BEYOND MATRIX COMPLETION

ANKUR MOITRA

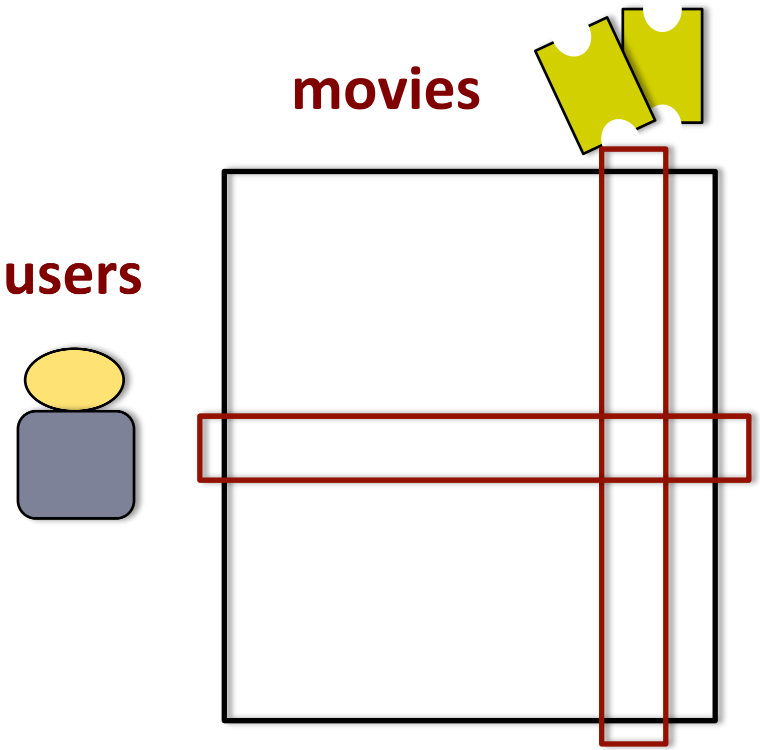
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Based on joint work with Boaz Barak (Harvard)

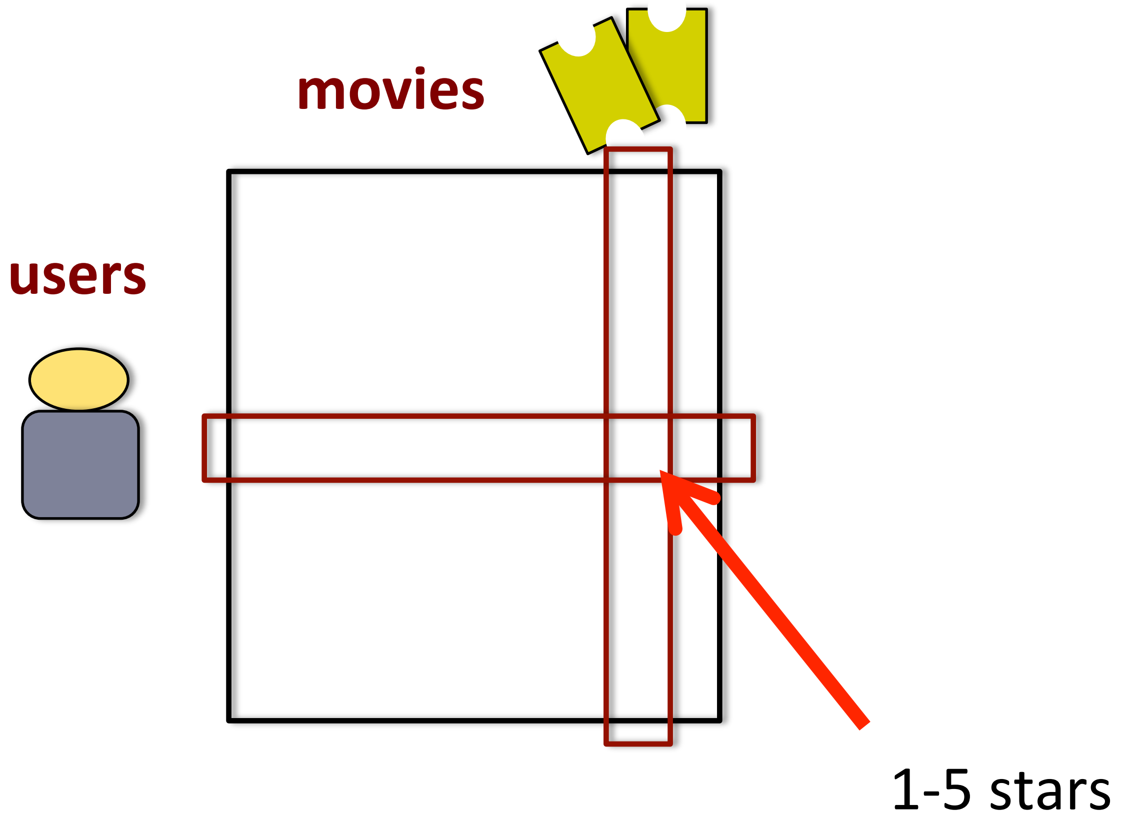
Part I:

Matrix completion

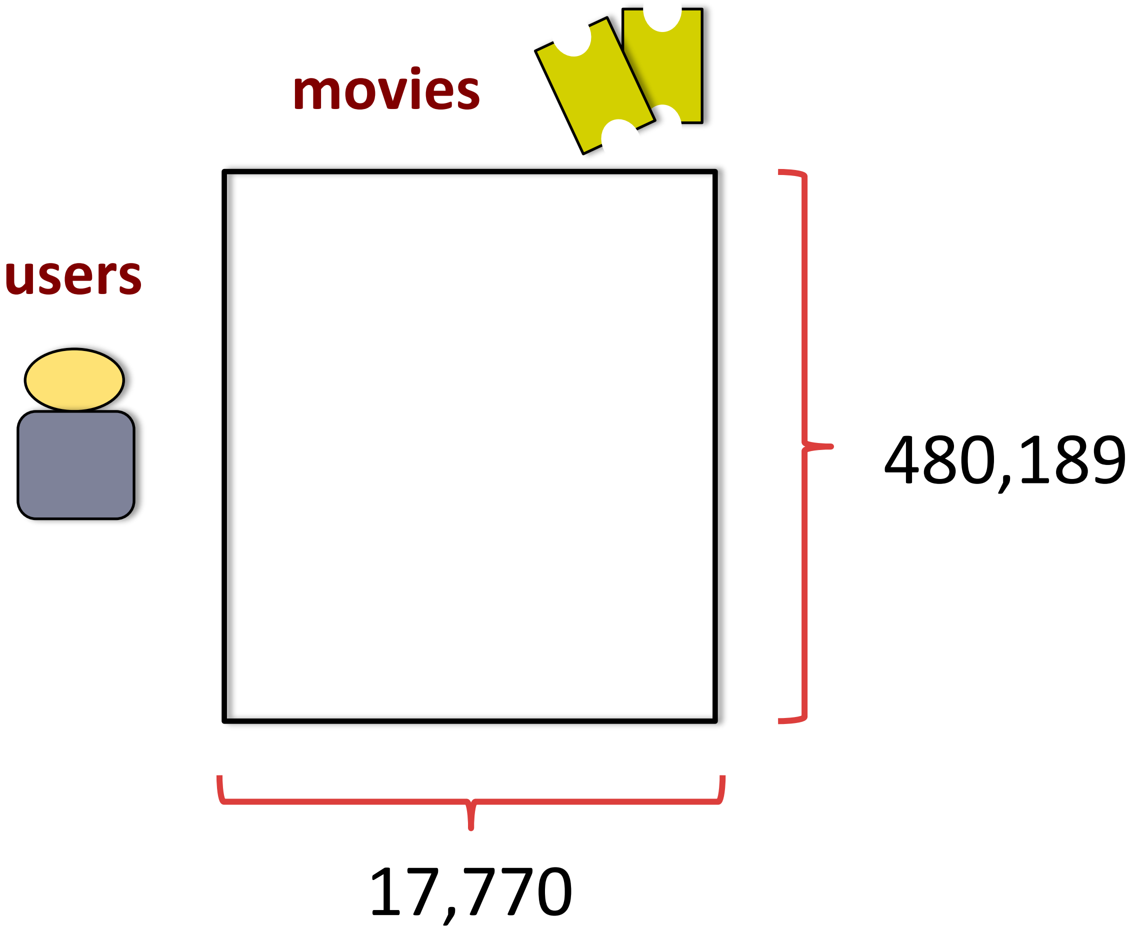
THE NETFLIX PROBLEM



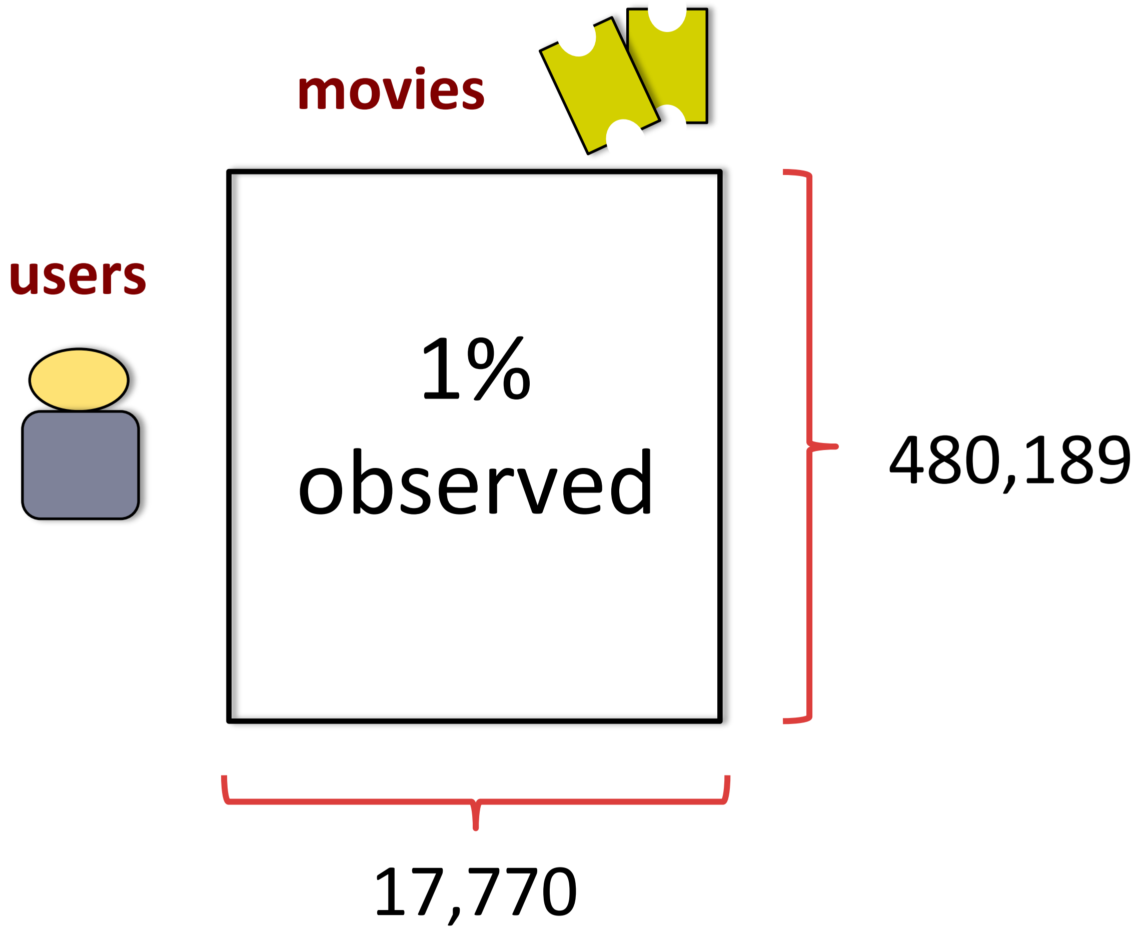
THE NETFLIX PROBLEM



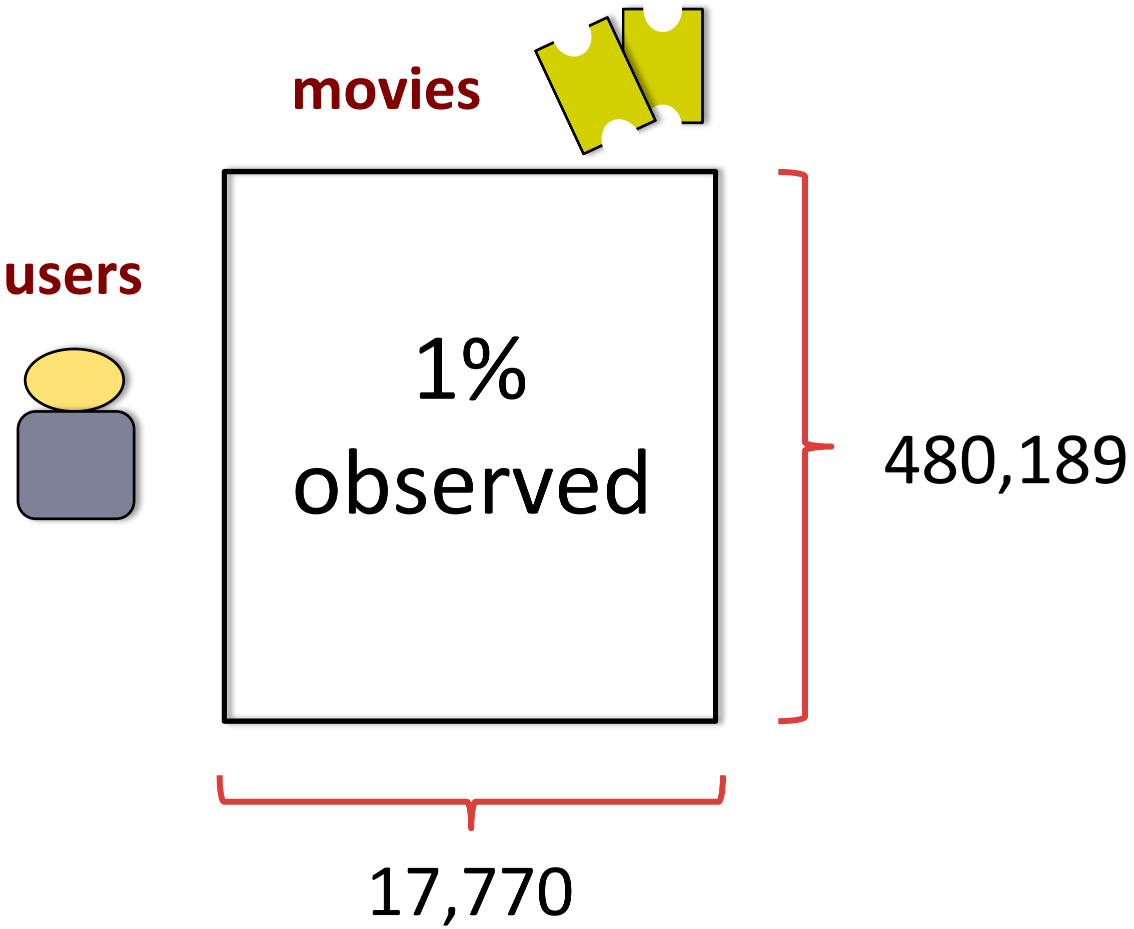
THE NETFLIX PROBLEM



THE NETFLIX PROBLEM



THE NETFLIX PROBLEM



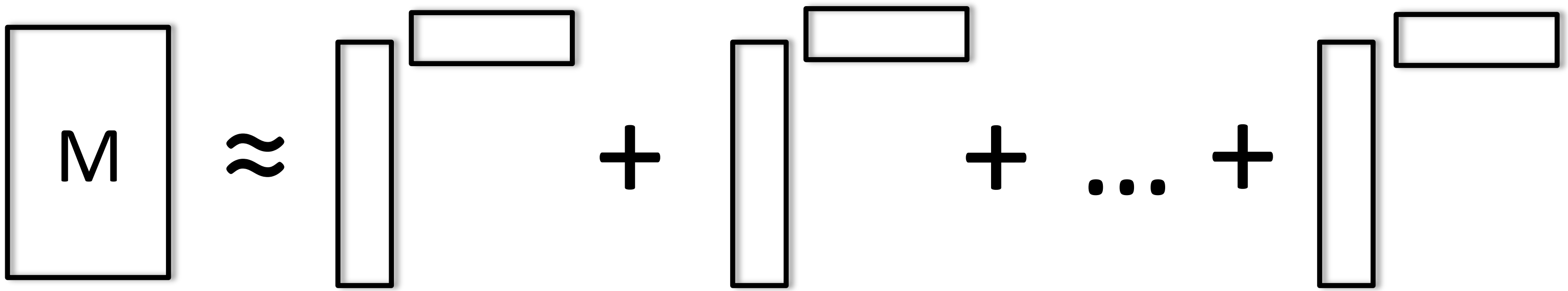
Can we (approximately) fill-in the missing entries?

MATRIX COMPLETION

Let M be an unknown, approximately low-rank matrix

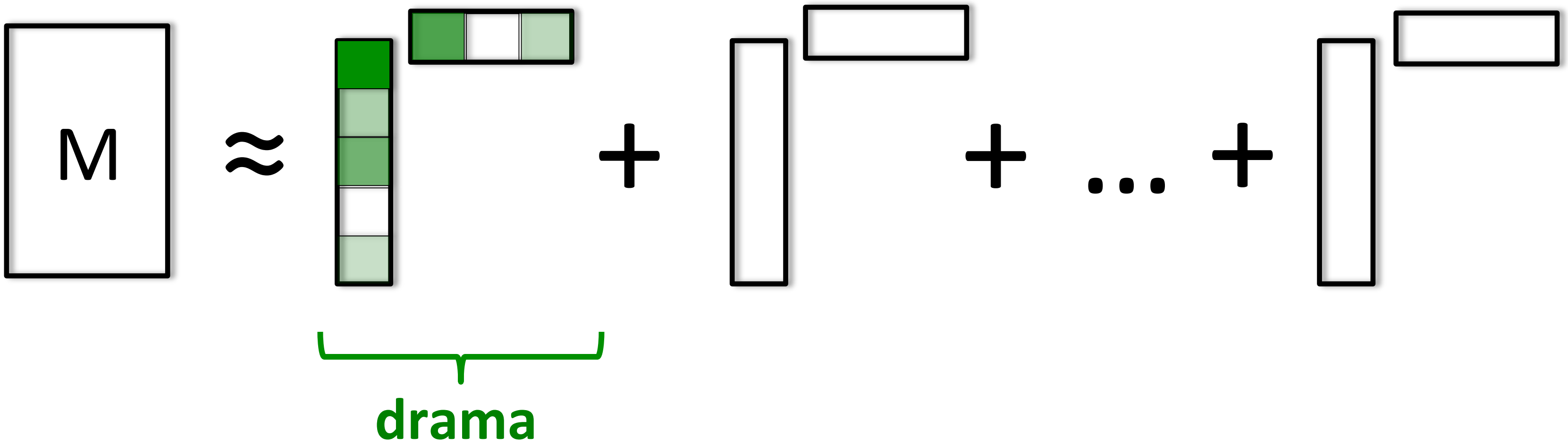
MATRIX COMPLETION

Let M be an unknown, approximately low-rank matrix



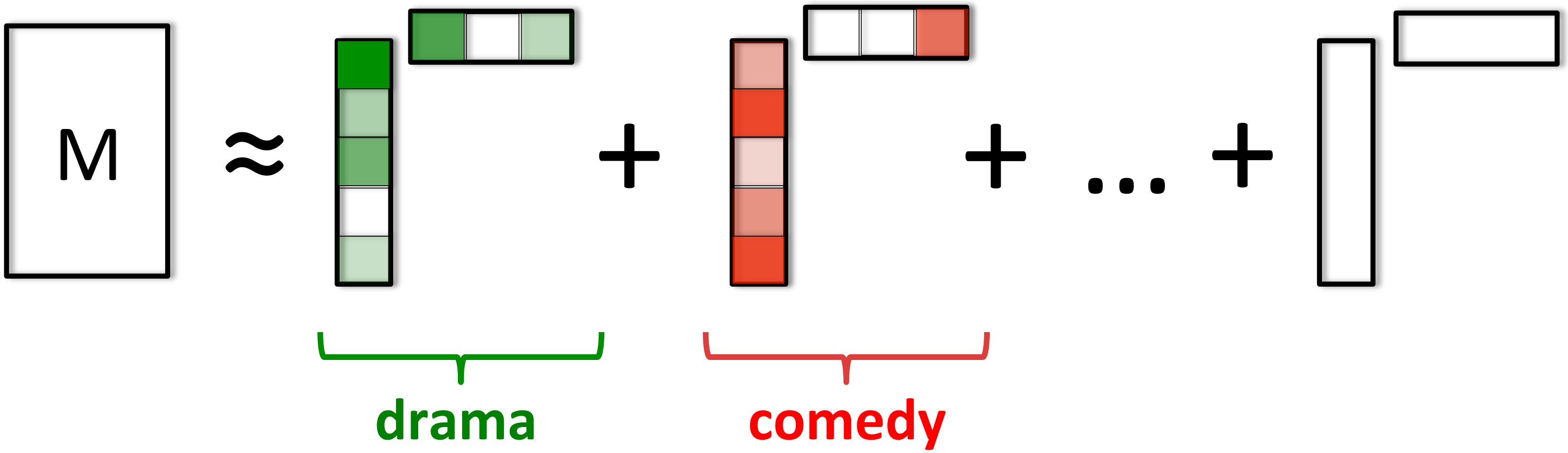
MATRIX COMPLETION

Let M be an unknown, approximately low-rank matrix



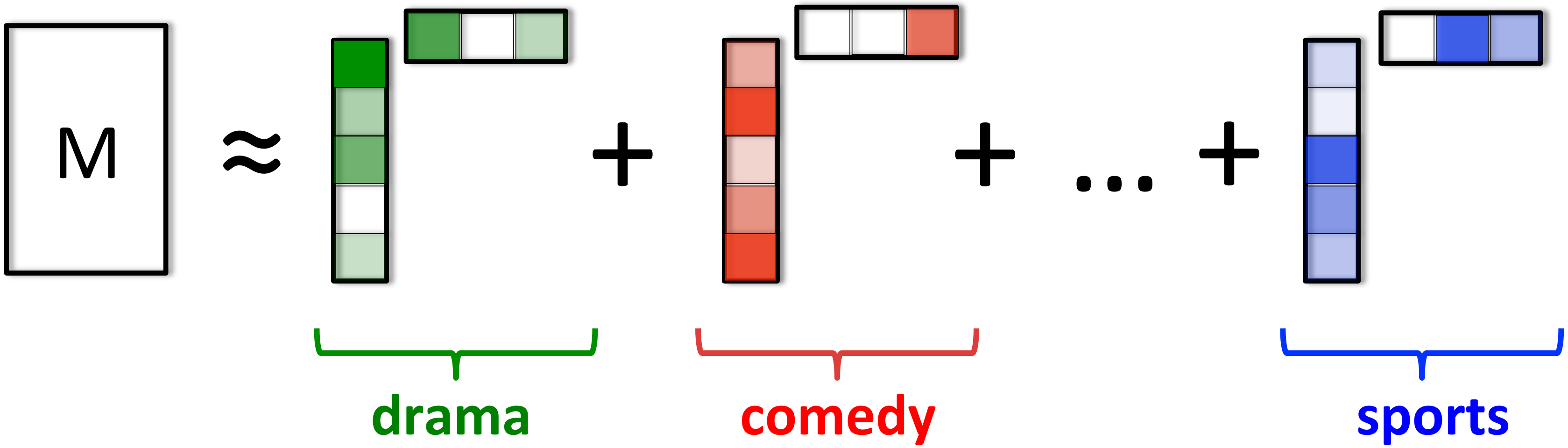
MATRIX COMPLETION

Let M be an unknown, approximately low-rank matrix



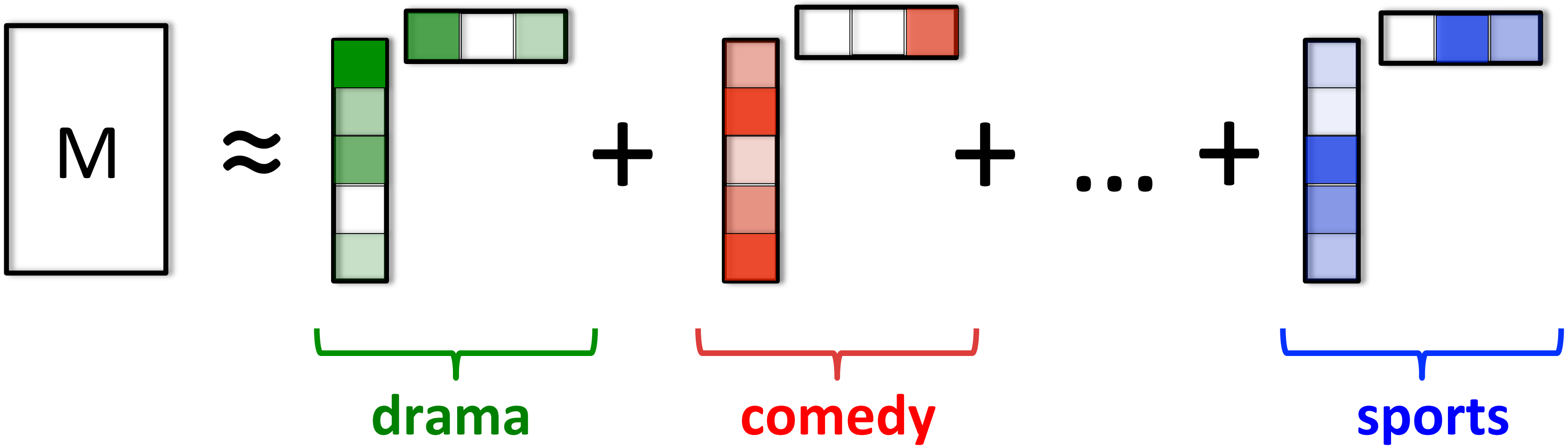
MATRIX COMPLETION

Let M be an unknown, approximately low-rank matrix



MATRIX COMPLETION

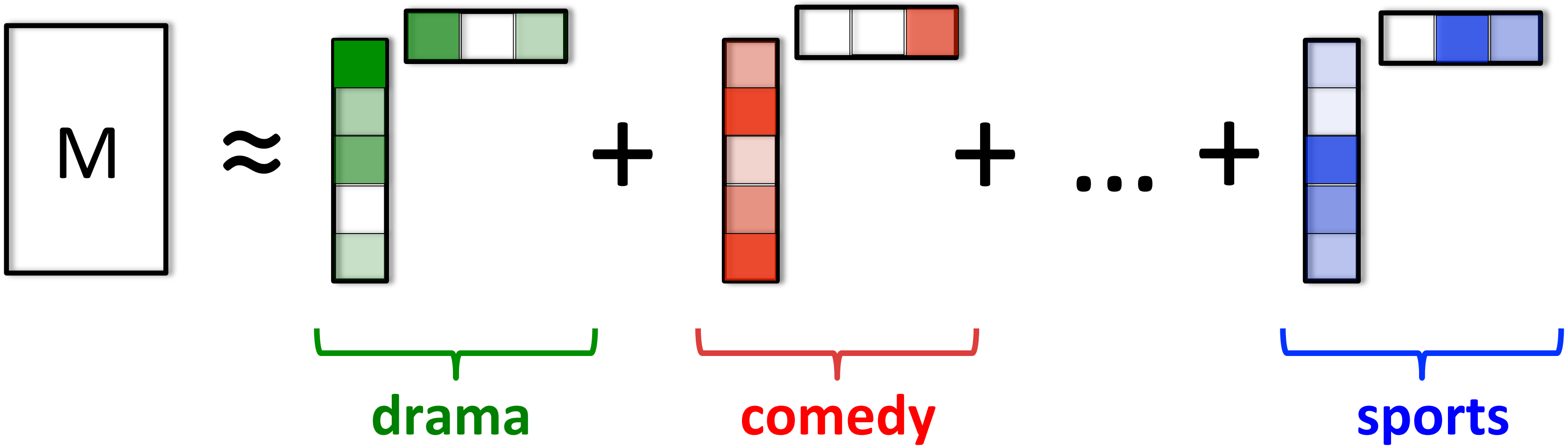
Let M be an unknown, approximately low-rank matrix



Model: we are given random observations $M_{i,j}$ for all $i,j \in \Omega$

MATRIX COMPLETION

Let M be an unknown, approximately low-rank matrix



Model: we are given random observations $M_{i,j}$ for all $i,j \in \Omega$

Is there an efficient algorithm to recover M?

MATRIX COMPLETION

The natural formulation is **non-convex**, and **NP-hard**

$$\min \operatorname{rank}(X) \;\; \text{s.t.} \;\; \frac{1}{|\Omega|} \sum_{(i,j)\in\Omega} |X_{i,j} - M_{i,j}| \leq \eta$$

MATRIX COMPLETION

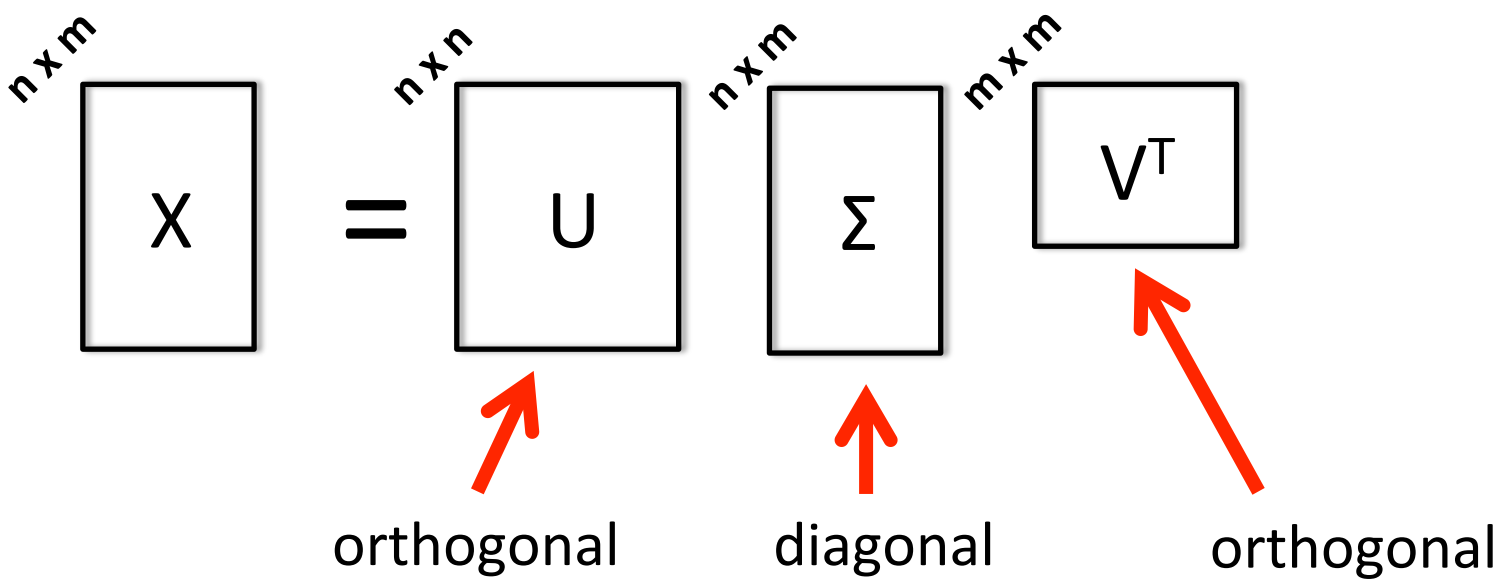
The natural formulation is **non-convex**, and **NP-hard**

$$\min \operatorname{rank}(X) \ \text{ s.t. } \frac{1}{|\Omega|} \sum_{(i,j)\in\Omega} |X_{i,j} - M_{i,j}| \leq \eta$$

There is a powerful, convex relaxation…

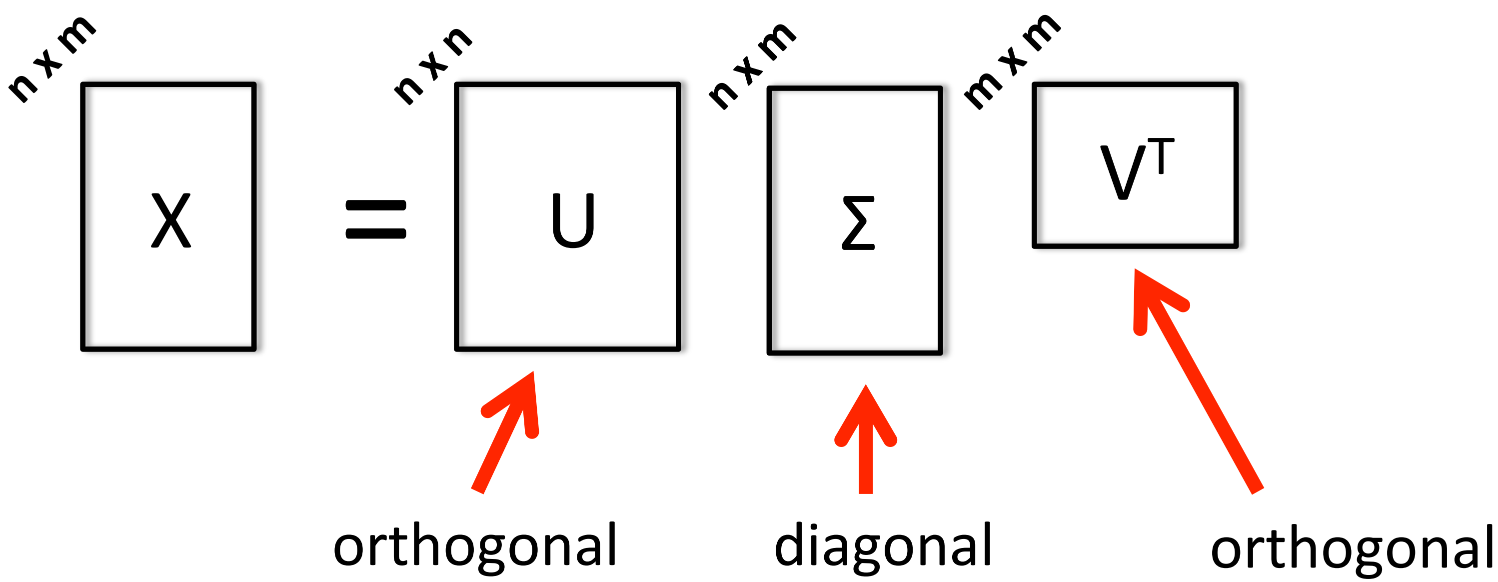
THE NUCLEAR NORM

Consider the **singular value decomposition** of X:



THE NUCLEAR NORM

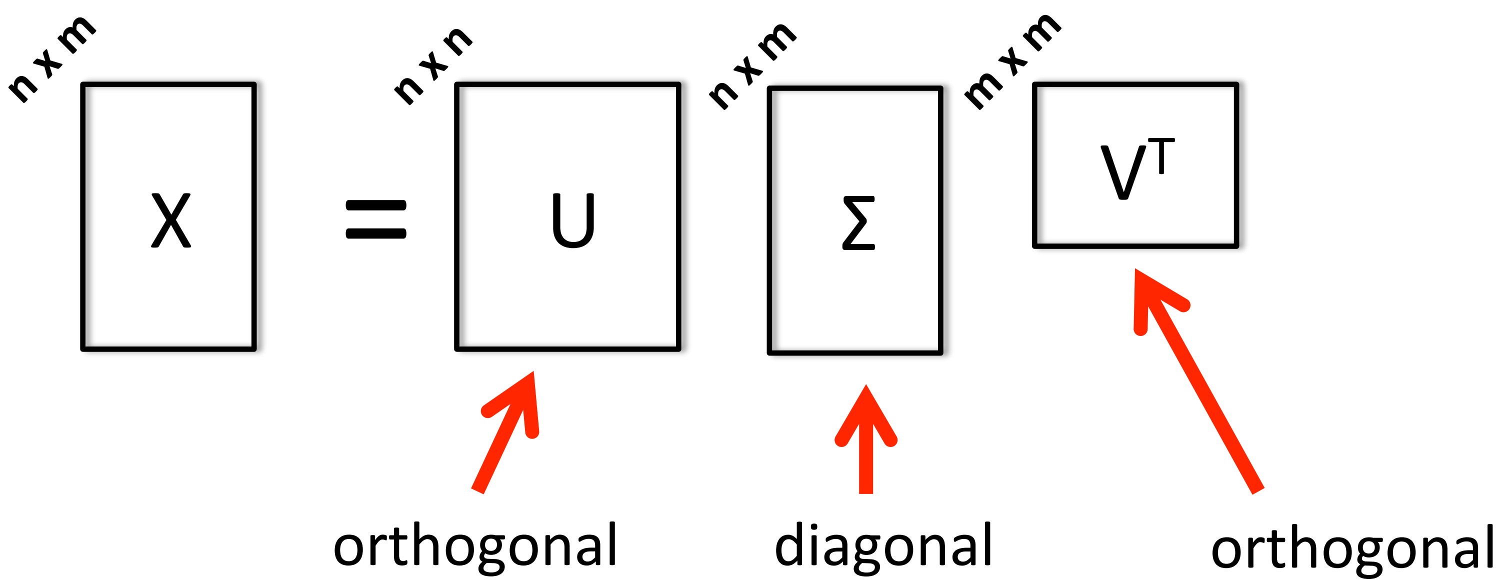
Consider the **singular value decomposition** of X:



Let $\sigma_1 \geq \sigma_2 \geq \ldots \sigma_r > \sigma_{r+1} = \ldots \sigma_m = 0$ be the singular values

THE NUCLEAR NORM

Consider the **singular value decomposition** of X:



Let $\sigma_1 \geq \sigma_2 \geq \ldots \sigma_r > \sigma_{r+1} = \ldots \sigma_m = 0$ be the singular values

Then rank(X) = r, and $||X||_* = \sigma_1 + \sigma_2 + \ldots + \sigma_r$ (**nuclear norm**)

This yields a convex relaxation, that can be solved efficiently:

$$\min \|X\|_* \text{ s.t. } \frac{1}{|\Omega|} \sum_{(i,j)\in\Omega} |X_{i,j} - M_{i,j}| \leq \eta \quad \textbf{(P)}$$

[Fazel], [Srebro, Shraibman], [Recht, Fazel, Parrilo], [Candes, Recht], [Candes, Tao], [Candes, Plan], [Recht],

This yields a convex relaxation, that can be solved efficiently:

$$\min \|X\|_* \text{ s.t. } \frac{1}{|\Omega|} \sum_{(i,j)\in\Omega} |X_{i,j} - M_{i,j}| \leq \eta \quad \textbf{(P)}$$

[Fazel], [Srebro, Shraibman], [Recht, Fazel, Parrilo], [Candes, Recht], [Candes, Tao], [Candes, Plan], [Recht],

Theorem: If M is n x n and has rank r, and is C-incoherent then **(P)** recovers M exactly from $C^6 nr\log^2 n$ observations

This yields a convex relaxation, that can be solved efficiently:

$$\min \|X\|_* \text{ s.t. } \frac{1}{|\Omega|} \sum_{(i,j)\in\Omega} |X_{i,j} - M_{i,j}| \leq \eta \qquad \textbf{(P)}$$

[Fazel], [Srebro, Shraibman], [Recht, Fazel, Parrilo], [Candes, Recht], [Candes, Tao], [Candes, Plan], [Recht],

Theorem: If M is n x n and has rank r, and is C-incoherent then **(P)** recovers M exactly from $C^6 nr\log^2 n$ observations

This is nearly optimal, since there are O(nr) parameters

Robust PCA [Candes et al.], [Chandrasekaran et al.], …

Can we recover a low rank matrix from sparse corruptions?

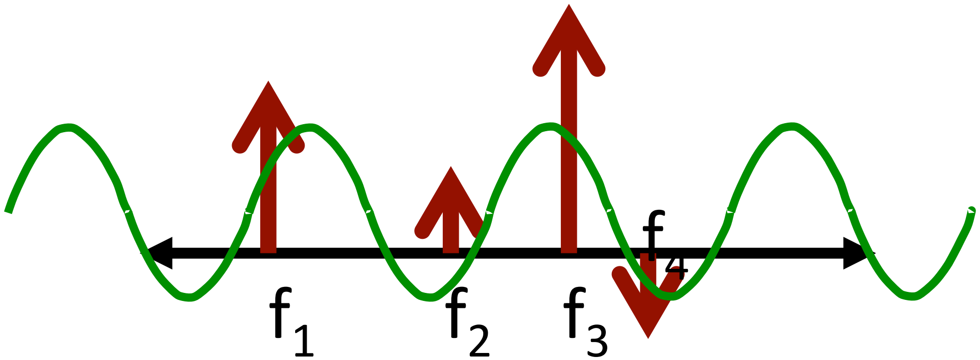
Robust PCA [Candes et al.], [Chandrasekaran et al.], …

Can we recover a low rank matrix from sparse corruptions?

Superresolution, compressed sensing off-the-grid

[Candes, Fernandez-Granda], [Tang et al.], …

Can we recover well-separated points from low-frequency measurements?



Part II:

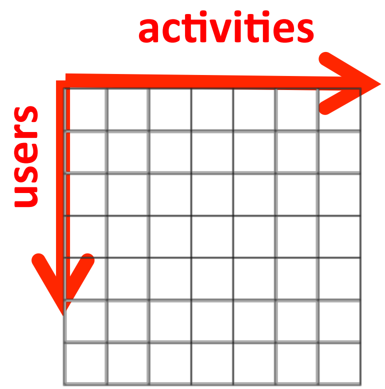
Higher order structure?

TENSOR COMPLETION

Can using **more than two** attributes can lead to better recommendations?

TENSOR COMPLETION

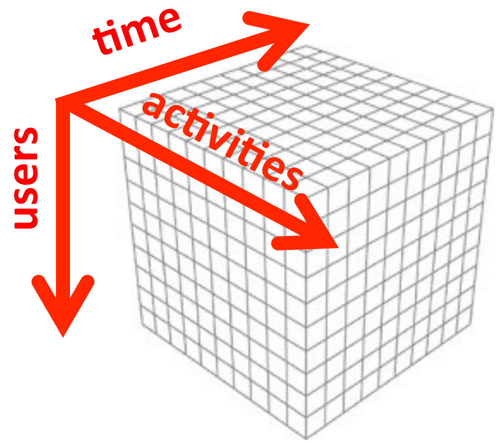
Can using **more than two** attributes can lead to better recommendations?



e.g. Groupon

TENSOR COMPLETION

Can using **more than two** attributes can lead to better recommendations?

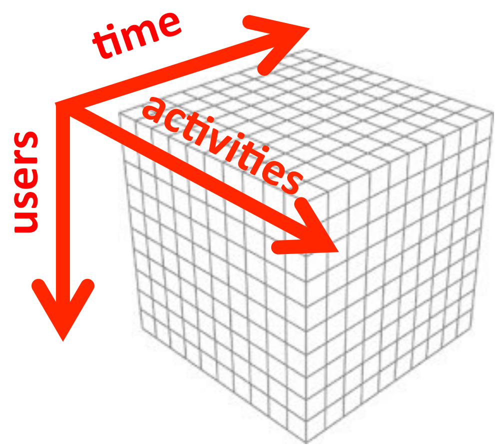


e.g. Groupon

time: season, time of day, weekday/weekend, etc

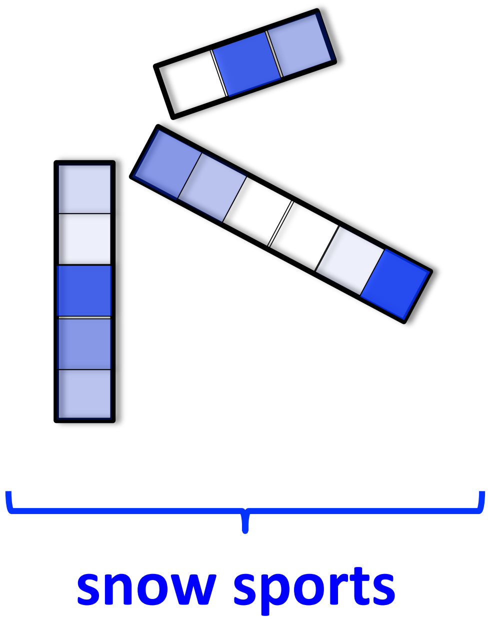
TENSOR COMPLETION

Can using **more than two** attributes can lead to better recommendations?



e.g. Groupon

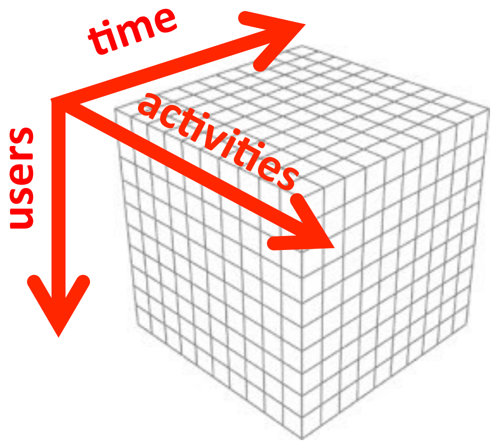
$$T \approx \sum_{i=1}^{r}$$



time: season, time of day, weekday/weekend, etc

TENSOR COMPLETION

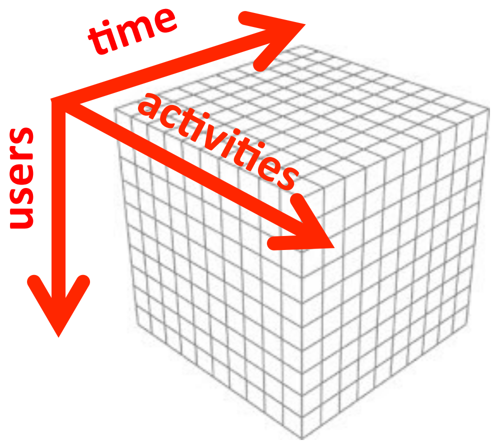
Can using **more than two** attributes can lead to better recommendations?



$$T \approx \sum_{i=1}^{r} a_i \otimes b_i \otimes c_i$$

TENSOR COMPLETION

Can using **more than two** attributes can lead to better recommendations?



$$T \approx \sum_{i=1}^{r} a_i \otimes b_i \otimes c_i$$

Can we (approximately) fill-in the missing entries?

THE TROUBLE WITH TENSORS

Natural approach (suggested by many authors):

$$\min \ \|X\|_* \ \text{s.t.} \ \frac{1}{|\Omega|}\sum_{(i,j,k)\in\Omega} |X_{i,j,k} - T_{i,j,k}| \leq \eta \quad \textbf{(P)}$$

tensor nuclear norm

THE TROUBLE WITH TENSORS

Natural approach (suggested by many authors):

$$\min \|X\|_* \text{ s.t. } \frac{1}{|\Omega|}\sum_{(i,j,k)\in\Omega} |X_{i,j,k} - T_{i,j,k}| \leq \eta \quad \textbf{(P)}$$

tensor nuclear norm

The tensor nuclear norm is **NP-hard** to compute!

[Gurvits], [Liu], [Harrow, Montanaro]

In fact, most of the linear algebra toolkit is **ill-posed**, or **computationally hard** for tensors…

In fact, most of the linear algebra toolkit is **ill-posed**, or **computationally hard** for tensors…

e.g. [Hillar, Lim] "Most Tensor Problems are NP-Hard"

In fact, most of the linear algebra toolkit is **ill-posed**, or **computationally hard** for tensors…

e.g. [Hillar, Lim] "Most Tensor Problems are NP-Hard"

Table I. Tractability of Tensor Problems

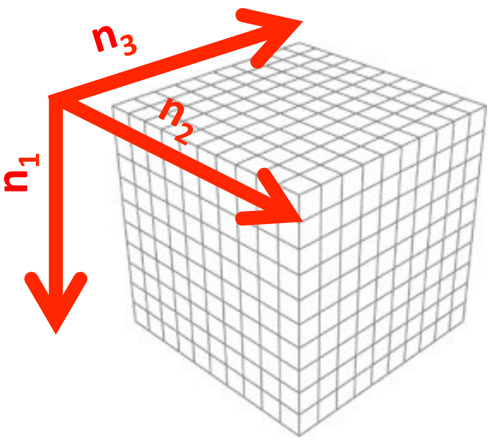
Problem	**Complexity**
Bivariate Matrix Functions over $\mathbb{R}$, $\mathbb{C}$	Undecidable (Proposition 12.2)
Bilinear System over $\mathbb{R}$, $\mathbb{C}$	NP-hard (Theorems 2.6, 3.7, 3.8)
Eigenvalue over $\mathbb{R}$	NP-hard (Theorem 1.3)
Approximating Eigenvector over $\mathbb{R}$	NP-hard (Theorem 1.5)
Symmetric Eigenvalue over $\mathbb{R}$	NP-hard (Theorem 9.3)
Approximating Symmetric Eigenvalue over $\mathbb{R}$	NP-hard (Theorem 9.6)
Singular Value over $\mathbb{R}$, $\mathbb{C}$	NP-hard (Theorem 1.7)
Symmetric Singular Value over $\mathbb{R}$	NP-hard (Theorem 10.2)
Approximating Singular Vector over $\mathbb{R}$, $\mathbb{C}$	NP-hard (Theorem 6.3)
Spectral Norm over $\mathbb{R}$	NP-hard (Theorem 1.10)
Symmetric Spectral Norm over $\mathbb{R}$	NP-hard (Theorem 10.2)
Approximating Spectral Norm over $\mathbb{R}$	NP-hard (Theorem 1.11)
Nonnegative Definiteness	NP-hard (Theorem 11.2)
Best Rank-1 Approximation	NP-hard (Theorem 1.13)
Best Symmetric Rank-1 Approximation	NP-hard (Theorem 10.2)
Rank over $\mathbb{R}$ or $\mathbb{C}$	NP-hard (Theorem 8.2)
Enumerating Eigenvectors over $\mathbb{R}$	#P-hard (Corollary 1.16)
Combinatorial Hyperdeterminant	NP-, #P-, VNP-hard (Theorems 4.1 , 4.2, Corollary 4.3)
Geometric Hyperdeterminant	Conjectures 1.9, 13.1
Symmetric Rank	Conjecture 13.2
Bilinear Programming	Conjecture 13.4
Bilinear Least Squares	Conjecture 13.5

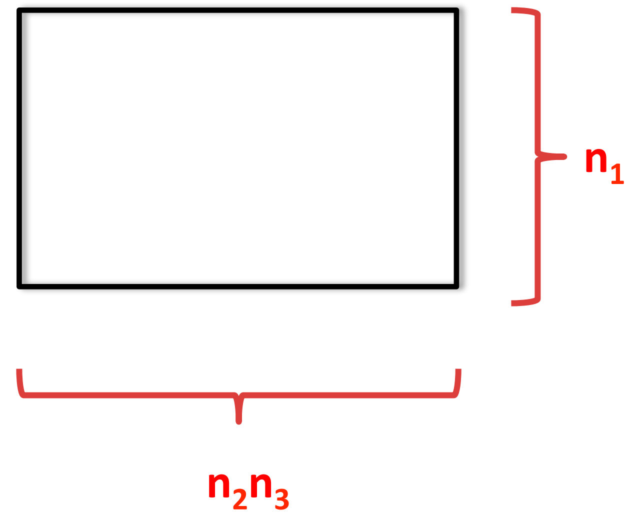
FLATTENING A TENSOR

Many tensor methods rely on **flattening**:

FLATTENING A TENSOR

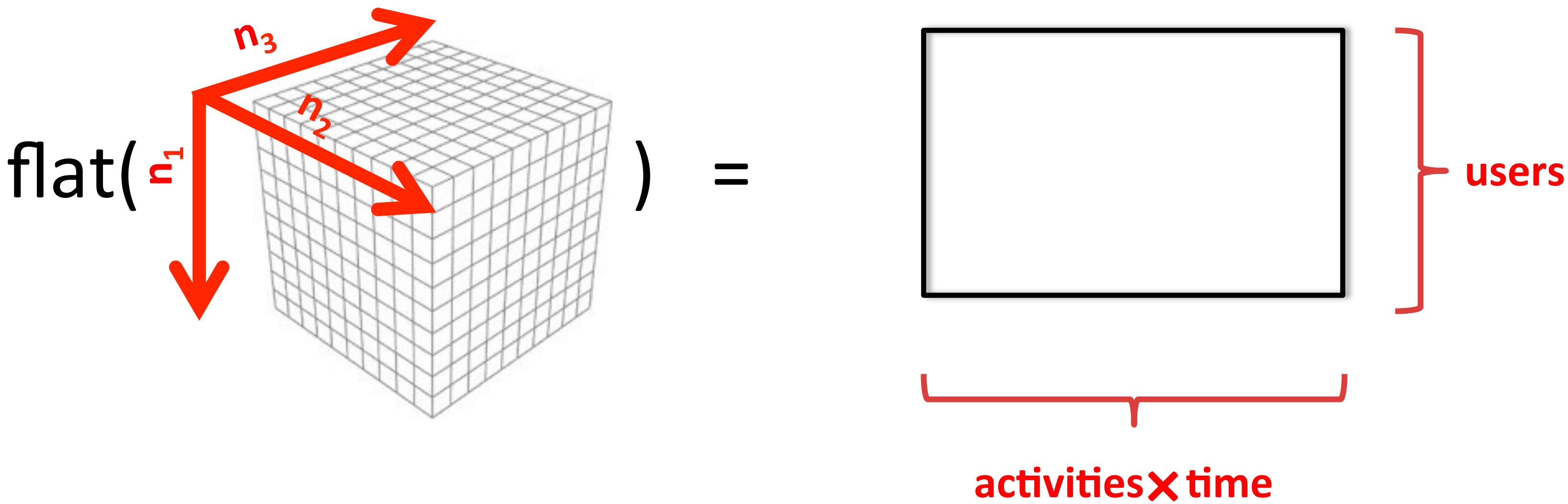
Many tensor methods rely on **flattening**:

flat(

) = 

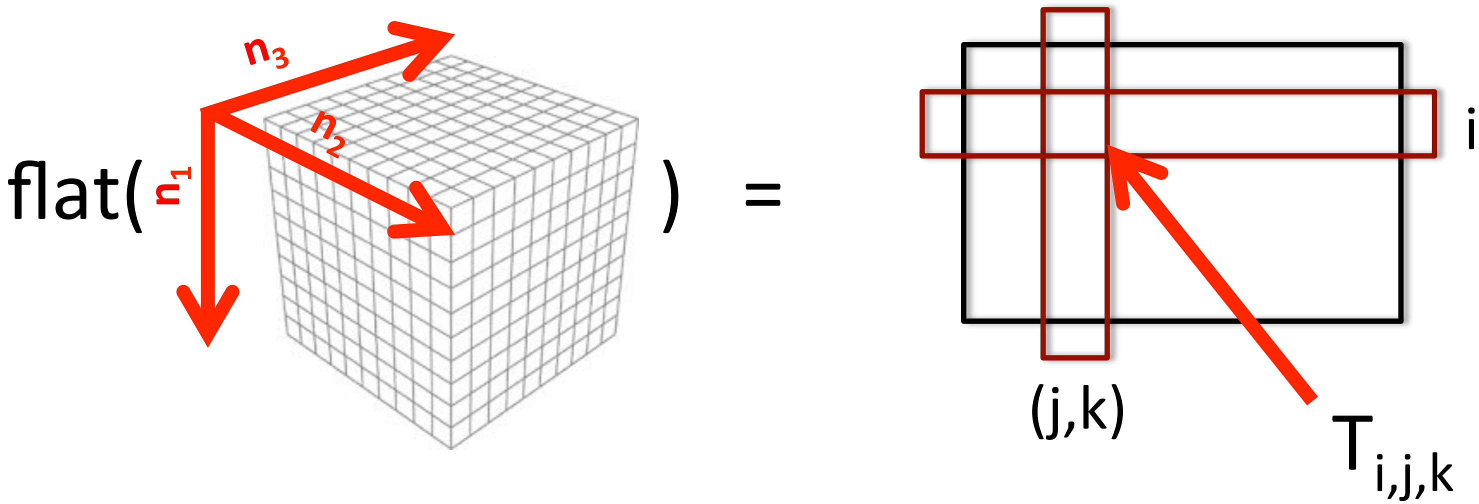
FLATTENING A TENSOR

Many tensor methods rely on **flattening**:



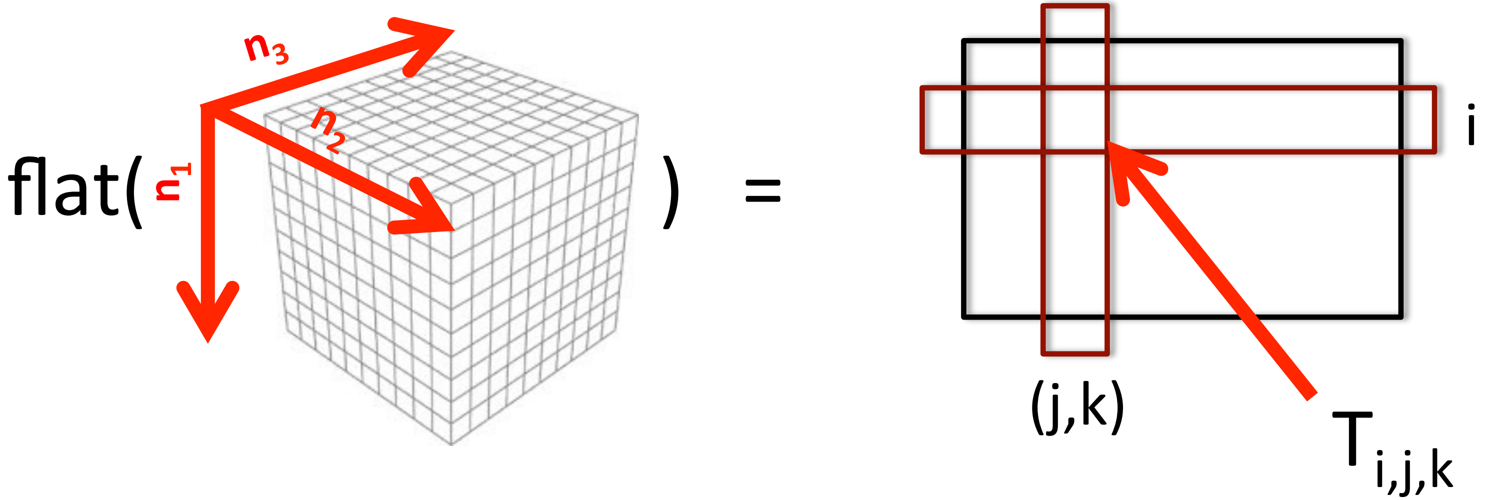
FLATTENING A TENSOR

Many tensor methods rely on **flattening**:



FLATTENING A TENSOR

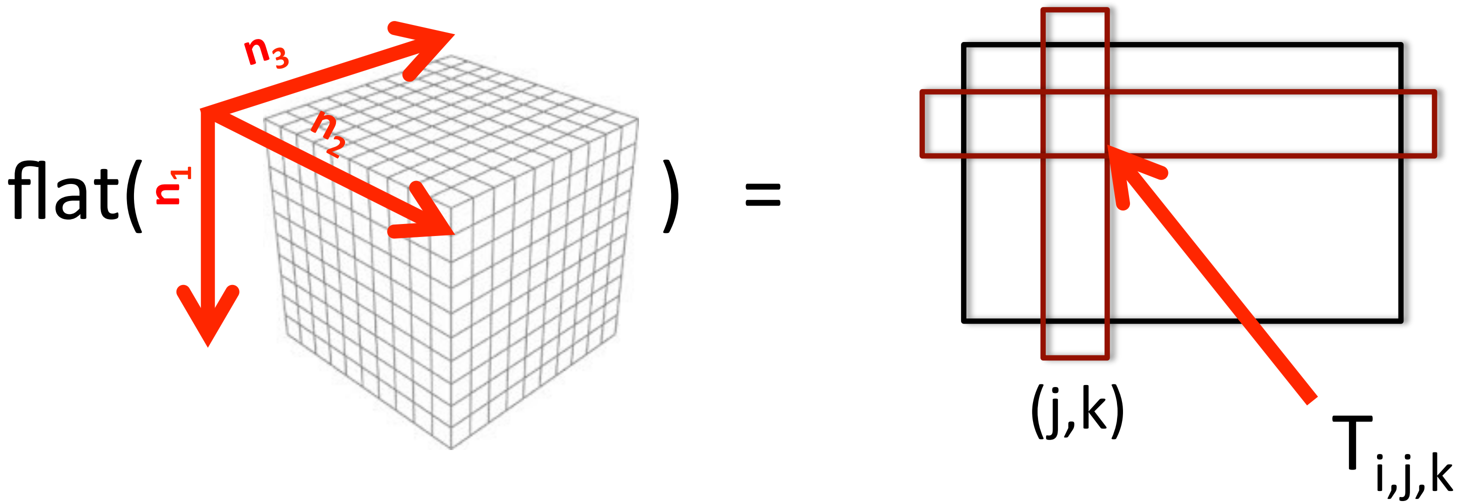
Many tensor methods rely on **flattening**:



This is a **rearrangement** of the entries, into a matrix, that does not increase its **rank**

FLATTENING A TENSOR

Many tensor methods rely on **flattening**:



$$\mathrm{flat}\left(\sum_{i=1}^{r} a_i \otimes b_i \otimes c_i\right) = \sum_{i=1}^{r} a_i \otimes \underbrace{\mathrm{vec}(b_i c_i^T)}_{n_2 n_3\text{-dimensional vector}}$$

Let $n_1 = n_2 = n_3 = n$

We would need $\widetilde{O}(n^2 r)$ observations to fill-in flat(T)

Let $n_1 = n_2 = n_3 = n$

We would need $\widetilde{O}(n^2 r)$ observations to fill-in flat(T)

There are many other variants of **flattening**, but with comparable guarantees

[Liu, Musialski, Wonka, Ye], [Gandy, Recht, Yamada], [Signoretto, De Lathauwer, Suykens], [Tomioko, Hayashi, Kashima], [Mu, Huang, Wright, Goldfarb], …

Let $n_1 = n_2 = n_3 = n$

We would need $\widetilde{O}(n^2 r)$ observations to fill-in flat(T)

There are many other variants of **flattening**, but with comparable guarantees

[Liu, Musialski, Wonka, Ye], [Gandy, Recht, Yamada], [Signoretto, De Lathauwer, Suykens], [Tomioko, Hayashi, Kashima], [Mu, Huang, Wright, Goldfarb], …

Can we beat flattening?

Let $n_1 = n_2 = n_3 = n$

We would need $\widetilde{O}(n^2 r)$ observations to fill-in flat(T)

There are many other variants of **flattening**, but with comparable guarantees

[Liu, Musialski, Wonka, Ye], [Gandy, Recht, Yamada], [Signoretto, De Lathauwer, Suykens], [Tomioko, Hayashi, Kashima], [Mu, Huang, Wright, Goldfarb], …

Can we beat flattening?

Can we make better predictions than we do by treating each **activity x time** as unrelated?

Part III:

Nearly optimal algorithms for noisy tensor completion

OUR RESULTS

Suppose we are given $|\Omega| = m$ noisy observations from T:

$$T = \sum_{i=1}^{r} \sigma_i \, a_i \otimes b_i \otimes c_i \; + \textbf{noise}$$

with $|\sigma_i|$, $|a_i|_\infty$, $|b_i|_\infty$, $|c_i|_\infty \leq C$

bdd by η

OUR RESULTS

Suppose we are given $|\Omega| = m$ noisy observations from T:

$$T = \sum_{i=1}^{r} \sigma_i \, a_i \otimes b_i \otimes c_i \; + \textbf{noise}$$

with $|\sigma_i|, \; |a_i|_\infty, \; |b_i|_\infty, \; |c_i|_\infty \leq C$

bdd by η

Theorem: There is an efficient algorithm that with prob $1-\delta$, outputs X with

$$\frac{1}{n^3} \sum_{i,j,k} |X_{i,j,k} - T_{i,j,k}| \leq C^3 r \sqrt{\frac{n^{3/2} \log^4 n}{m}} + 2C^3 r \sqrt{\frac{\ln(2/\delta)}{m}} + 2\eta$$

SOME REMARKS

Theorem: There is an efficient algorithm that with prob 1-δ, outputs X with

$$\frac{1}{n^3}\sum_{i,j,k} |X_{i,j,k} - T_{i,j,k}| \leq C^3 r\sqrt{\frac{n^{3/2}\log^4 n}{m}} + 2C^3 r\sqrt{\frac{\ln(2/\delta)}{m}} + 2\eta$$

SOME REMARKS

Theorem: There is an efficient algorithm that with prob 1-δ, outputs X with

$$\frac{1}{n^3}\sum_{i,j,k} |X_{i,j,k} - T_{i,j,k}| \leq C^3 r\sqrt{\frac{n^{3/2}\log^4 n}{m}} + 2C^3 r\sqrt{\frac{\ln(2/\delta)}{m}} + 2\eta$$

In many settings, 1-o(1) fraction of entries of T are at least $r^{1/2}$/polylog(n) in magnitude

SOME REMARKS

Theorem: There is an efficient algorithm that with prob 1-δ, outputs X with

$$\frac{1}{n^3}\sum_{i,j,k} |X_{i,j,k} - T_{i,j,k}| \leq C^3 r\sqrt{\frac{n^{3/2}\log^4 n}{m}} + 2C^3 r\sqrt{\frac{\ln(2/\delta)}{m}} + 2\eta$$

In many settings, 1-o(1) fraction of entries of T are at least $r^{1/2}$/polylog(n) in magnitude

When $m = \widetilde{\Omega}(n^{3/2}r)$, average error is asymptotically smaller and hence $X_{i,j,k} = (1\pm o(1))T_{i,j,k}$ for 1-o(1) fraction of entries

SOME REMARKS

Theorem: There is an efficient algorithm that with prob 1-δ, outputs X with

$$\frac{1}{n^3}\sum_{i,j,k} |X_{i,j,k} - T_{i,j,k}| \leq C^3 r\sqrt{\frac{n^{3/2}\log^4 n}{m}} + 2C^3 r\sqrt{\frac{\ln(2/\delta)}{m}} + 2\eta$$

In many settings, 1-o(1) fraction of entries of T are at least $r^{1/2}$/polylog(n) in magnitude

When m = $\widetilde{\Omega}(n^{3/2}r)$, average error is asymptotically smaller and hence $X_{i,j,k} = (1\pm o(1))T_{i,j,k}$ for 1-o(1) fraction of entries

"Almost all of the entries, almost entirely correct"

SOME REMARKS

Theorem: There is an efficient algorithm that with prob 1-δ, outputs X with

$$\frac{1}{n^3}\sum_{i,j,k} |X_{i,j,k} - T_{i,j,k}| \leq C^3 r\sqrt{\frac{n^{3/2}\log^4 n}{m}} + 2C^3 r\sqrt{\frac{\ln(2/\delta)}{m}} + 2\eta$$

SOME REMARKS

Theorem: There is an efficient algorithm that with prob 1-δ, outputs X with

$$\frac{1}{n^3}\sum_{i,j,k} |X_{i,j,k} - T_{i,j,k}| \leq C^3 r \sqrt{\frac{n^{3/2}\log^4 n}{m}} + 2C^3 r \sqrt{\frac{\ln(2/\delta)}{m}} + 2\eta$$

For $r = n^{3/2-\delta}$ (**highly overcomplete**), we only need to observe an $n^{-\delta}$ fraction of the entries

SOME REMARKS

Theorem: There is an efficient algorithm that with prob 1-δ, outputs X with

$$\frac{1}{n^3}\sum_{i,j,k} |X_{i,j,k} - T_{i,j,k}| \leq C^3 r\sqrt{\frac{n^{3/2}\log^4 n}{m}} + 2C^3 r\sqrt{\frac{\ln(2/\delta)}{m}} + 2\eta$$

For $r = n^{3/2-\delta}$ (**highly overcomplete**), we only need to observe an $n^{-\delta}$ fraction of the entries

Algorithms for decomposing such tensors given all the entries need stronger (e.g. factors are **random**) assumptions

LOWER BOUNDS

Not only is the **tensor nuclear norm** hard to compute, but…

LOWER BOUNDS

Not only is the **tensor nuclear norm** hard to compute, but…

Noisy tensor completion with m observations → Refute random 3-SAT with m clauses

LOWER BOUNDS

Not only is the **tensor nuclear norm** hard to compute, but…

Noisy tensor completion with m observations → Refute random 3-SAT with m clauses

Believed to be hard,
If $m = n^{3/2-\delta}$

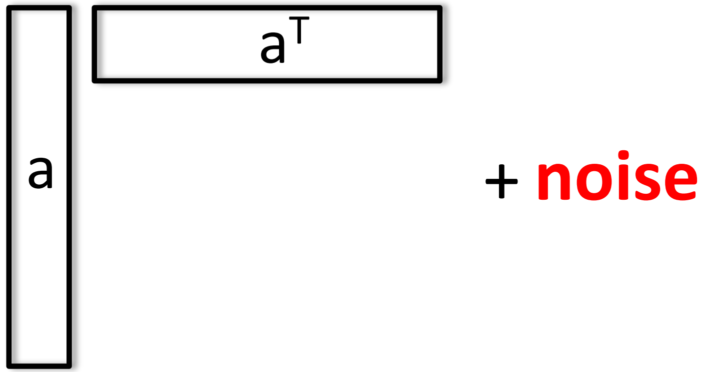
Part IV:

Matrix completion revisited: Connections to random CSPs

Can we distinguish between low-rank and random?

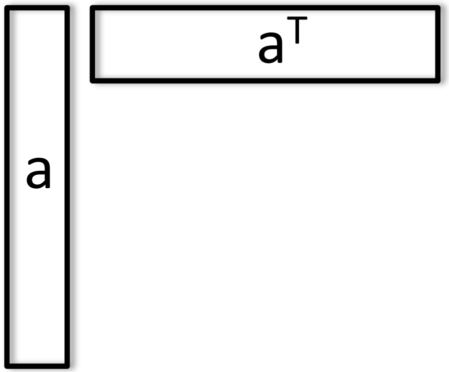
Can we distinguish between low-rank and random?

Case #1: Approximately low-rank



Can we distinguish between low-rank and random?

Case #1: Approximately low-rank



+ **noise**

For each $(i,j) \in \Omega$

$$M_{i,j} = \begin{cases} a_i a_j & \text{w/ probability } 3/4 \\ \text{random } \pm 1 & \text{w/ probability } 1/4 \end{cases}$$

where each $a_i = \pm 1$

Can we distinguish between low-rank and random?

Can we distinguish between low-rank and random?

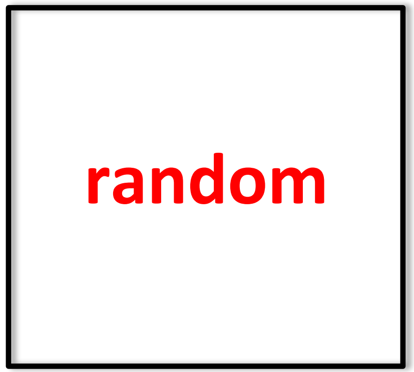
Case #2: Random

random

For each $(i,j) \in \Omega$, $M_{i,j}$ = random ± 1

Can we distinguish between low-rank and random?

Case #2: Random



For each $(i,j) \in \Omega$, $M_{i,j}$ = random ± 1

In **Case #1** the entries are (somewhat) predictable, but in **Case #2** they are completely **unpredictable**

There are two very different communities that (essentially) attacked this same distinguishing problem:

There are two very different communities that (essentially) attacked this same distinguishing problem:

The community working on **matrix completion**

There are two very different communities that (essentially) attacked this same distinguishing problem:

The community working on **matrix completion**

The community working on **refuting random CSPs**

AN INTERPRETATION

We can interpret:

$$(i_1, j_1; \boldsymbol{\sigma_1}), (i_2, j_2; \boldsymbol{\sigma_2}), \ldots, (i_m, j_m; \boldsymbol{\sigma_m})$$

±1 r.v.

as a random 2-XOR formula ψ

AN INTERPRETATION

We can interpret:

$$(i_1, j_1; \boldsymbol{\sigma_1}), (i_2, j_2; \boldsymbol{\sigma_2}), \ldots, (i_m, j_m; \boldsymbol{\sigma_m})$$

±1 r.v.

as a random 2-XOR formula ψ

In particular each observation/fctn value maps to a clause:

$$(i, j, \boldsymbol{\sigma}) \longrightarrow \underbrace{v_i \bullet v_j}_{\text{variables}} = \underbrace{\sigma}_{\text{constraint}}$$

AN INTERPRETATION

We can interpret:

$$(i_1, j_1; \boldsymbol{\sigma_1}), (i_2, j_2; \boldsymbol{\sigma_2}), \ldots, (i_m, j_m; \boldsymbol{\sigma_m})$$

±1 r.v.

as a random 2-XOR formula ψ (and vice-versa)

In particular each observation/fctn value maps to a clause:

$$(i, j, \boldsymbol{\sigma}) \longrightarrow \underbrace{v_i \bullet v_j}_{\text{variables}} = \underbrace{\sigma}_{\text{constraint}}$$

STRONG REFUTATION

We will say that an algorithm **strongly refutes*** random 2-XOR with m clauses if:

STRONG REFUTATION

We will say that an algorithm **strongly refutes*** random 2-XOR with m clauses if:

(1) On any 2-XOR formula ψ, it outputs **val** where:

$$\mathrm{OPT}(\psi) \leq \mathrm{val}(\psi)$$

STRONG REFUTATION

We will say that an algorithm **strongly refutes*** random 2-XOR with m clauses if:

(1) On any 2-XOR formula ψ, it outputs **val** where:

$$\underbrace{\mathrm{OPT}(\psi)}_{\text{largest fraction of clauses of }\psi\text{ that can be satisfied}} \leq \mathrm{val}(\psi)$$

STRONG REFUTATION

We will say that an algorithm **strongly refutes*** random 2-XOR with m clauses if:

(1) On any 2-XOR formula ψ, it outputs **val** where:

$$\mathrm{OPT}(\psi) \le \mathrm{val}(\psi)$$

STRONG REFUTATION

We will say that an algorithm **strongly refutes*** random 2-XOR with m clauses if:

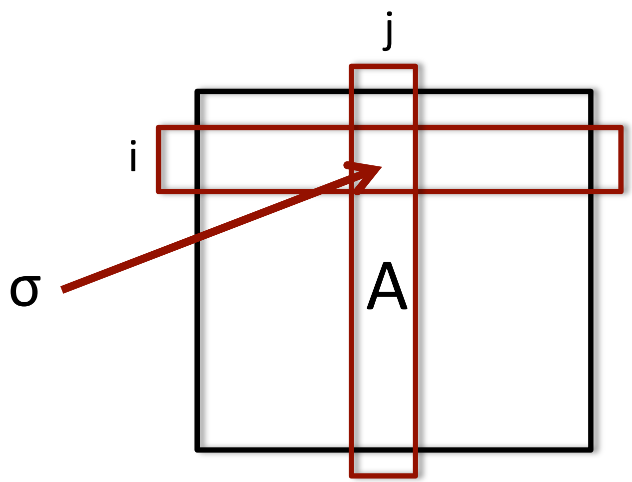
(1) On any 2-XOR formula ψ, it outputs **val** where:

$$\mathrm{OPT}(\psi) \leq \mathrm{val}(\psi)$$

(2) With high probability (for random ψ with m clauses):

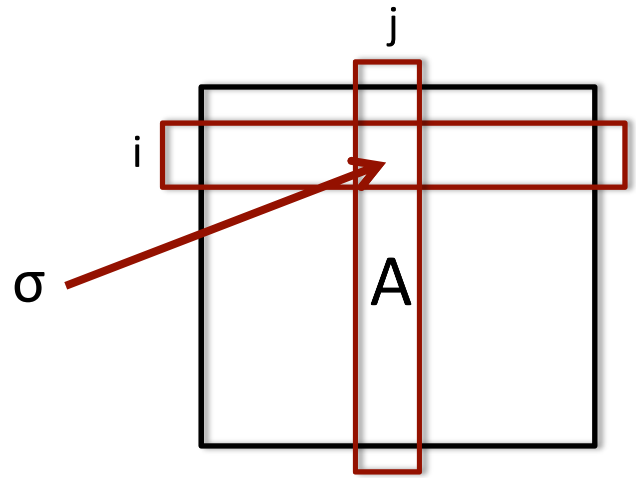
$$\mathrm{val}(\psi) = \frac{1}{2} + o(1)$$

Lemma: If $(i_1, j_1; \boldsymbol{\sigma_1}), \ldots, (i_m, j_m; \boldsymbol{\sigma_m}) \leftrightarrow \psi$ then



$$\frac{2\,\mathrm{OPT}(\psi) - 1}{n} \leq \frac{1}{m} \|A\|_2$$

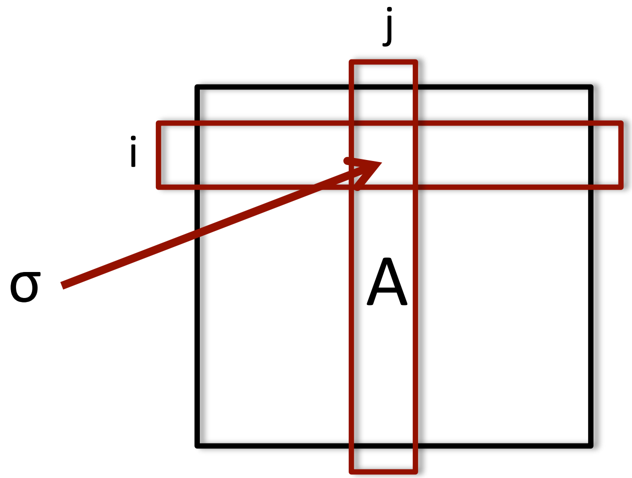
Lemma: If $(i_1, j_1; \boldsymbol{\sigma_1}), \ldots, (i_m, j_m; \boldsymbol{\sigma_m}) \leftrightarrow \psi$ then



$$\frac{2\,\mathrm{OPT}(\psi) - 1}{n} \leq \frac{1}{m}\left\|A\right\|_2$$

Proof: Map the assignment to a unit vector so that $x_i = \pm 1/\sqrt{n}$ and take the quadratic form on A ■

Lemma: If $(i_1, j_1; \boldsymbol{\sigma_1}), \ldots, (i_m, j_m; \boldsymbol{\sigma_m}) \leftrightarrow \psi$ then

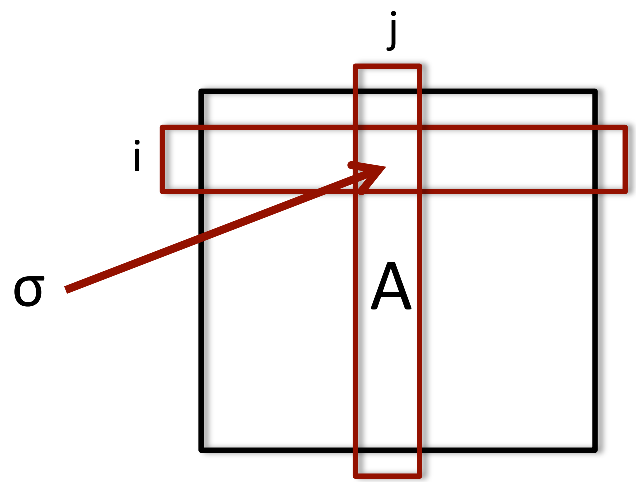


$$\frac{2\,\mathrm{OPT}(\psi) - 1}{n} \leq \frac{1}{m} \|A\|_2$$

Proof: Map the assignment to a unit vector so that $x_i = \pm 1/\sqrt{n}$ and take the quadratic form on A $\blacksquare$

$$\frac{1}{m}\|A\| \sim \sqrt{\frac{1}{mn}}$$

Lemma: If $(i_1, j_1; \boldsymbol{\sigma_1}), \ldots, (i_m, j_m; \boldsymbol{\sigma_m}) \leftrightarrow \psi$ then

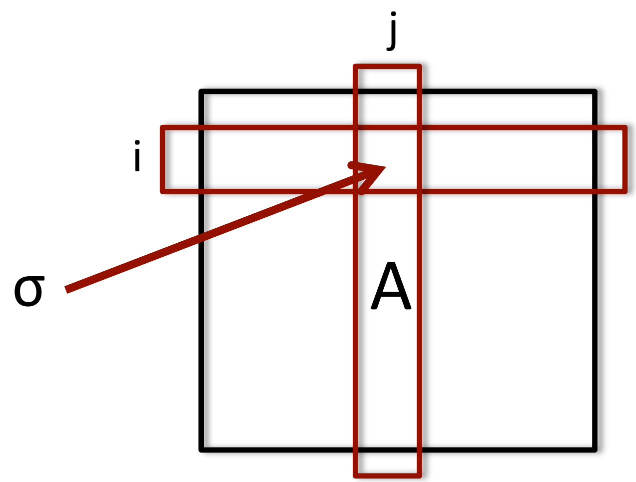


$$\frac{2\,\mathrm{OPT}(\psi) - 1}{n} \leq \frac{1}{m} \|A\|_2$$

Proof: Map the assignment to a unit vector so that $x_i = \pm 1/\sqrt{n}$ and take the quadratic form on A ∎

$$\frac{1}{m}\|A\| \sim \sqrt{\frac{1}{mn}} \xrightarrow{m = \omega(n)} \mathrm{OPT}(\psi) \leq \frac{1}{2} + o(1)$$

Lemma: If $(i_1, j_1; \boldsymbol{\sigma_1}), \ldots, (i_m, j_m; \boldsymbol{\sigma_m}) \leftrightarrow \psi$ then



$$\frac{2\,\mathrm{OPT}(\psi) - 1}{n} \le \frac{1}{m} \|A\|_2$$

Proof: Map the assignment to a unit vector so that $x_i = \pm 1/\sqrt{n}$ and take the quadratic form on A $\blacksquare$

$$\frac{1}{m}\|A\| \sim \sqrt{\frac{1}{mn}} \xrightarrow{m = \omega(n)} \mathrm{OPT}(\psi) \le \frac{1}{2} + o(1)$$

This solves the strong refutation problem…

There are two very different communities that (essentially) attacked this same distinguishing problem:

The community working on **matrix completion**

The community working on **refuting random CSPs**

There are two very different communities that (essentially) attacked this same distinguishing problem:

The community working on **matrix completion**

The community working on **refuting random CSPs**

The **same** spectral bound implies:

(1) An algorithm for strongly refuting random 2-XOR

There are two very different communities that (essentially) attacked this same distinguishing problem:

The community working on **matrix completion**

The community working on **refuting random CSPs**

The **same** spectral bound implies:

(1) An algorithm for strongly refuting random 2-XOR

(2) An algorithm for the distinguishing problem

There are two very different communities that (essentially) attacked this same distinguishing problem:

The community working on **matrix completion**

The community working on **refuting random CSPs**

The **same** spectral bound implies:

(1) An algorithm for strongly refuting random 2-XOR

(2) An algorithm for the distinguishing problem

(3) Generalization bounds for the nuclear norm

It also yields bounds on how well the solution to the convex program generalizes **[Srebro, Shraibman]** …

$$\min \|X\|_* \text{ s.t. } \frac{1}{|\Omega|} \sum_{(i,j)\in\Omega} |X_{i,j} - M_{i,j}| \le \eta$$

It also yields bounds on how well the solution to the convex program generalizes **[Srebro, Shraibman]** …

$$\min \|X\|_* \text{ s.t. } \frac{1}{|\Omega|} \sum_{(i,j)\in\Omega} |X_{i,j} - M_{i,j}| \leq \eta$$

An approach through **statistical learning theory**:

It also yields bounds on how well the solution to the convex program generalizes **[Srebro, Shraibman]** …

$$\min \|X\|_* \text{ s.t. } \frac{1}{|\Omega|} \sum_{(i,j)\in\Omega} |X_{i,j} - M_{i,j}| \leq \eta$$

An approach through **statistical learning theory**:

empirical error: $\frac{1}{|\Omega|} \sum_{(i,j)\in\Omega} |X_{i,j} - M_{i,j}|$

It also yields bounds on how well the solution to the convex program generalizes **[Srebro, Shraibman]** …

$$\min \|X\|_* \text{ s.t. } \frac{1}{|\Omega|} \sum_{(i,j)\in\Omega} |X_{i,j} - M_{i,j}| \leq \eta$$

An approach through **statistical learning theory**:

empirical error: $\frac{1}{|\Omega|} \sum_{(i,j)\in\Omega} |X_{i,j} - M_{i,j}|$ **($\leq \eta$)**

It also yields bounds on how well the solution to the convex program generalizes **[Srebro, Shraibman]** …

$$\min \|X\|_* \text{ s.t. } \frac{1}{|\Omega|} \sum_{(i,j)\in\Omega} |X_{i,j} - M_{i,j}| \leq \eta$$

An approach through **statistical learning theory**:

empirical error: $$\frac{1}{|\Omega|} \sum_{(i,j)\in\Omega} |X_{i,j} - M_{i,j}|$$ **(≤ η)**

prediction error: $$\frac{1}{n^2} \sum |X_{i,j} - M_{i,j}|$$

It also yields bounds on how well the solution to the convex program generalizes **[Srebro, Shraibman]** …

Then if we let

$$\mathcal{K} = \{\, X \text{ s.t. } ||X||_* \leq 1 \} = \text{conv}\{\, ab^T \text{ s.t. } ||a||, ||b|| \leq 1 \}$$

It also yields bounds on how well the solution to the convex program generalizes **[Srebro, Shraibman]** …

Then if we let

$$\mathcal{K} = \{ X \text{ s.t. } ||X||_* \leq 1 \} = \text{conv}\{ ab^T \text{ s.t. } ||a||, ||b|| \leq 1 \}$$

generalization error:

$$\sup_{X \in \mathcal{K}} \left| \underset{(\text{on } \Omega)}{\textbf{empirical error}} (X) - \textbf{prediction error} (X) \right|$$

It also yields bounds on how well the solution to the convex program generalizes **[Srebro, Shraibman]** …

Then if we let

$$\mathcal{K} = \{ X \text{ s.t. } ||X||_* \leq 1 \} = \text{conv}\{ ab^T \text{ s.t. } ||a||, ||b|| \leq 1 \}$$

generalization error:

$$\sup_{X \in \mathcal{K}} \Big| \underset{\text{(on } \Omega)}{\textbf{empirical error}\ (X)} - \textbf{prediction error}\ (X) \Big|$$

Theorem:

"generalization error ≤ best agreement with random function"
(on Ω)

It also yields bounds on how well the solution to the convex program generalizes **[Srebro, Shraibman]** …

Then if we let

$$\mathcal{K} = \{ X \text{ s.t. } ||X||_* \leq 1 \} = \text{conv}\{ ab^T \text{ s.t. } ||a||, ||b|| \leq 1 \}$$

generalization error:

$$\sup_{X \in \mathcal{K}} \left| \underset{(\text{on } \Omega)}{\textbf{empirical error}\ (X)} - \textbf{prediction error}\ (X) \right|$$

Theorem:

$$\text{“generalization error} \leq \underbrace{\underset{(\text{on } \Omega)}{\text{best agreement with random function}}}_{\textbf{Rademacher complexity}}\text{”}$$

It also yields bounds on how well the solution to the convex program generalizes **[Srebro, Shraibman]** …

More precisely:

$$\underbrace{\sup_{X \in \mathcal{K}} \frac{1}{m} \left| \sum_{a=1} \sigma_a X_{i_a, j_a} \right|}_{}$$

Rademacher complexity ($R^m(\mathcal{K})$)

It also yields bounds on how well the solution to the convex program generalizes **[Srebro, Shraibman]** …

More precisely:

$$\underbrace{\sup_{X \in \mathcal{K}} \frac{1}{m} \left| \sum_{a=1} \sigma_a X_{i_a, j_a} \right|}_{\textbf{Rademacher complexity } (R^m(\mathcal{K}))} = \frac{1}{m} || A ||$$

It also yields bounds on how well the solution to the convex program generalizes **[Srebro, Shraibman]** …

More precisely:

$$\underbrace{\sup_{X \in \mathcal{K}} \frac{1}{m} \left| \sum_{a=1}^{m} \sigma_a X_{i_a, j_a} \right|}_{\textbf{Rademacher complexity } (R^m(\mathcal{K}))} = \frac{1}{m} \| A \|$$

$$\frac{1}{m} \| A \| \sim \sqrt{\frac{1}{mn}}$$

It also yields bounds on how well the solution to the convex program generalizes **[Srebro, Shraibman]** …

More precisely:

$$\underbrace{\sup_{X \in \mathcal{K}} \frac{1}{m} \left| \sum_{a=1} \sigma_a X_{i_a, j_a} \right|}_{\textbf{Rademacher complexity } (R^m(\mathcal{K}))} = \frac{1}{m} \| A \|$$

$$\frac{1}{m} \| A \| \sim \sqrt{\frac{1}{mn}} \xrightarrow{m = \omega(n)} R^m(\mathcal{K}) = o\left(\frac{1}{n}\right)$$

There are two very different communities that (essentially) attacked this same distinguishing problem:

The community working on **matrix completion**

The community working on **refuting random CSPs**

Noisy matrix completion with m observations $\xrightarrow{\text{Rademacher Complexity}}$ **Strongly refute*** random 2-XOR/2-SAT with m clauses

*Want an algorithm that certifies a formula is far from satisfiable

There are two very different communities that (essentially) attacked this same distinguishing problem:

The community working on **matrix completion**

The community working on **refuting random CSPs**

Noisy **tensor** completion with m observations $\xrightarrow[\text{Rademacher Complexity}]{}$ **Strongly refute*** random **3-XOR/3-SAT** with m clauses

*Want an algorithm that certifies a formula is far from satisfiable

There are two very different communities that (essentially) attacked this same distinguishing problem:

The community working on **matrix completion**

The community working on **refuting random CSPs**

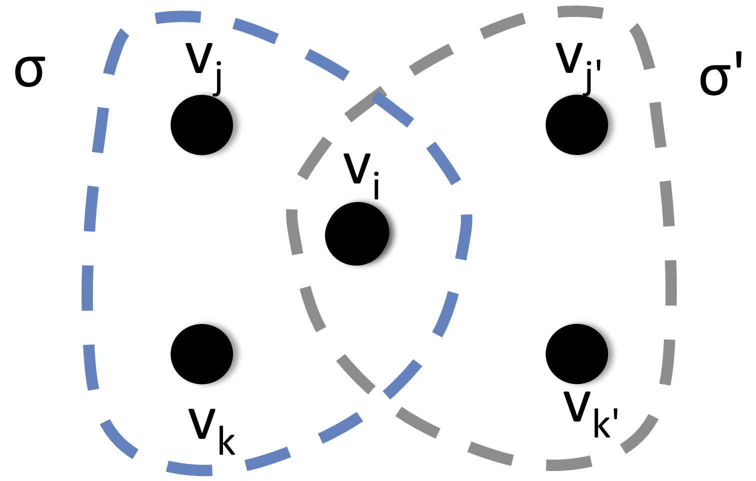
Noisy **tensor** completion with m observations → **Rademacher Complexity** → **Strongly refute*** random **3-XOR/3-SAT** with m clauses

[Coja-Oghlan, Goerdt, Lanka]

*Want an algorithm that certifies a formula is far from satisfiable

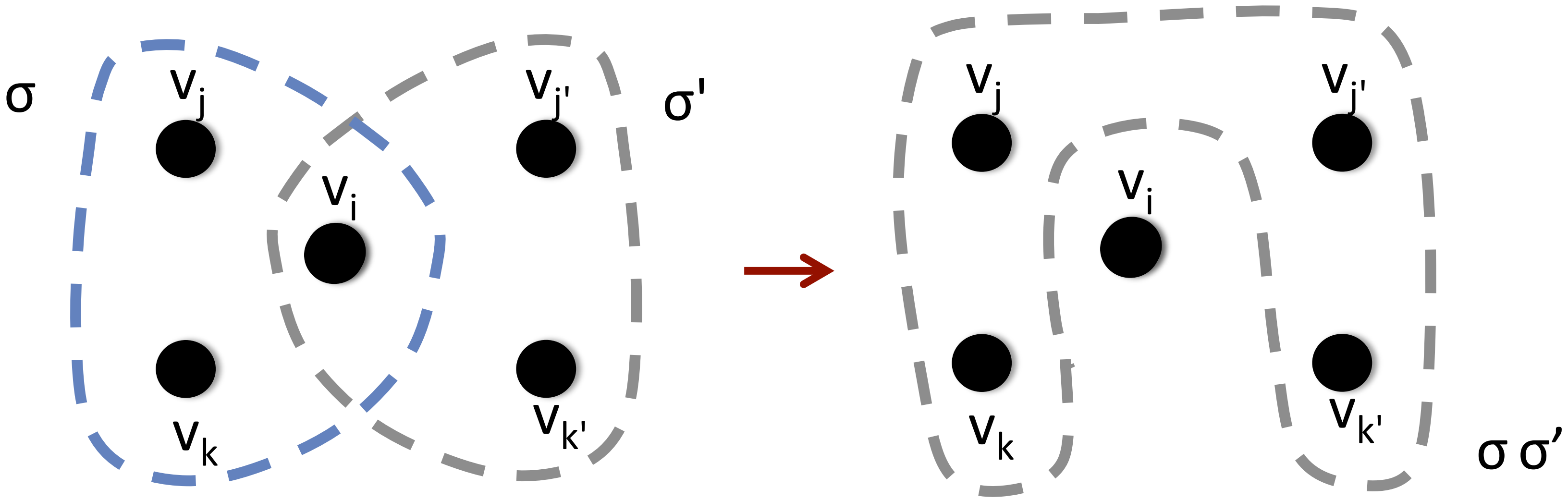
[Coja-Ohghlan, Goerdt, Lanka]: Reduce 3-XOR to 4-XOR

$$\left(v_i \cdot v_j \cdot v_k = \sigma\right) \cdot \left(v_i \cdot v_{j'} \cdot v_{k'} = \sigma'\right)$$



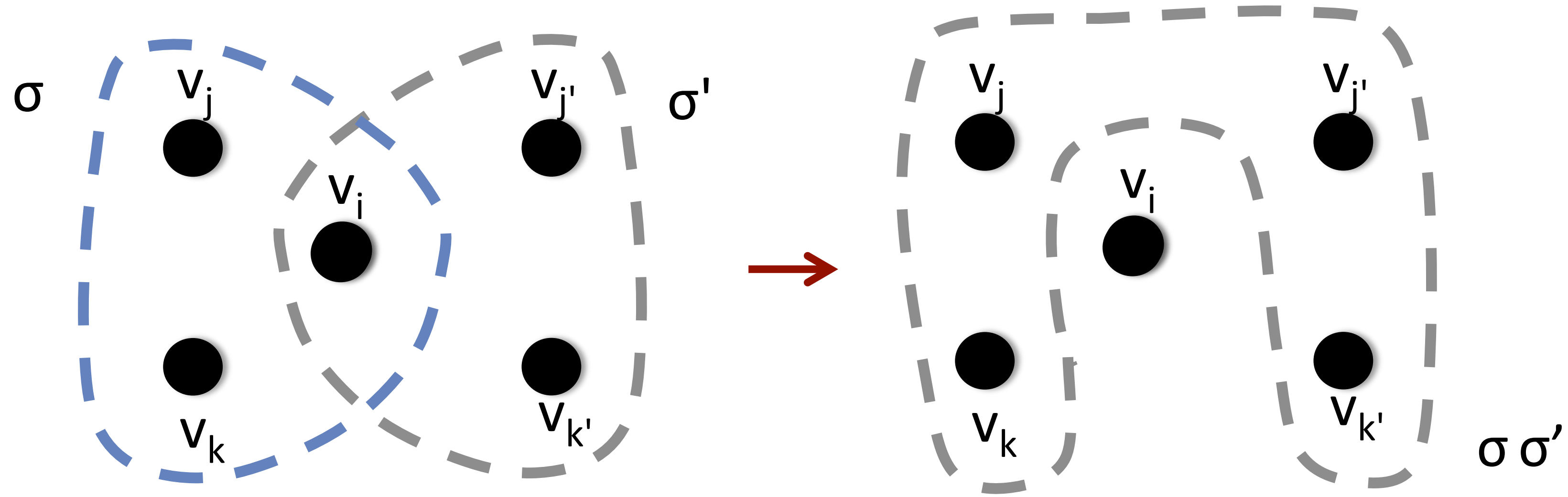
[Coja-Ohglan, Goerdt, Lanka]: Reduce 3-XOR to 4-XOR

$$(v_i \bullet v_j \bullet v_k = \sigma) \bullet (v_i \bullet v_{j'} \bullet v_{k'} = \sigma') \rightarrow (v_j \bullet v_k \bullet v_{j'} \bullet v_{k'} = \sigma\,\sigma')$$



[Coja-Ohglan, Goerdt, Lanka]: Reduce 3-XOR to 4-XOR

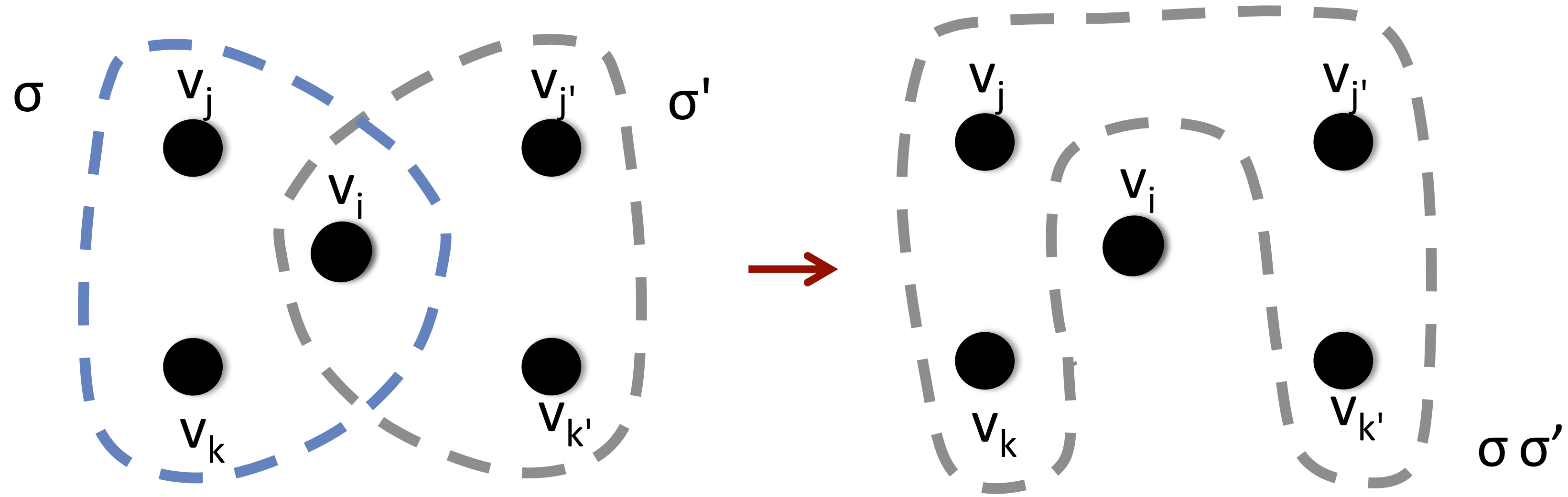
$$(v_i \bullet v_j \bullet v_k = \sigma) \bullet (v_i \bullet v_{j'} \bullet v_{k'} = \sigma') \longrightarrow (v_j \bullet v_k \bullet v_{j'} \bullet v_{k'} = \sigma\sigma')$$



This yields $n^5p^2 = n^2 \log^{O(1)} n$ clauses

[Coja-Ohglan, Goerdt, Lanka]: Reduce 3-XOR to 4-XOR

$$(v_i \bullet v_j \bullet v_k = \sigma) \bullet (v_i \bullet v_{j'} \bullet v_{k'} = \sigma') \longrightarrow (v_j \bullet v_k \bullet v_{j'} \bullet v_{k'} = \sigma\sigma')$$

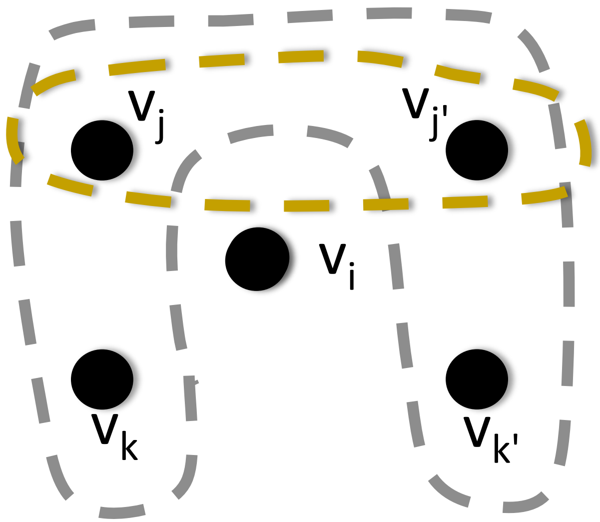


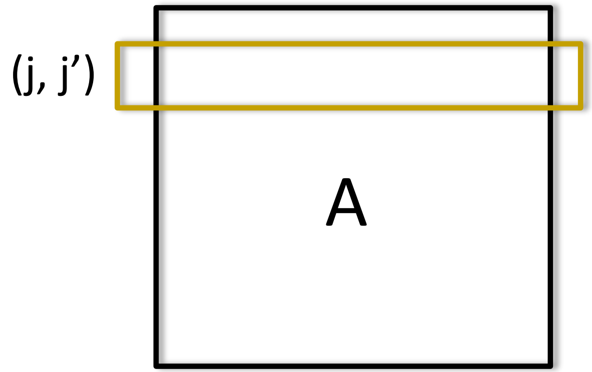
This yields $n^5p^2 = n^2 \log^{O(1)} n$ clauses

Warning: The 4-XOR clauses are not independent!

[Coja-Ohglan, Goerdt, Lanka]: Reduce 3-XOR to 4-XOR

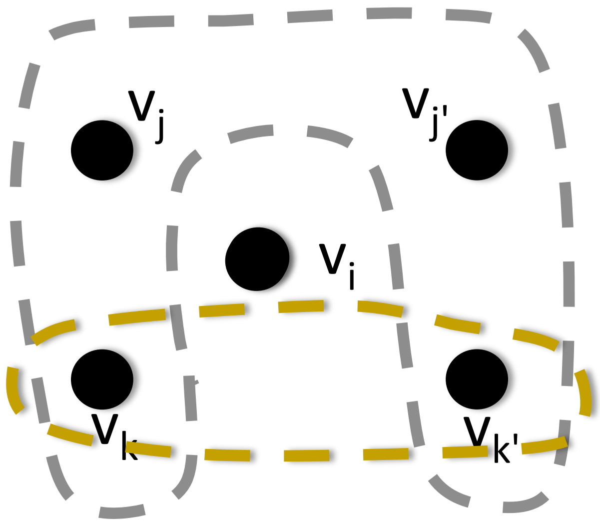
$$(v_i \cdot v_j \cdot v_k = \sigma) \cdot (v_i \cdot v_{j'} \cdot v_{k'} = \sigma') \longrightarrow (v_j \cdot v_k \cdot v_{j'} \cdot v_{k'} = \sigma\sigma')$$

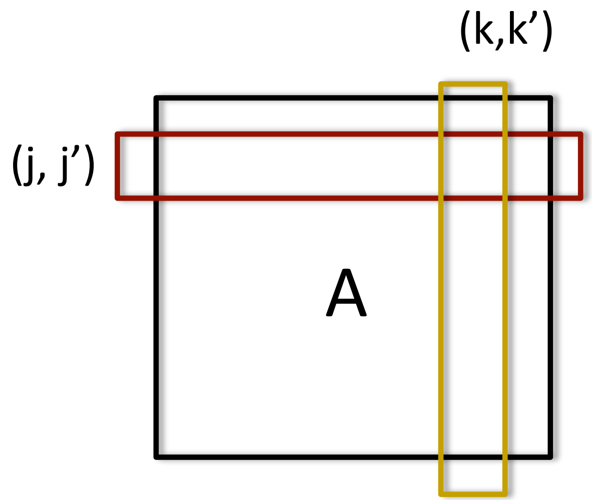




[Coja-Ohglan, Goerdt, Lanka]: Reduce 3-XOR to 4-XOR

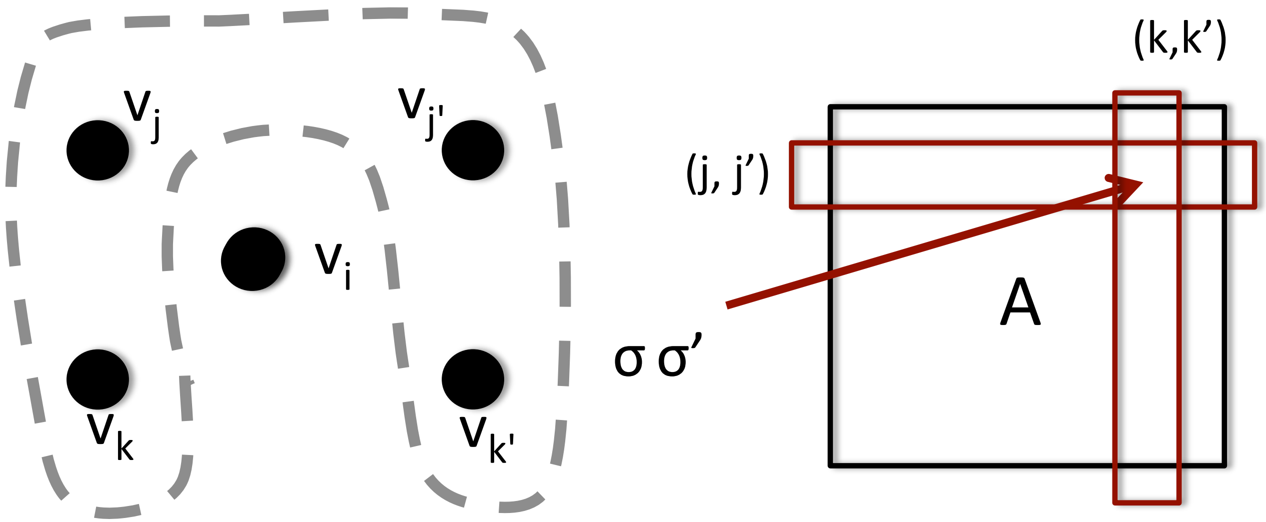
$$(v_i \bullet v_j \bullet v_k = \sigma) \bullet (v_i \bullet v_{j'} \bullet v_{k'} = \sigma') \longrightarrow (v_j \bullet v_k \bullet v_{j'} \bullet v_{k'} = \sigma\sigma')$$





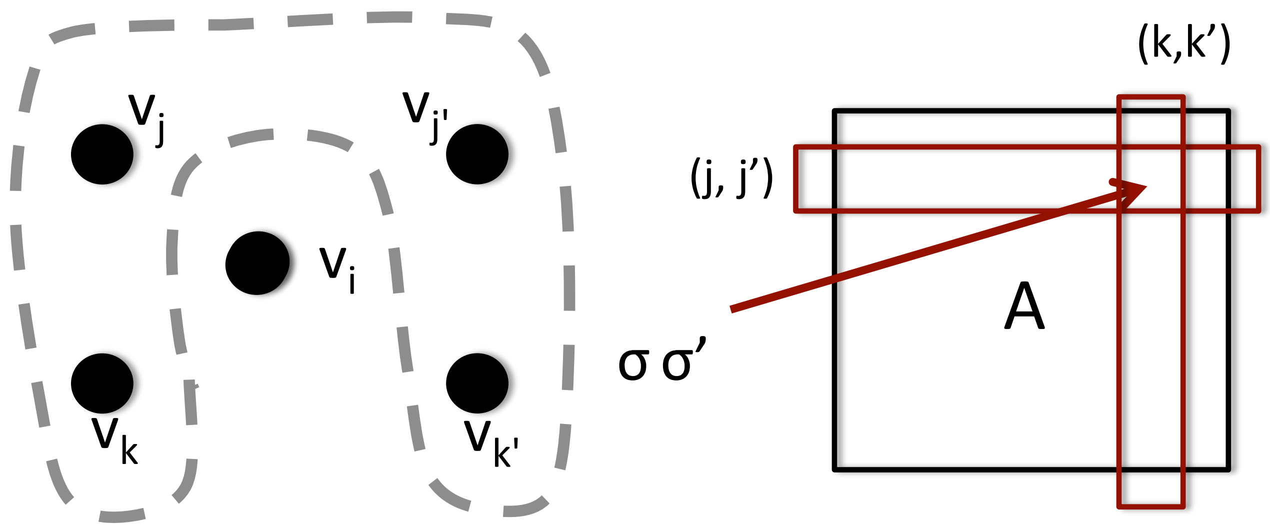
[Coja-Ohglan, Goerdt, Lanka]: Reduce 3-XOR to 4-XOR

$$(v_i \bullet v_j \bullet v_k = \sigma) \bullet (v_i \bullet v_{j'} \bullet v_{k'} = \sigma') \longrightarrow (v_j \bullet v_k \bullet v_{j'} \bullet v_{k'} = \sigma \sigma')$$



[Coja-Ohglan, Goerdt, Lanka]: Reduce 3-XOR to 4-XOR

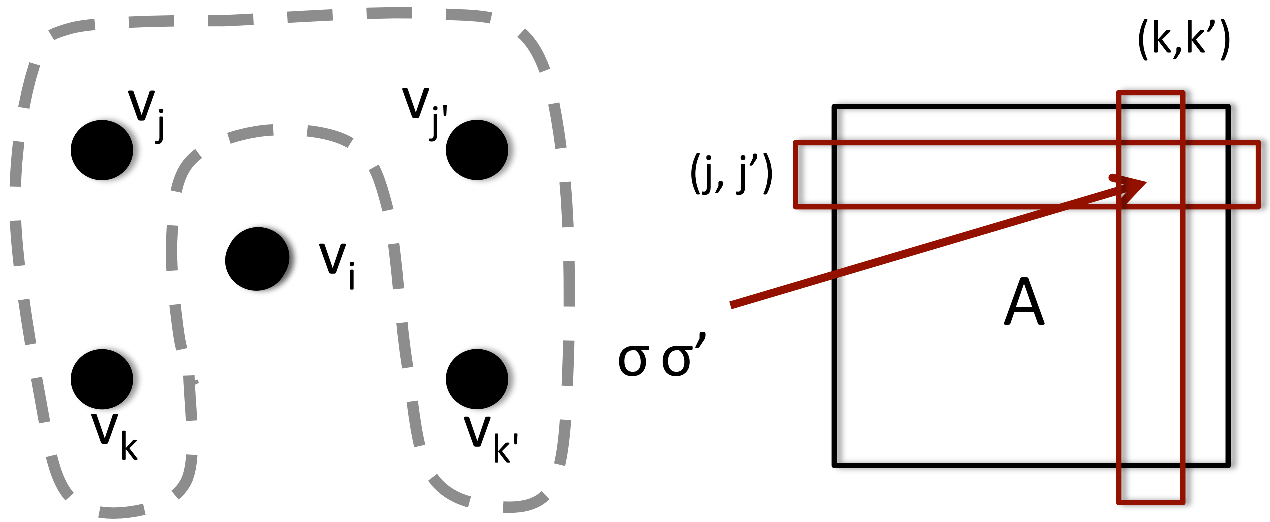
$$(v_i \cdot v_j \cdot v_k = \sigma) \cdot (v_i \cdot v_{j'} \cdot v_{k'} = \sigma') \longrightarrow (v_j \cdot v_k \cdot v_{j'} \cdot v_{k'} = \sigma\sigma')$$



Hence the paired variables for the rows (and colns) come from different clauses!

[Coja-Ohglan, Goerdt, Lanka]: Reduce 3-XOR to 4-XOR

$$(v_i \bullet v_j \bullet v_k = \sigma) \bullet (v_i \bullet v_{j'} \bullet v_{k'} = \sigma') \longrightarrow (v_j \bullet v_k \bullet v_{j'} \bullet v_{k'} = \sigma\sigma')$$



+ trace method

Hence the paired variables for the rows (and colns) come from different clauses!

There are two very different communities that (essentially) attacked this same distinguishing problem:

The community working on **matrix completion**

The community working on **refuting random CSPs**

Noisy **tensor** completion with m observations $\xrightarrow[\text{Complexity}]{\text{Rademacher}}$ **Strongly refute*** random **3-XOR/3-SAT** with m clauses

[Coja-Oghlan, Goerdt, Lanka]

*Want an algorithm that certifies a formula is far from satisfiable

There are two very different communities that (essentially) attacked this same distinguishing problem:

The community working on **matrix completion**

The community working on **refuting random CSPs**

Noisy **tensor** completion with m observations ← **Embedding in SOS** **Strongly refute*** random **3-XOR/3-SAT** with m clauses

[Coja-Oghlan, Goerdt, Lanka]

We then embed this algorithm into the **sixth** level of the sum-of-squares hierarchy, to get a relaxation for tensor prediction

*Want an algorithm that certifies a formula is far from satisfiable

GENERALIZATION BOUNDS

Suppose we are given $|\Omega| = m$ noisy observations $T_{i,j,k} \pm \eta$, and the factors of T are C-incoherent:

Theorem: There is an efficient algorithm that with prob 1-δ, outputs X with

$$\frac{1}{n^3}\sum_{i,j,k} |X_{i,j,k} - T_{i,j,k}| \leq C^3 r\sqrt{\frac{n^{3/2}\log^4 n}{m}} + 2C^3 r\sqrt{\frac{\ln(2/\delta)}{m}} + 2\eta$$

GENERALIZATION BOUNDS

Suppose we are given $|\Omega| = m$ noisy observations $T_{i,j,k} \pm \eta$, and the factors of T are C-incoherent:

Theorem: There is an efficient algorithm that with prob 1-δ, outputs X with

$$\frac{1}{n^3}\sum_{i,j,k} |X_{i,j,k} - T_{i,j,k}| \leq C^3 r\sqrt{\frac{n^{3/2}\log^4 n}{m}} + 2C^3 r\sqrt{\frac{\ln(2/\delta)}{m}} + 2\eta$$

This comes from giving an efficiently computable norm $\|\cdot\|_K$ whose Rademacher complexity is asymptotically smaller than the trivial bound whenever $m=\Omega(n^{3/2}\log^4 n)$

SUMMARY

New Algorithm:

We gave an algorithm for 3rd-order tensor prediction that uses $m = n^{3/2} r \log^4 n$ observations

SUMMARY

New Algorithm:

We gave an algorithm for 3rd-order tensor prediction that uses $m = n^{3/2} r \log^4 n$ observations

---

An Inefficient Algorithm: (via **tensor nuclear norm**)

There is an inefficient algorithm that use $m = nr\log n$ observations

SUMMARY

New Algorithm:

We gave an algorithm for 3^{rd}-order tensor prediction that uses $m = n^{3/2} r \log^4 n$ observations

An Inefficient Algorithm: (via **tensor nuclear norm**)

There is an inefficient algorithm that use $m = nr\log n$ observations

A Phase Transition:

Even for n^{δ} rounds of the powerful sum-of-squares hierarchy, no norm solves tensor prediction with $m = n^{3/2-\delta} r$ observations

Epilogue:

New directions in computational vs. statistical tradeoffs

DISCUSSION

Convex programs are unreasonably effective for linear inverse problems!

DISCUSSION

Convex programs are unreasonably effective for linear inverse problems!

But we gave simple **linear inverse problems** that exhibit striking **gaps** between efficient and inefficient estimators

DISCUSSION

Convex programs are unreasonably effective for linear inverse problems!

But we gave simple **linear inverse problems** that exhibit striking **gaps** between efficient and inefficient estimators

Where else are there computational vs statistical tradeoffs?

DISCUSSION

Convex programs are unreasonably effective for linear inverse problems!

But we gave simple **linear inverse problems** that exhibit striking **gaps** between efficient and inefficient estimators

Where else are there computational vs statistical tradeoffs?

New Direction: Explore computational vs. statistical tradeoffs through the powerful **sum-of-squares** hierarchy