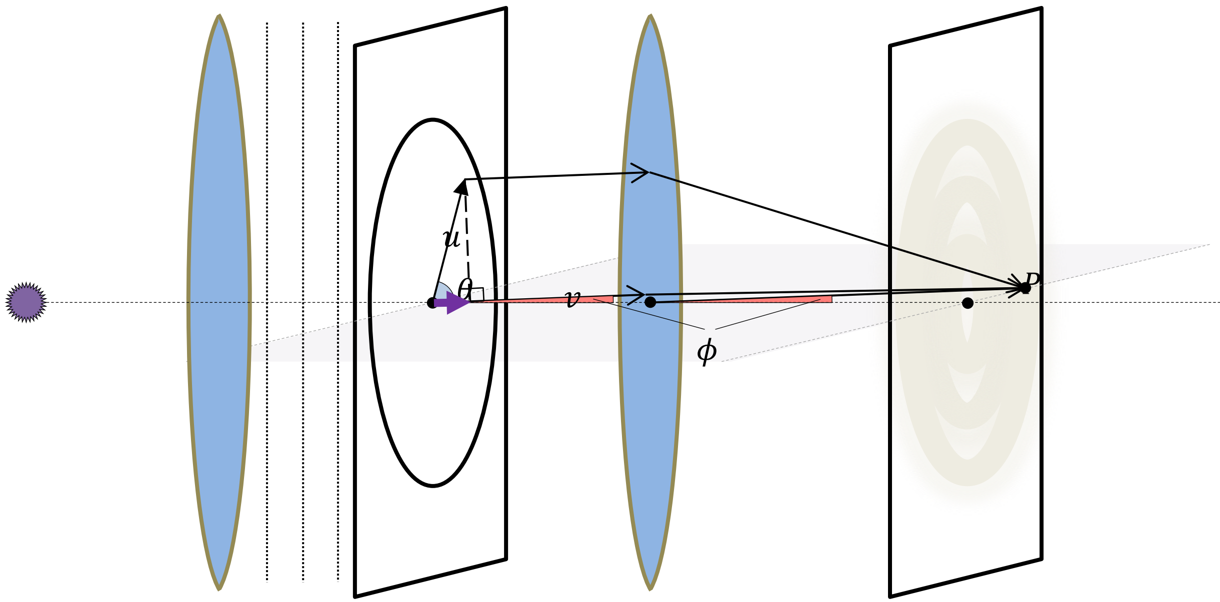
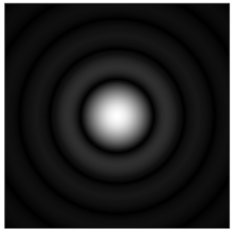


The normalized intensity is called an **Airy disk**

THE PHYSICS OF DIFFRACTION

Definition: The Airy disk is the function

$$I(x) = \frac{1}{\pi\sigma^2}\left(\frac{2J_1(\|x\|_2/\sigma)}{\|x\|_2/\sigma}\right)^2 \longleftrightarrow$$



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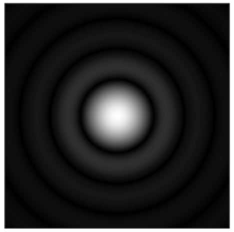

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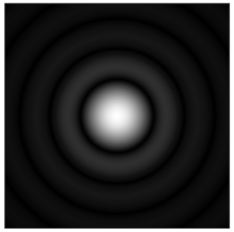

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First explicitly computed by Sir George Biddell Airy in 1835

THE DIFFRACTION LIMIT

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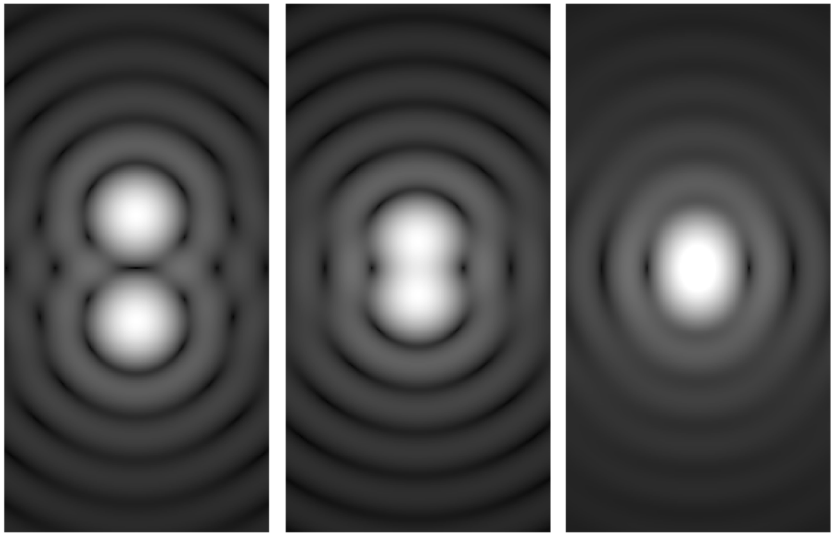

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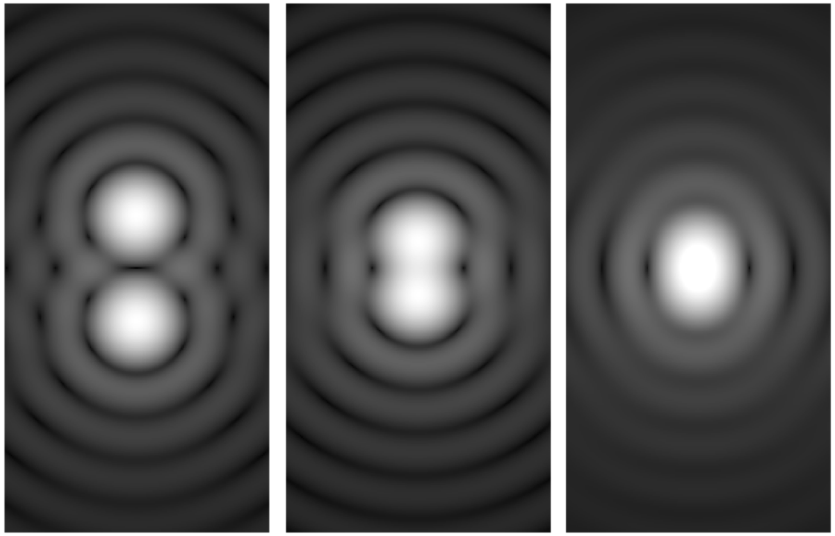
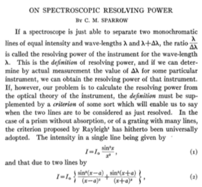


Main Question: Are there statistical/algorithmic limitations to how accurately we can estimate a mixture of Airy disks?

THE DIFFRACTION LIMIT

In particular, how should the **minimum separation** that you can resolve depend on the parameters of the optical system?



ON SPECTROSCOPIC RESOLVING POWER

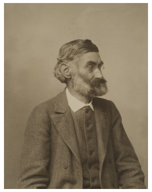
By C. M. SPARROW

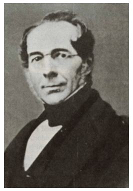
If a spectroscope is just able to separate two monochromatic lines of equal intensity and wave-lengths λ and $\lambda+\Delta\lambda$, the ratio $\frac{\lambda}{\Delta\lambda}$ is called the resolving power of the instrument for the wave-length λ. This is the *definition* of resolving power, and if we can determine by actual measurement the value of $\Delta\lambda$ for some particular instrument, we can obtain the resolving power of that instrument. If, however, our problem is to calculate the resolving power from the optical theory of the instrument, the *definition* must be supplemented by a *criterion* of some sort which will enable us to say when the two lines are to be considered as just resolved. In the case of a prism without absorption, or of a grating with many lines, the criterion proposed by Rayleigh[1] has hitherto been universally adopted. The intensity in a single line being given by

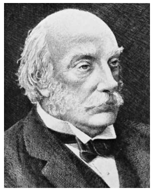
$$I=I_0\frac{\sin^2 x}{x^2}, \tag{1}$$

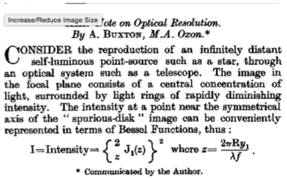
and that due to two lines by

$$I=I_0\left\{\frac{\sin^2(x-a)}{(x-a)^2}+\frac{\sin^2(x+a)}{(x+a)^2}\right\}, \tag{2}$$









Increase/Reduce Image Size

Note on Optical Resolution.

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CONSIDER the reproduction of an infinitely distant self-luminous point-source such as a star, through an optical system such as a telescope. The image in the focal plane consists of a central concentration of light, surrounded by light rings of rapidly diminishing intensity. The intensity at a point near the symmetrical axis of the "spurious-disk" image can be conveniently represented in terms of Bessel Functions, thus:

$$\mathrm{I}=\text{Intensity}=\left\{\frac{2}{z}\mathrm{J}_1(z)\right\}^2 \quad \text{where } z=\frac{2\pi \mathrm{R}y_1}{\lambda f}.$$

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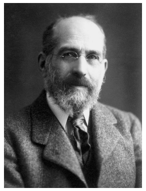

Sparrow	Abbe	Dawes	Houston	Rayleigh	Buxton	Schuster
0.94	1	1.02	1.03	1.22	1.46	2.44

Pairwise separation ($\times\pi\sigma$)

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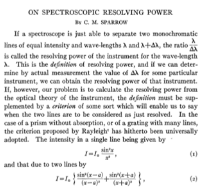

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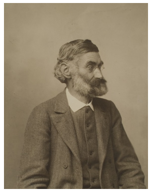

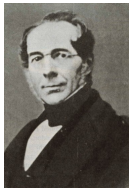

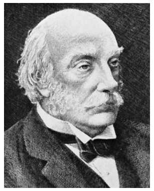

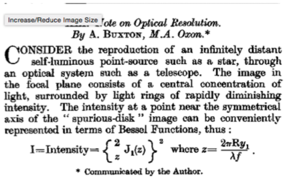

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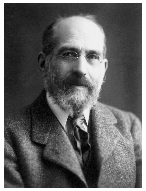

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Which, if any, of these criteria is the right one?

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- The Diffraction Limit as an Inverse Problem
- The Lost Art of Debate
- Rigorous Foundations and Visualizations

Part II: Learning Mixtures of Airy Disks

- Deconvolution via the Fourier Transform
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Part III: Connections to Mixtures of Gaussians

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> “ **It is obvious that the undulation condition should set an upper limit to the resolving power … My own observations on this point have been checked by a number of friends and colleagues.** ” Carroll Sparrow, 1918

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Can we put the diffraction limit on a rigorous foundation?

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Many arguments for the existence of a diffraction limit stem from reasoning about mixtures of two Airy disks --- but there is no fundamental limitation to what can be resolved in this setting!

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Conversely there are $(1-\epsilon)\gamma_-\pi\sigma$-separated mixtures of k Airy disks that require exponentially many samples to learn

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With any reasonable physical setup (finite exposure times, finite precision in recording locations of photons) there really is a fundamental limit to resolving many point sources

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2014 Nobel Prize in Chemistry!

Super-resolution through stimulated emission

Eric Betzig, Stefan Hell, William Moerner

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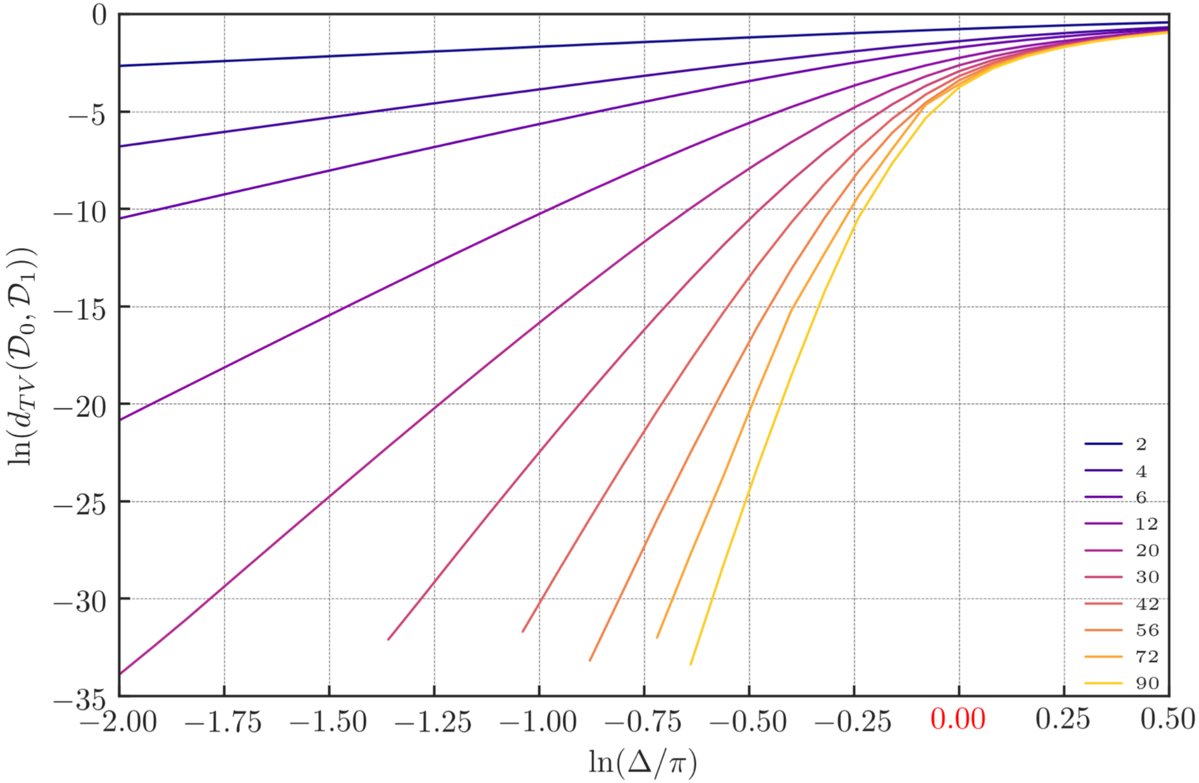

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Now can we **remove** the $\widehat{I}(\omega)$ term, at least in the region where it is nonzero?

Lemma: For any $\|\omega\| < \frac{1}{\pi\sigma}$ we can simulate noisy access to the exponential sum

$$f(\omega) = \sum_{j=1}^{k} \lambda_j e^{-2\pi i \langle \mu_j, \omega \rangle}$$

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This is achieved via a simple procedure:

- Draw samples to construct an empirical estimate of $\rho(x)$
- Form a Kernel Density Estimate (i.e. smooth by convolving with a small variance Gaussian)
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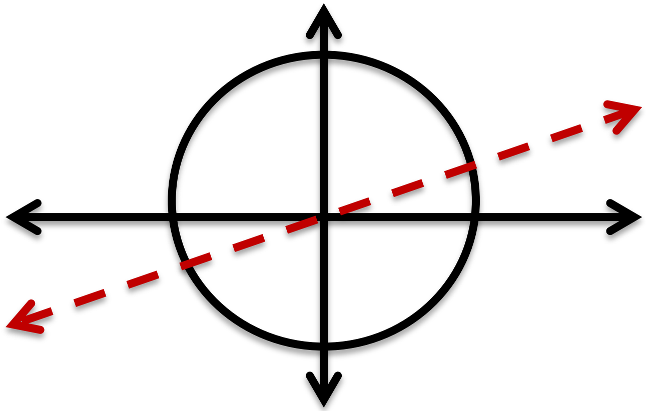
Now, can we estimate the centers from the exponential sum?

REDUCING TO ONE DIMENSION

What if we only query $f(\omega)$ on a line?



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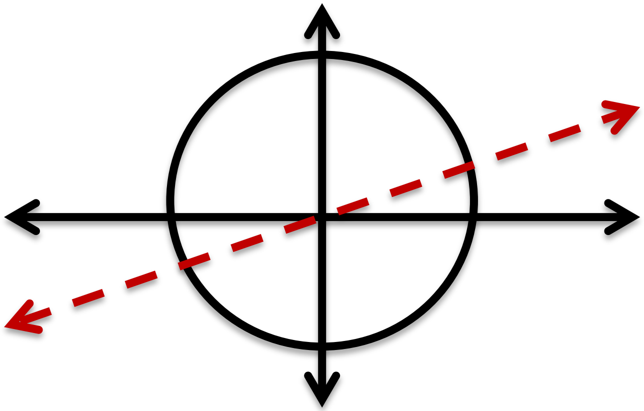
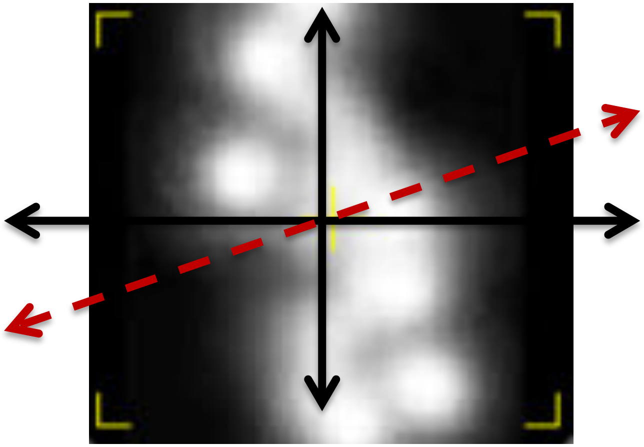


Equivalent to **projecting** the Airy disks onto a line



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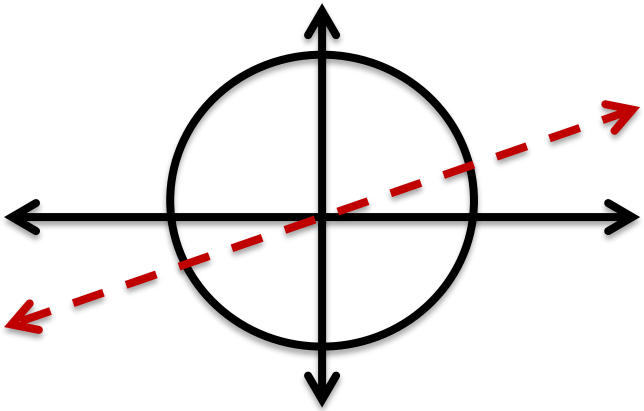

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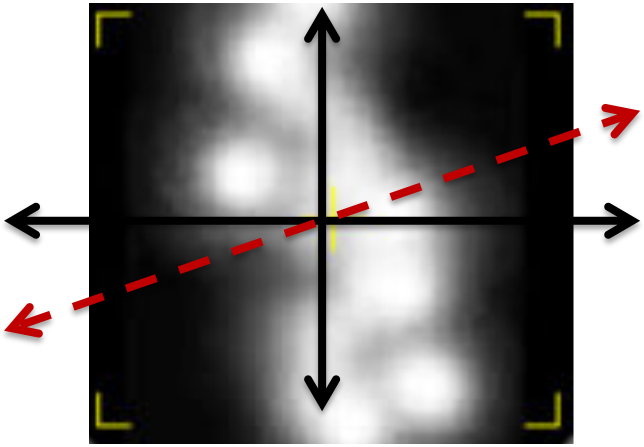

This could decrease the separation and make resolution harder, but let's figure out what our queries look like, mathematically

Fact: Suppose we sample $f(\omega)$ at the sequence of points on a line

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Then our vector of measurements can be expressed as

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_k \\ \alpha_1^2 & \alpha_2^2 & \cdots & \alpha_k^2 \\ \vdots & \vdots & & \vdots \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$$

Fact: Suppose we sample $f(\omega)$ at the sequence of points on a line

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where $\alpha_j = e^{-2\pi i \langle \mu_j, b \rangle}$ and $c_j = \lambda_j e^{-2\pi i a}$

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Lemma [Moitra '15]: The stability of the Matrix Pencil Method depends on the condition number of the Vandermonde matrix

Lemma: If we choose a line to restrict to at random and the measurements are finely spaced then

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Then repeat for new lines and piece together the estimates

BETTER BOUNDS?

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The approach was based on a 1-D extremal function from number theory called the **Beurling-Selberg Majorant** and is the best smooth approximation to the sgn function

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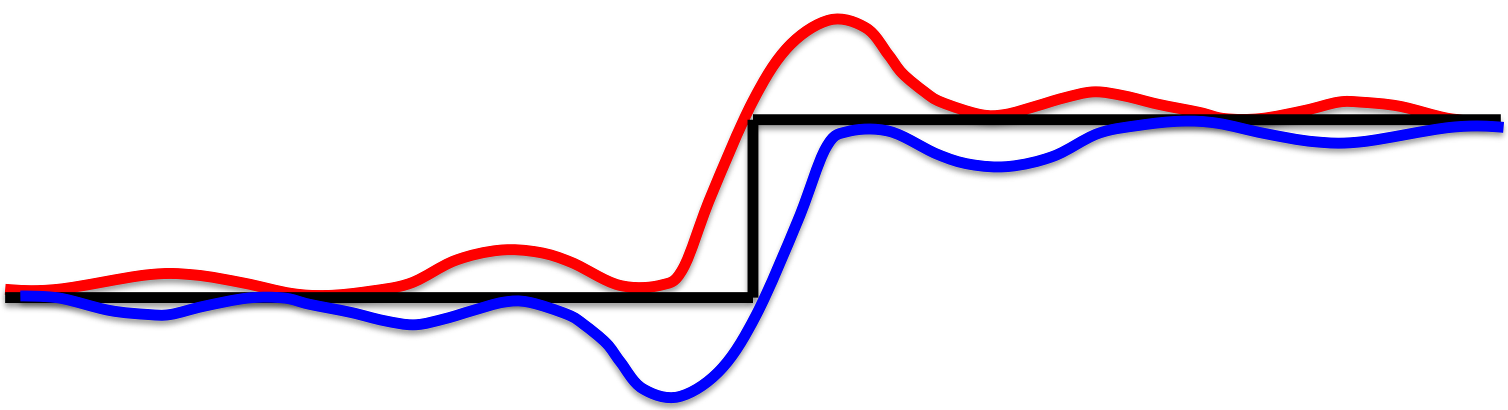

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Sometimes projection just doesn't work!

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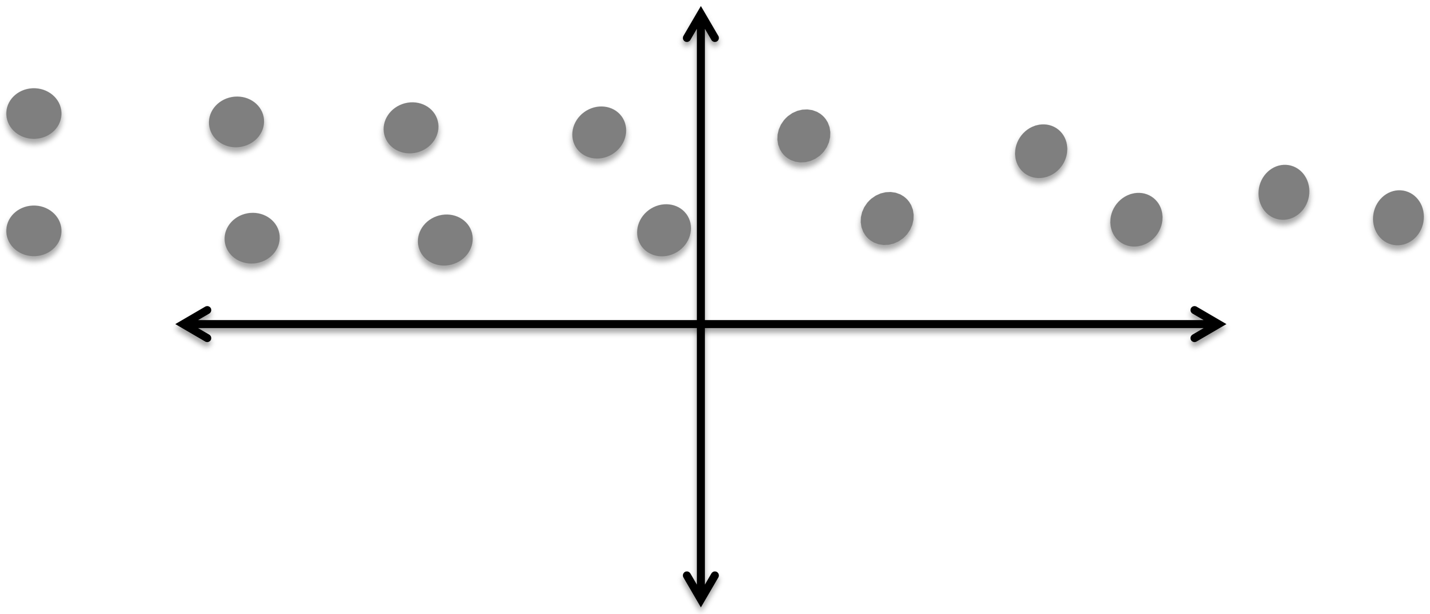

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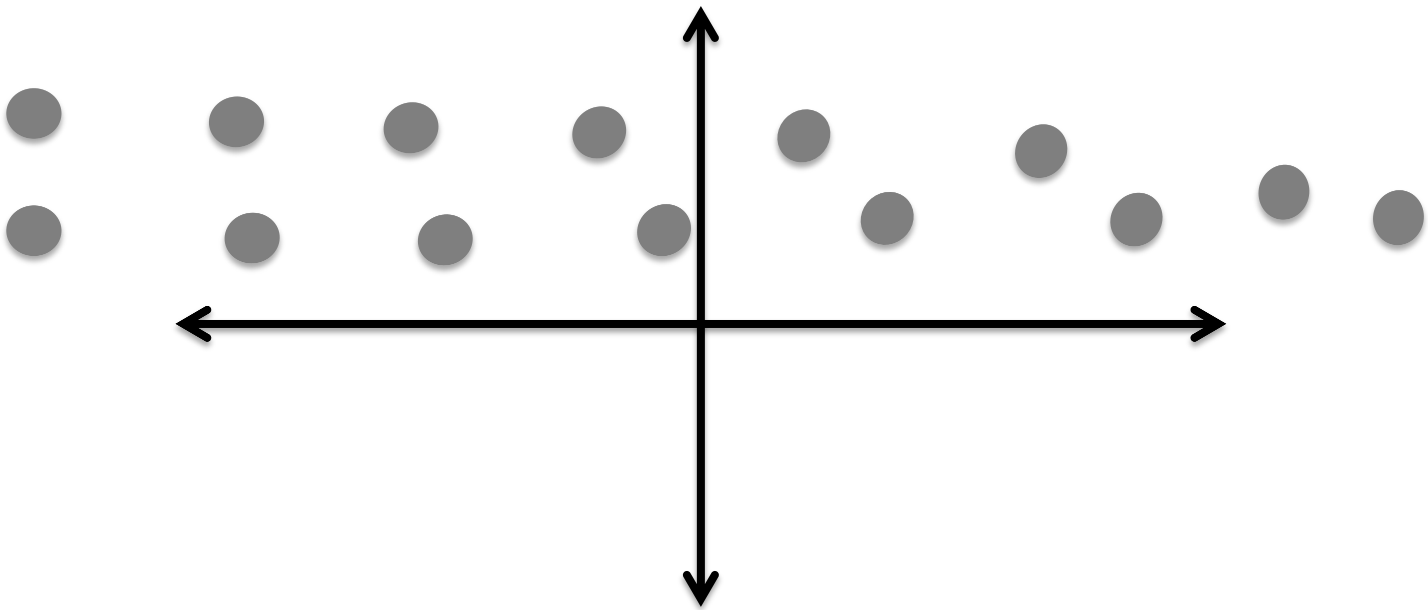

It's not just a failure of the technique! In fact the true threshold for 2-D problem is bounded away from that of the 1-D problem

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These bounds arise through the study of de Branges spaces of entire functions

[Huang, Kakade '15]: Introduced a tensor method for recovering exponential sums whose analysis depends

$$\begin{bmatrix} e^{-2\pi i\langle \mu_1,\omega_1\rangle} & e^{-2\pi i\langle \mu_1,\omega_2\rangle} & \cdots & e^{-2\pi i\langle \mu_1,\omega_m\rangle} \\ e^{-2\pi i\langle \mu_2,\omega_1\rangle} & e^{-2\pi i\langle \mu_2,\omega_2\rangle} & & e^{-2\pi i\langle \mu_2,\omega_m\rangle} \\ \vdots & & \ddots & \vdots \\ e^{-2\pi i\langle \mu_k,\omega_1\rangle} & e^{-2\pi i\langle \mu_k,\omega_2\rangle} & & e^{-2\pi i\langle \mu_k,\omega_m\rangle} \end{bmatrix}$$

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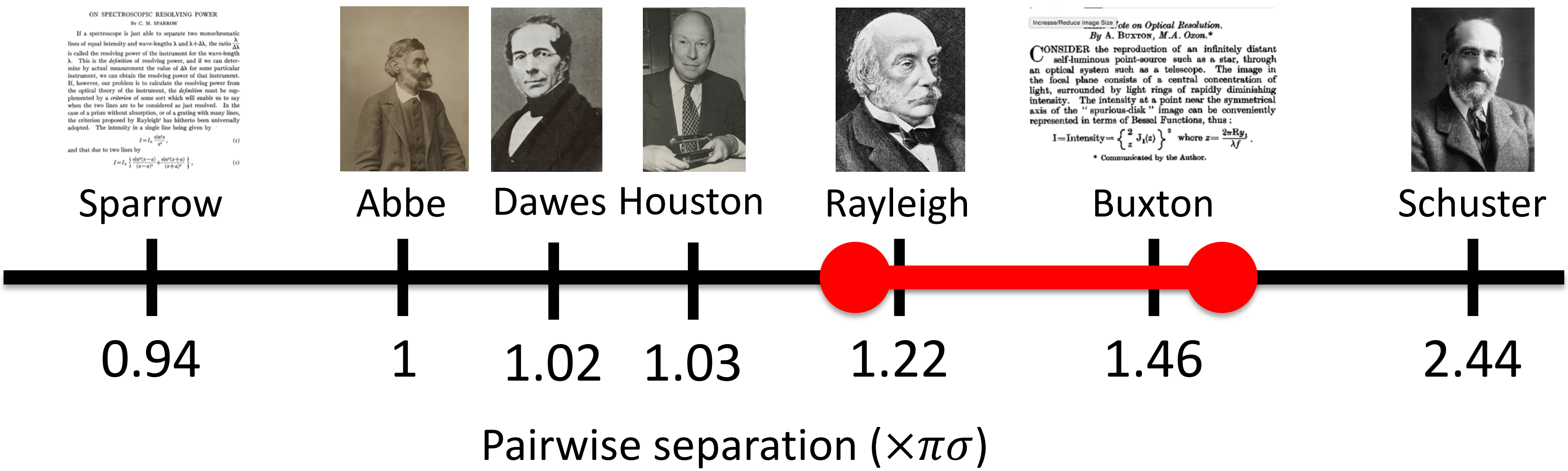
We can use the 2-D extremal functions to show that random ω_j's from the $B(0, \gamma_+)$ have bounded condition number whp

SUMMARY

With all due apologies to Carroll Sparrow's friends and colleagues:



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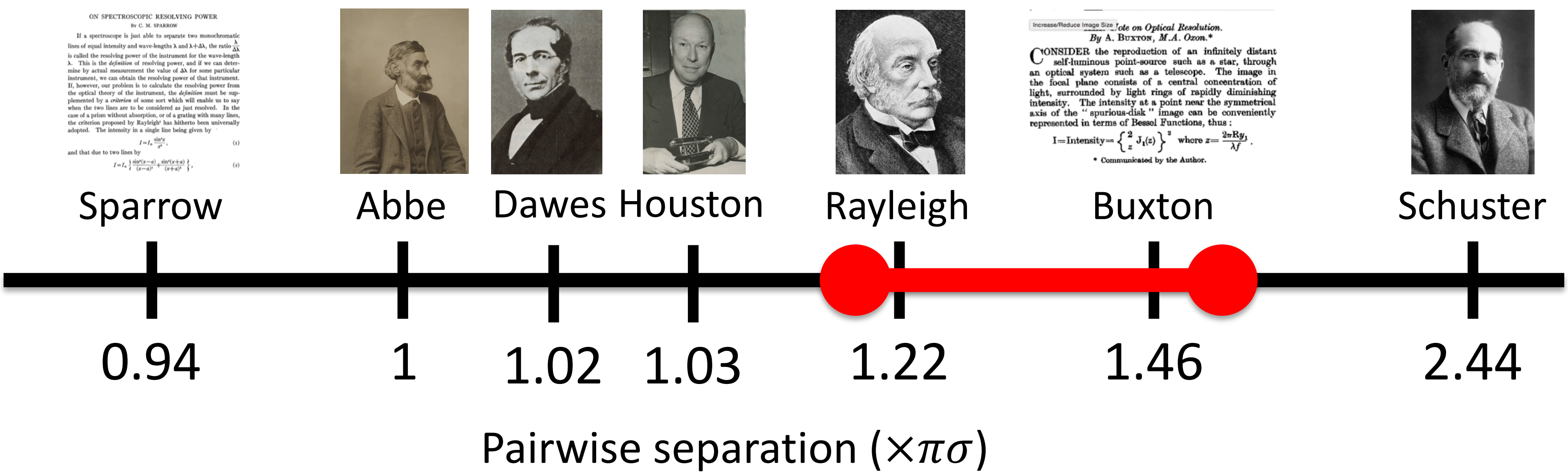

Finding the sharp diffraction limit remains a challenging problem in harmonic analysis

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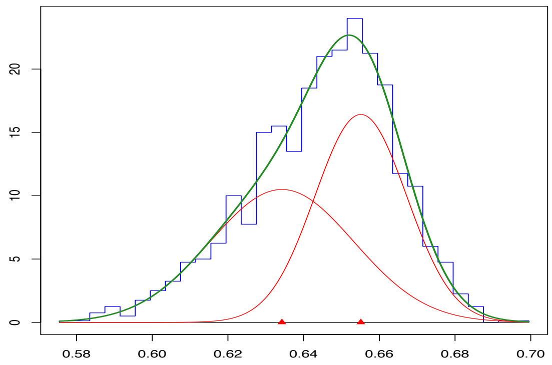

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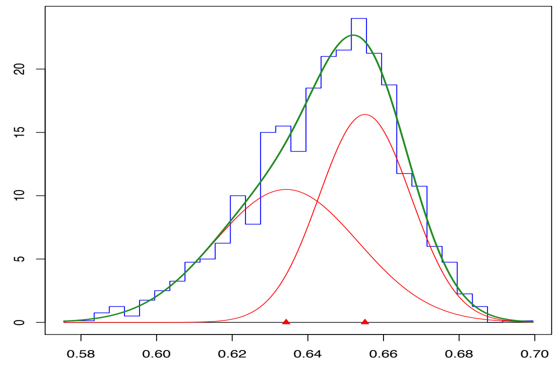

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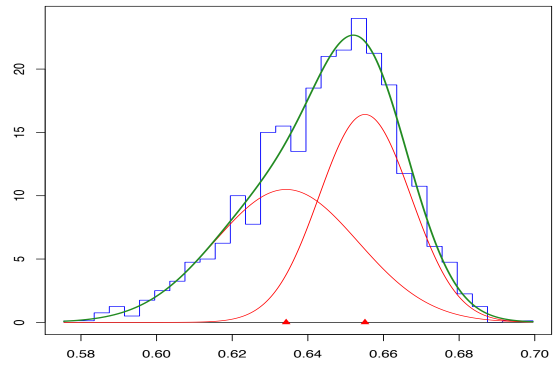

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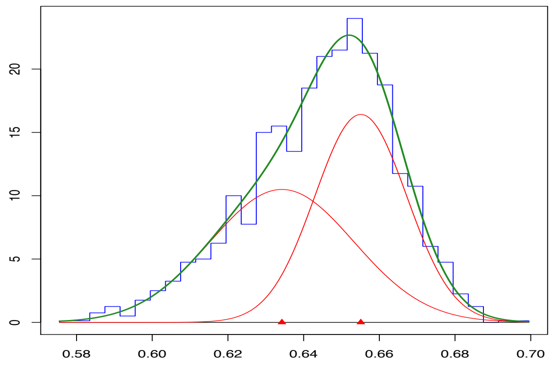

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Do tools from theoretical machine learning have more to say about provable algorithms for inverse problems in science?

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- The diffraction limit is an inverse problem
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- **Are other inverse problems in the Sciences amenable to tools from theoretical ML?**

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Thanks! Any Questions?