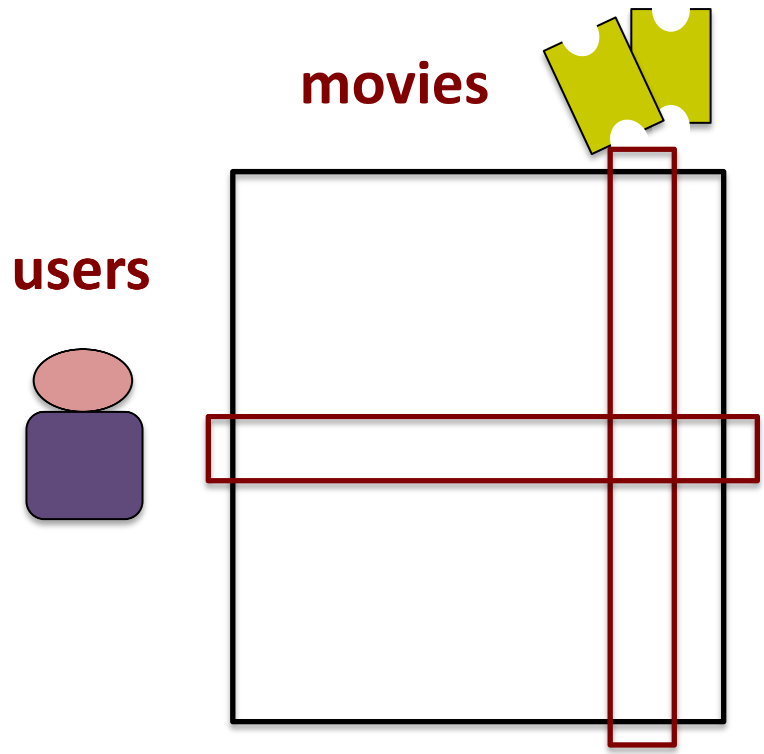
Tensor Completion and Refuting Random CSPs

Ankur Moitra (MIT)

IPAM Tutorial, Part 2

THE NETFLIX PROBLEM



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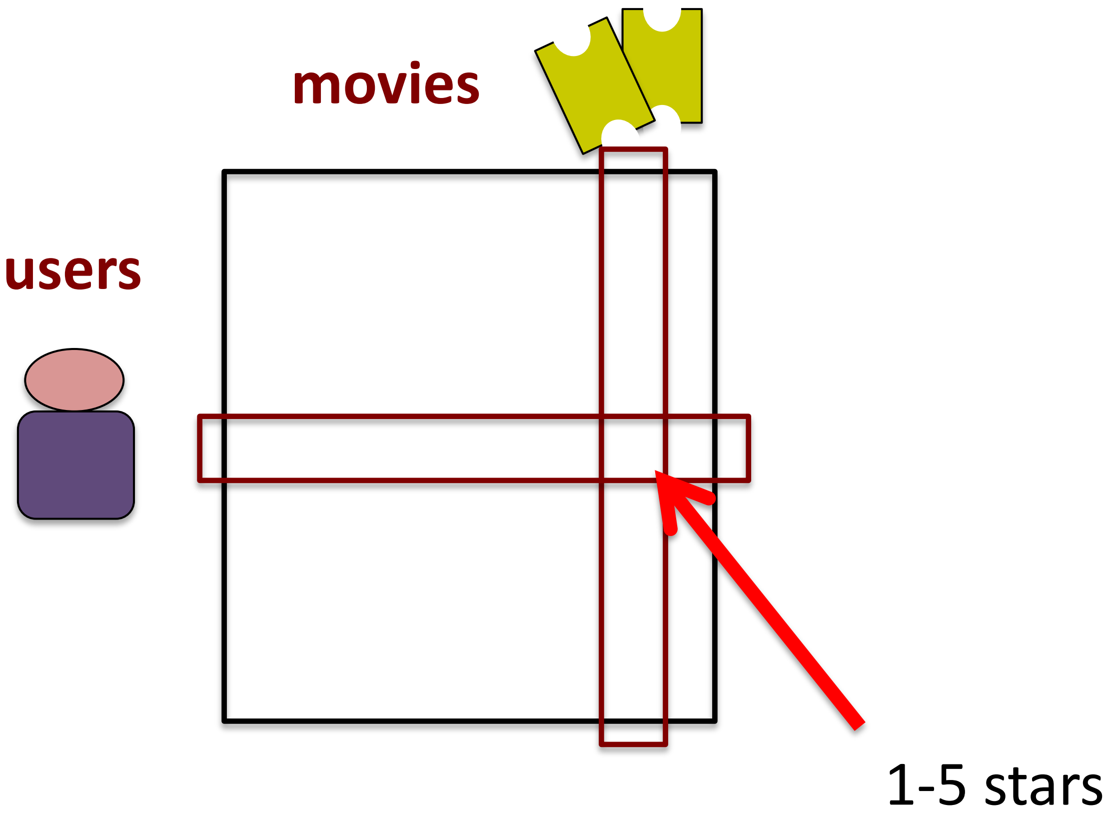

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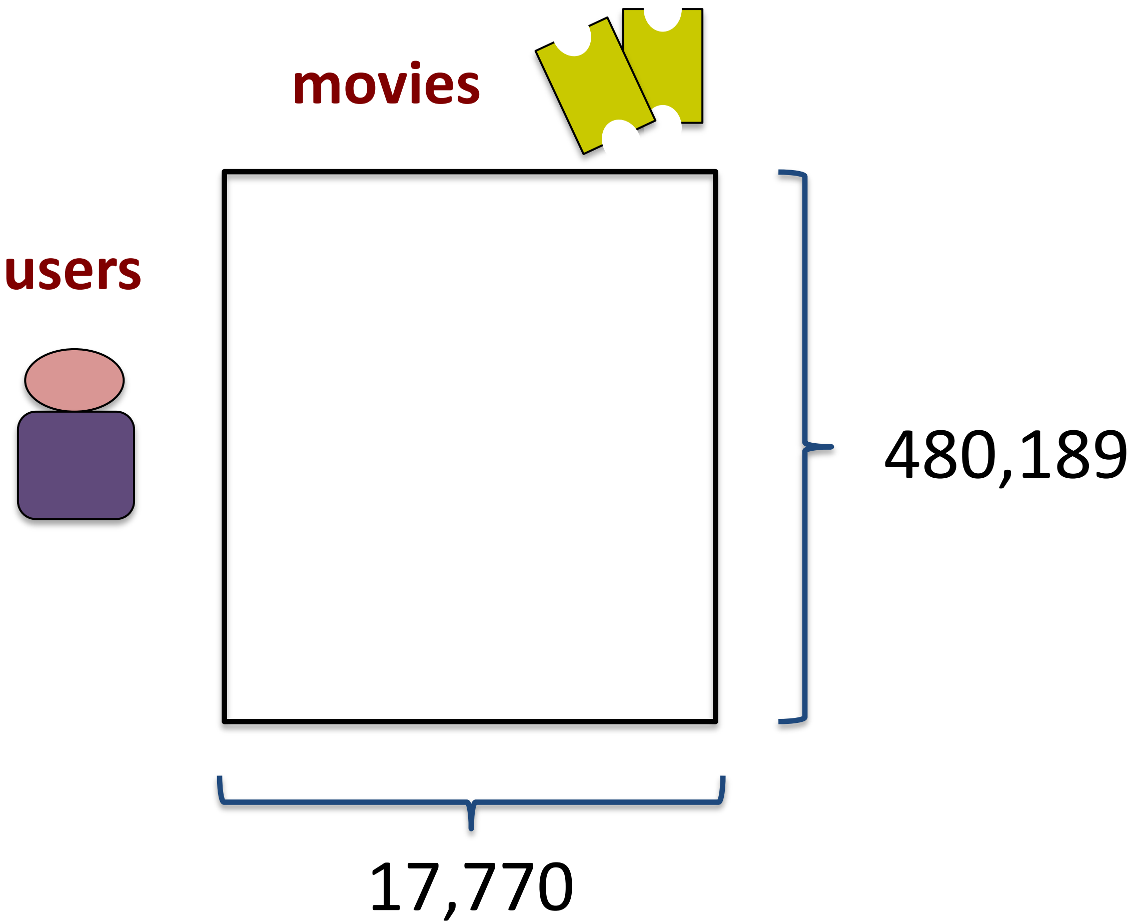

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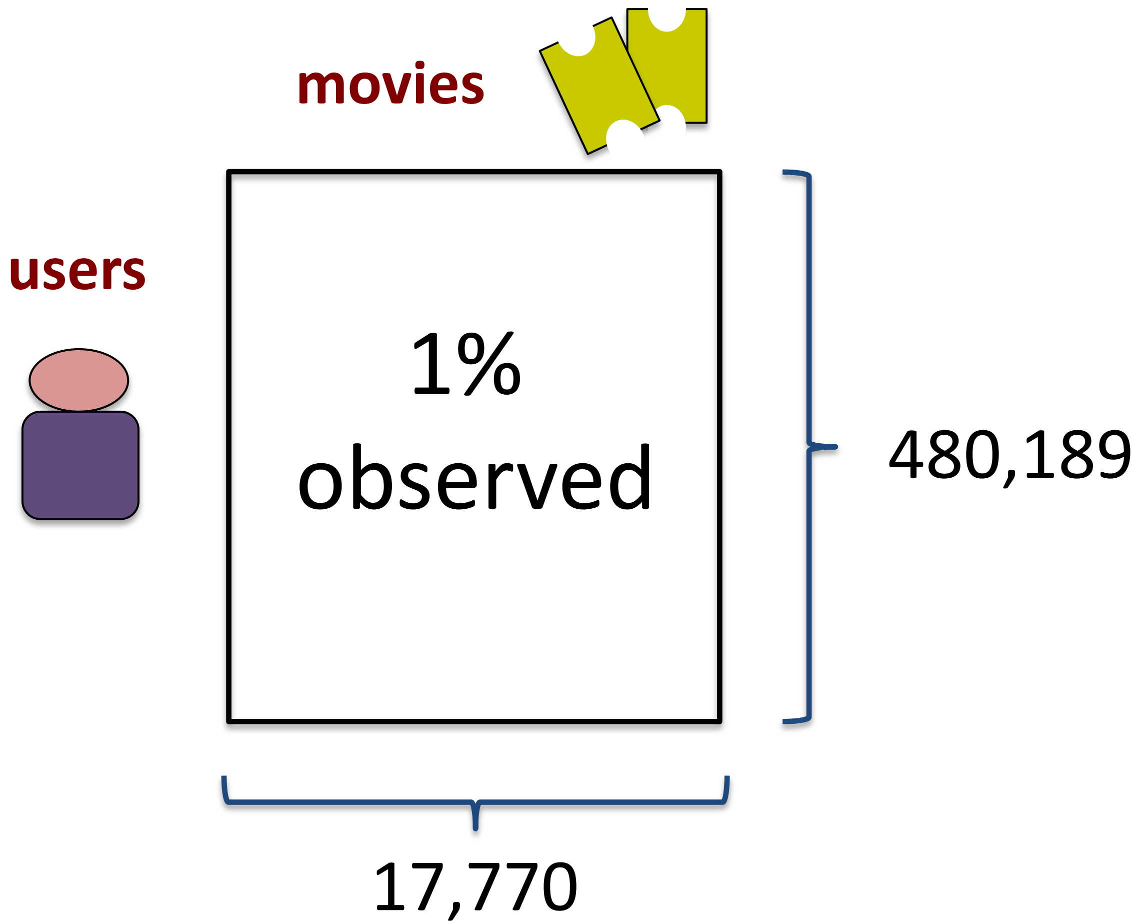

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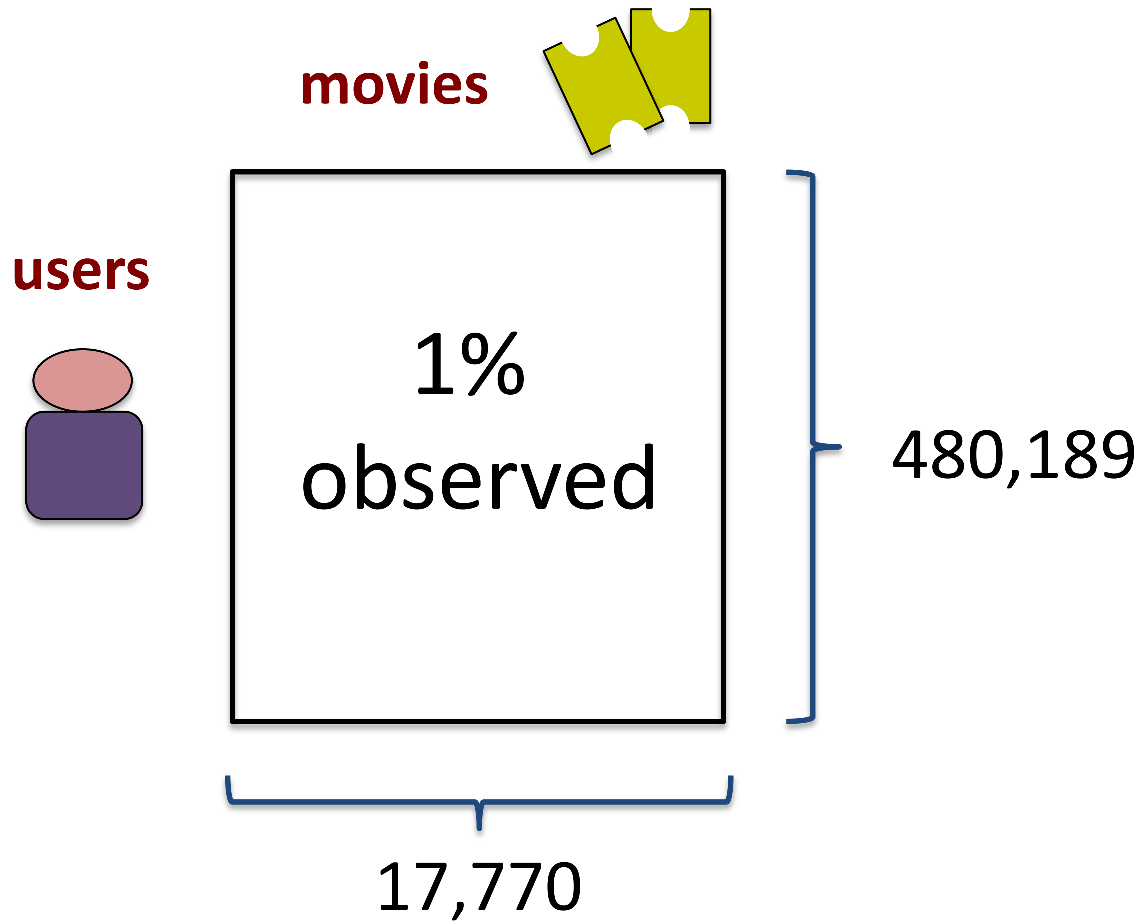

Can we (approximately) fill-in the missing entries?

MATRIX COMPLETION

Let M be an unknown, approximately low-rank matrix

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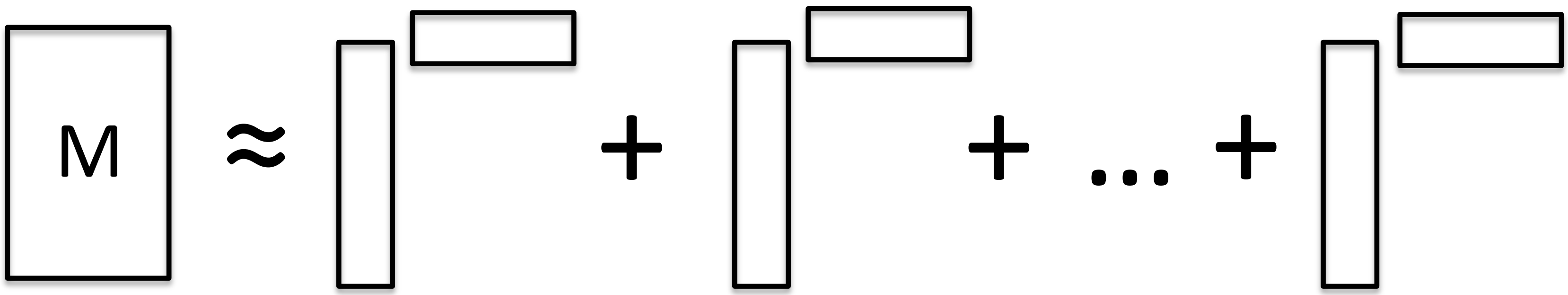

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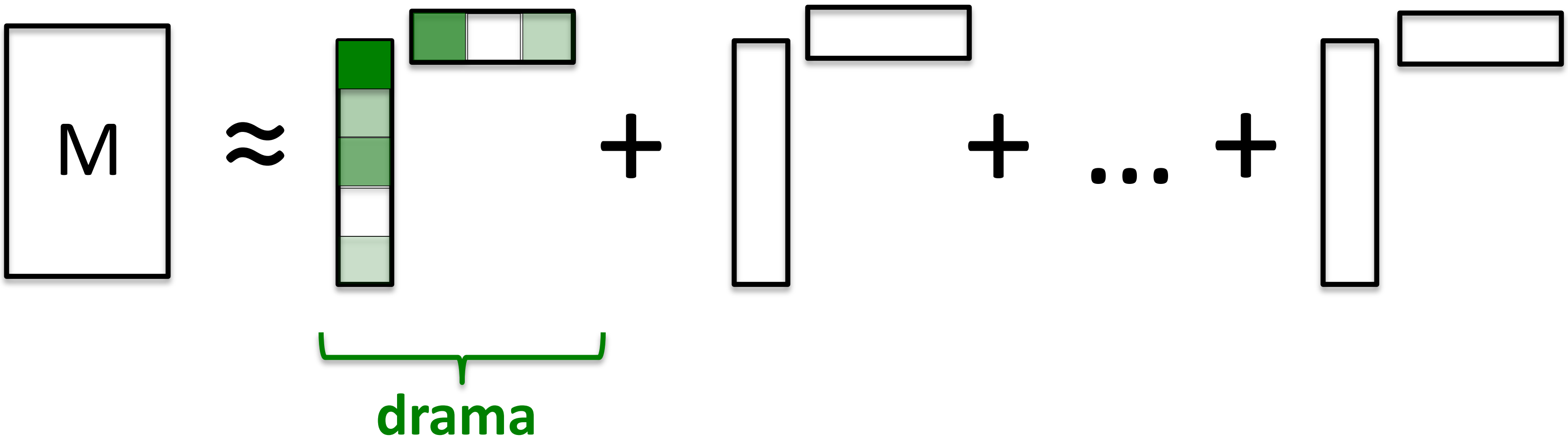

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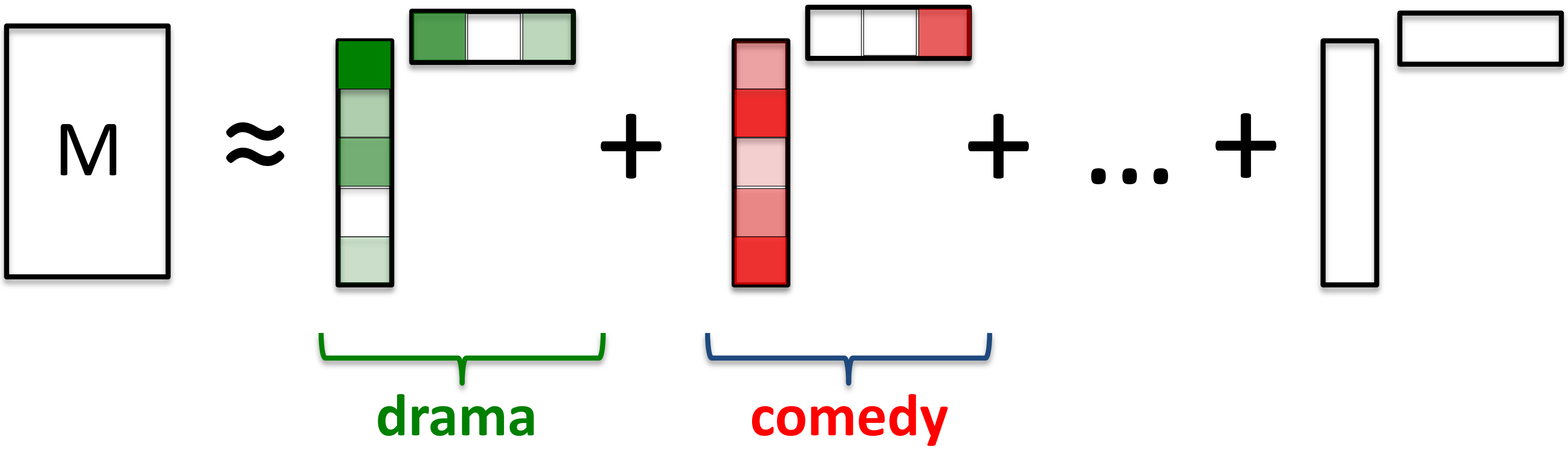

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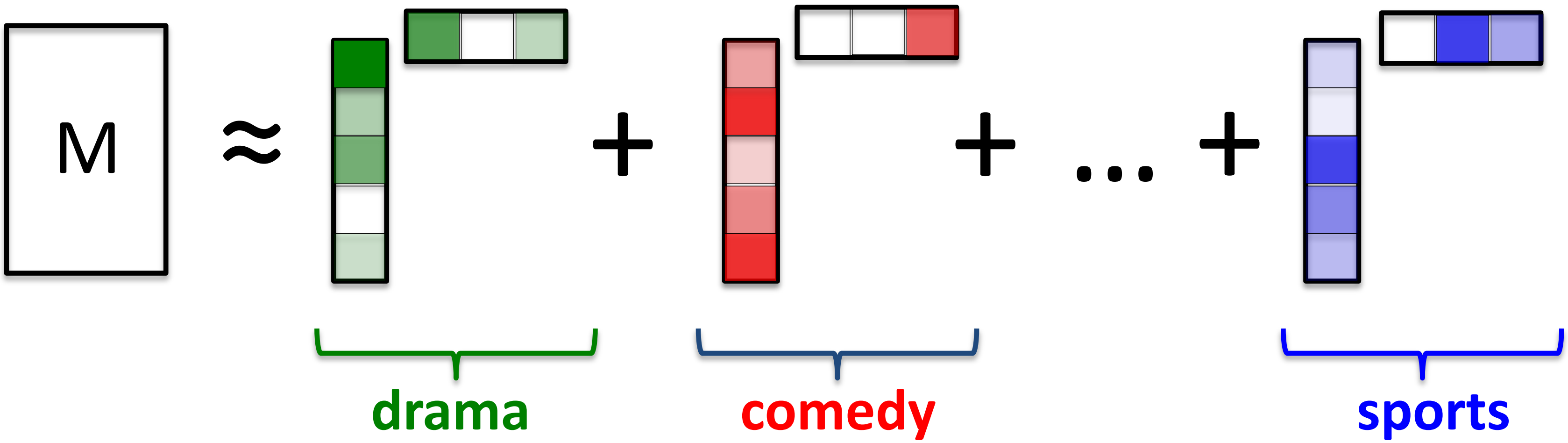

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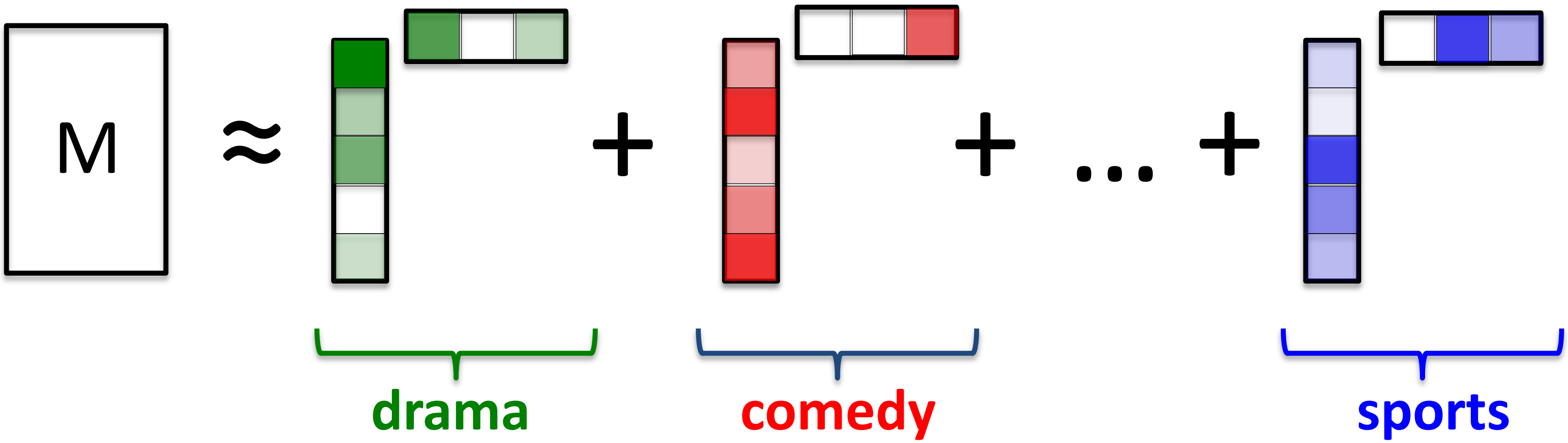

Model: we are given random observations $M_{i,j}$ for all $i,j \in \Omega$

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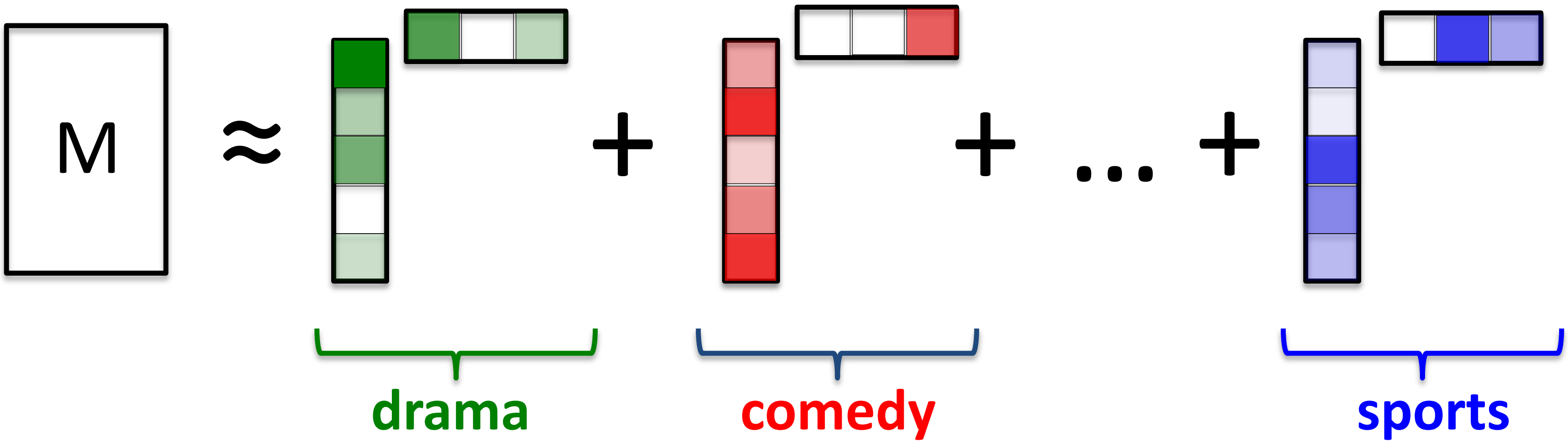

Model: we are given random observations $M_{i,j}$ for all $i,j \in \Omega$

Is there an efficient algorithm to recover M?

MATRIX COMPLETION

The natural formulation is **non-convex**, and **NP-hard**

$$\min \operatorname{rank}(X) \quad \text{s.t.} \quad \frac{1}{|\Omega|} \sum_{(i,j)\in\Omega} |X_{i,j} - M_{i,j}| \le \eta$$

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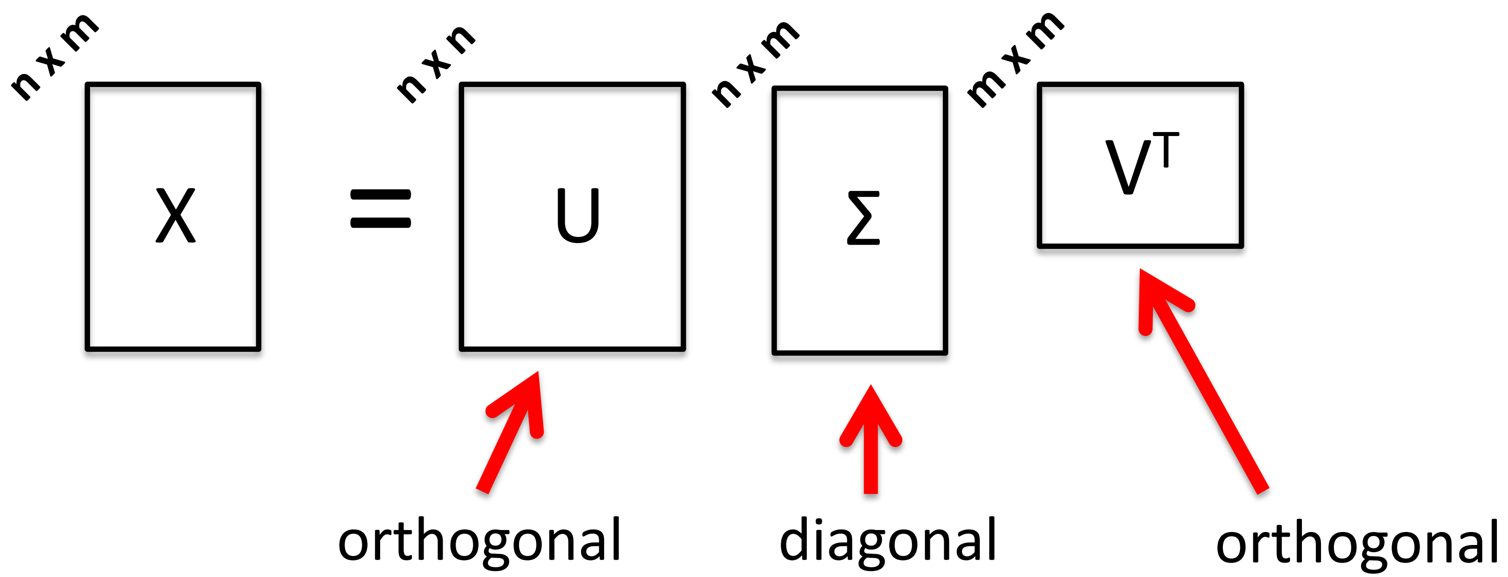
There is a powerful, convex relaxation…

THE NUCLEAR NORM

Consider the **singular value decomposition** of X:



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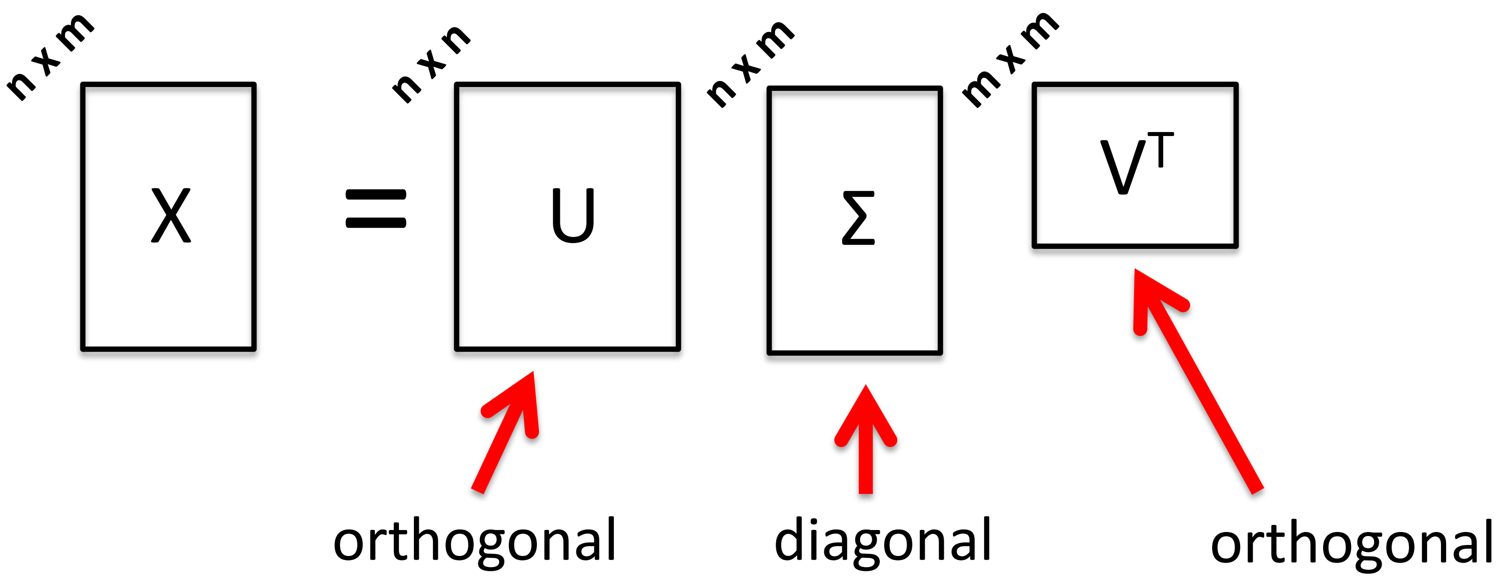

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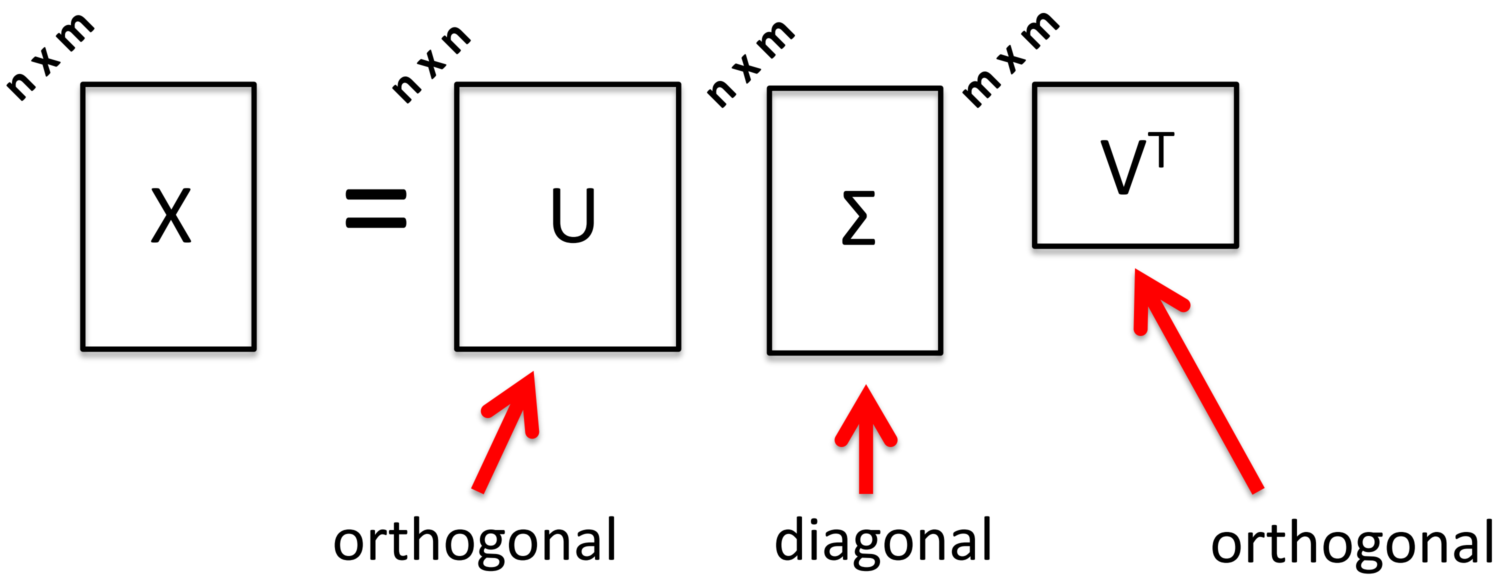

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Then $\mathrm{rank}(X) = r$, and $\|X\|_* = \sigma_1 + \sigma_2 + \ldots + \sigma_r$ (**nuclear norm**)

This yields a convex relaxation, that can be solved efficiently:

$$\min \|X\|_* \text{ s.t. } \frac{1}{|\Omega|} \sum_{(i,j)\in\Omega} |X_{i,j} - M_{i,j}| \leq \eta \qquad \textbf{(P)}$$

[Fazel], [Srebro, Shraibman], [Recht, Fazel, Parrilo], [Candes, Recht], [Candes, Tao], [Candes, Plan], [Recht],

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Theorem: If M is n x n and has rank r, and is C-incoherent then **(P)** recovers M exactly from $C^6 n r \log^2 n$ observations

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This is nearly optimal, since there are O(nr) parameters

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- The Netflix Problem and Matrix Completion
- Tensor Completion

Part II: To Random CSPs and Back

- A Distinguishing Problem
- Random CSPs and Strong Refutation
- Rademacher Complexity

Part III: Resolution

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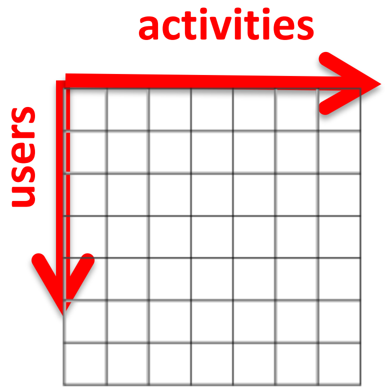

e.g. Groupon

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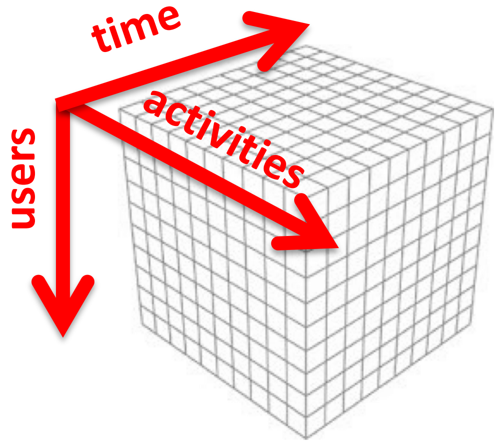

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time: season, time of day, weekday/weekend, etc

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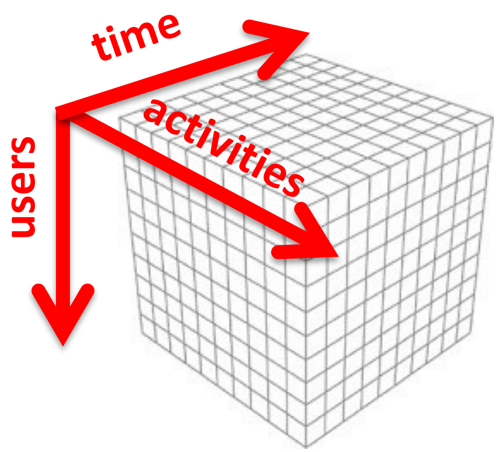

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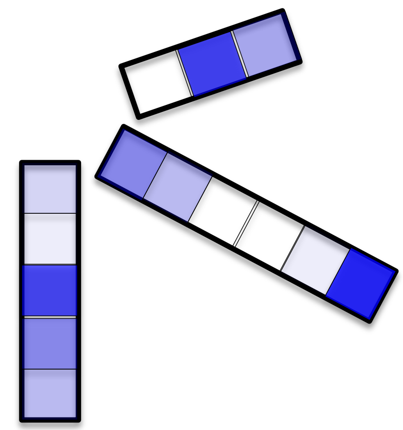
$$T \approx \sum_{i=1}^{r}$$



snow sports

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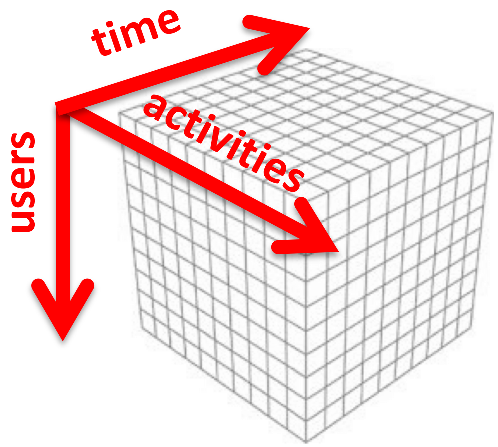

$$T \approx \sum_{i=1}^{r} a_i \otimes b_i \otimes c_i$$

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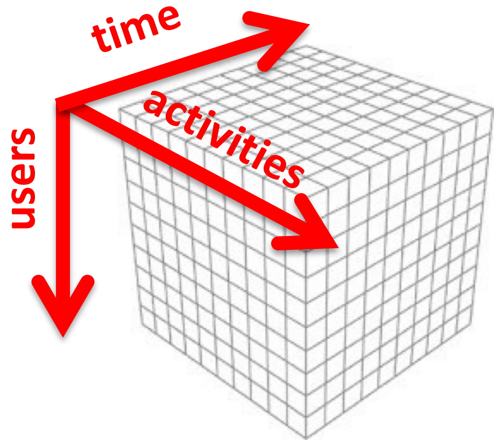

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THE TROUBLE WITH TENSORS

Natural approach (suggested by many authors):

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tensor nuclear norm

The tensor nuclear norm is **NP-hard** to compute!

[Gurvits], [Liu], [Harrow, Montanaro]

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Table I. Tractability of Tensor Problems

Problem	**Complexity**
Bivariate Matrix Functions over $\mathbb{R}$, $\mathbb{C}$	Undecidable (Proposition 12.2)
Bilinear System over $\mathbb{R}$, $\mathbb{C}$	NP-hard (Theorems 2.6, 3.7, 3.8)
Eigenvalue over $\mathbb{R}$	NP-hard (Theorem 1.3)
Approximating Eigenvector over $\mathbb{R}$	NP-hard (Theorem 1.5)
Symmetric Eigenvalue over $\mathbb{R}$	NP-hard (Theorem 9.3)
Approximating Symmetric Eigenvalue over $\mathbb{R}$	NP-hard (Theorem 9.6)
Singular Value over $\mathbb{R}$, $\mathbb{C}$	NP-hard (Theorem 1.7)
Symmetric Singular Value over $\mathbb{R}$	NP-hard (Theorem 10.2)
Approximating Singular Vector over $\mathbb{R}$, $\mathbb{C}$	NP-hard (Theorem 6.3)
Spectral Norm over $\mathbb{R}$	NP-hard (Theorem 1.10)
Symmetric Spectral Norm over $\mathbb{R}$	NP-hard (Theorem 10.2)
Approximating Spectral Norm over $\mathbb{R}$	NP-hard (Theorem 1.11)
Nonnegative Definiteness	NP-hard (Theorem 11.2)
Best Rank-1 Approximation	NP-hard (Theorem 1.13)
Best Symmetric Rank-1 Approximation	NP-hard (Theorem 10.2)
Rank over $\mathbb{R}$ or $\mathbb{C}$	NP-hard (Theorem 8.2)
Enumerating Eigenvectors over $\mathbb{R}$	#P-hard (Corollary 1.16)
Combinatorial Hyperdeterminant	NP-, #P-, VNP-hard (Theorems 4.1 , 4.2, Corollary 4.3)
Geometric Hyperdeterminant	Conjectures 1.9, 13.1
Symmetric Rank	Conjecture 13.2
Bilinear Programming	Conjecture 13.4
Bilinear Least Squares	Conjecture 13.5

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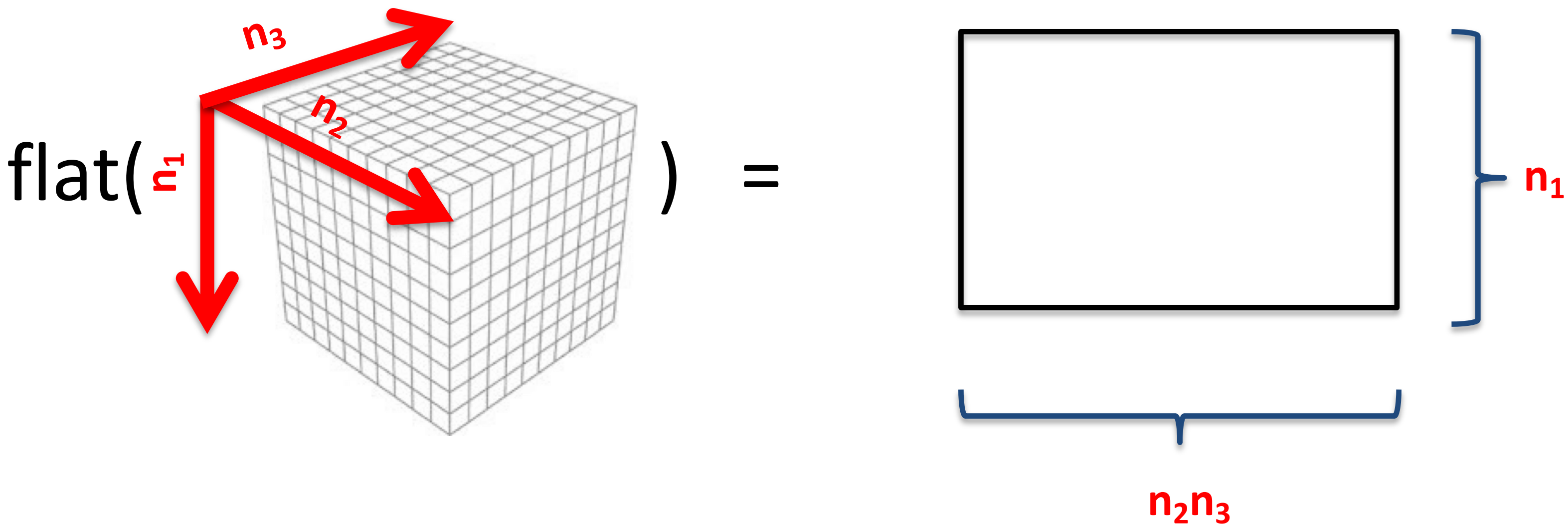

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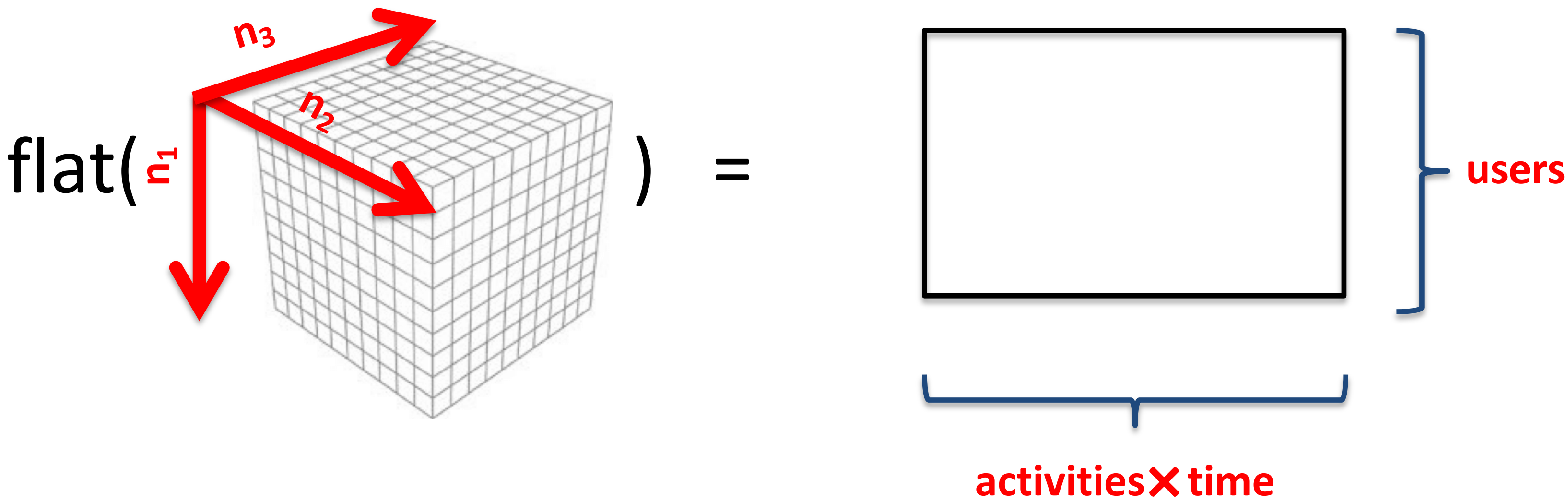

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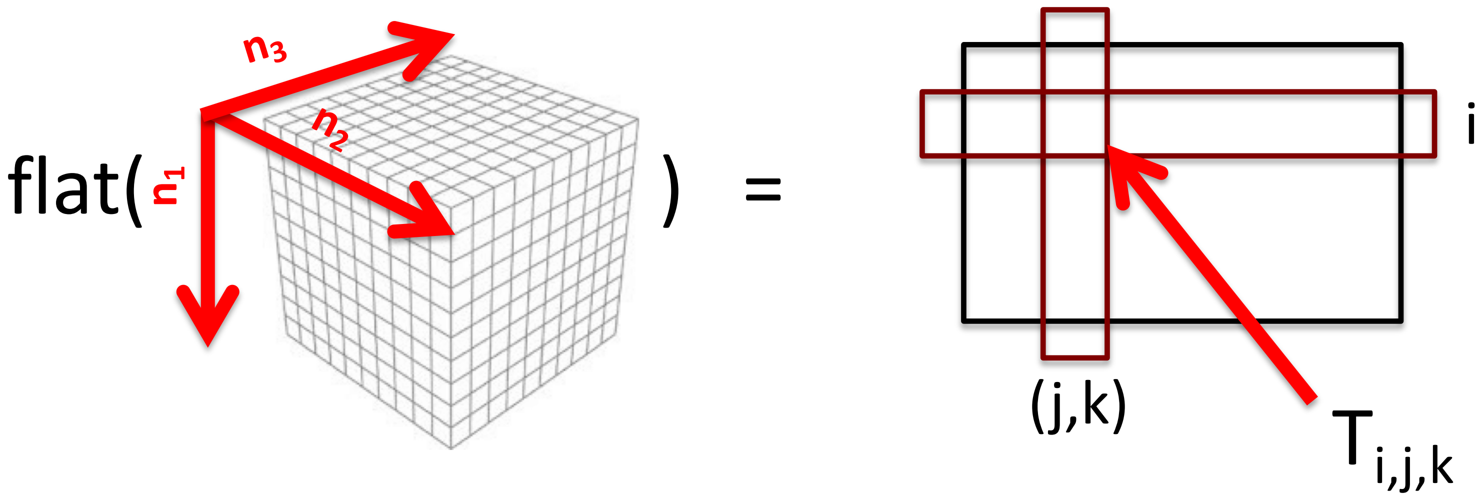

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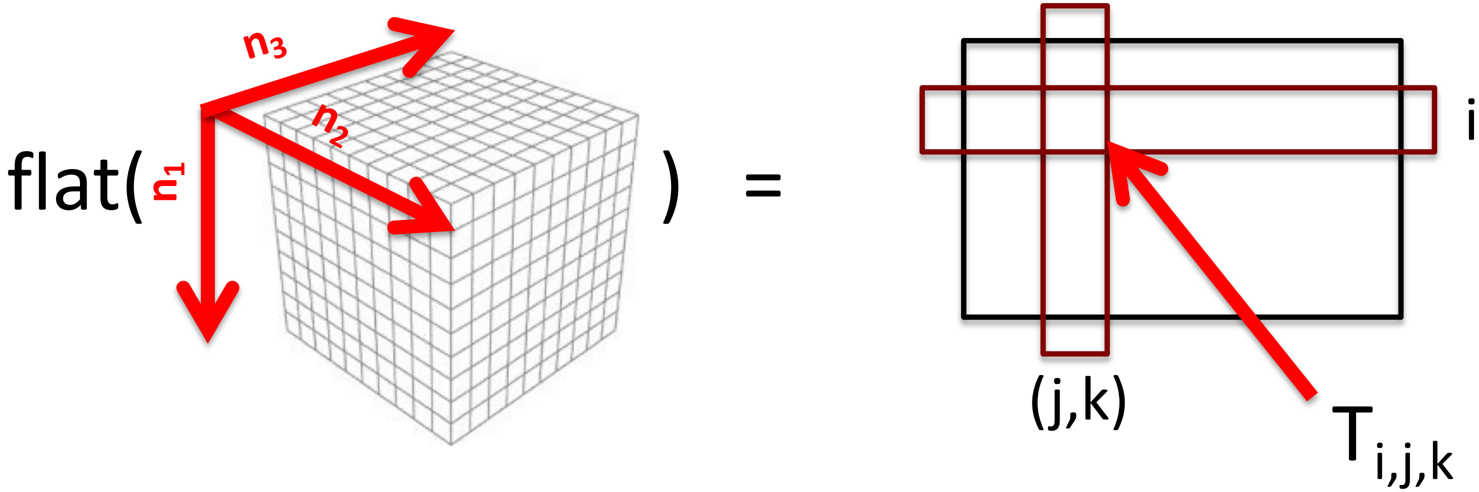

This is a **rearrangement** of the entries, into a matrix, that does not increase its **rank**

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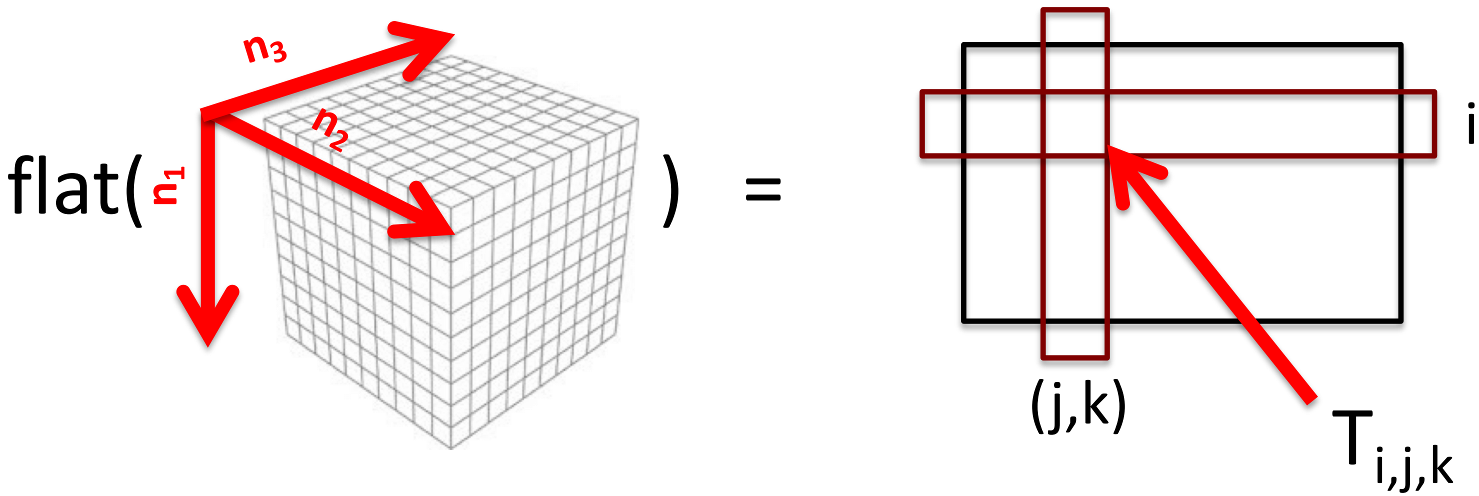

$$\mathrm{flat}\left(\sum_{i=1}^{r} a_i \otimes b_i \otimes c_i\right) = \sum_{i=1}^{r} a_i \otimes \underbrace{\mathrm{vec}(b_i c_i^T)}_{n_2 n_3\text{-dimensional vector}}$$

Let $n_1 = n_2 = n_3 = n$

We would need $\tilde{O}(n^2 r)$ observations to fill-in flat(T)

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There are many other variants of **flattening**, but with comparable guarantees

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Can we beat flattening?

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Can we beat flattening?

Can we make better predictions than we do by treating each **activity x time** as unrelated?

BEATING FLATTENING

Suppose we are given $|\Omega| = m$ noisy observations from T:

$$T = \sum_{i=1}^{r} \sigma_i \, a_i \otimes b_i \otimes c_i + \textbf{noise}$$

with $|\sigma_i|$, $|a_i|_\infty$, $|b_i|_\infty$, $|c_i|_\infty \leq C$

bdd by η

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Theorem [Barak, Moitra]: There is an efficient algorithm that with prob 1-δ, outputs X with

$$\frac{1}{n^3} \sum_{i,j,k} |X_{i,j,k} - T_{i,j,k}| \leq C^3 r \sqrt{\frac{n^{3/2} \log^4 n}{m}} + 2C^3 r \sqrt{\frac{\ln(2/\delta)}{m}} + 2\eta$$

LOWER BOUNDS

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Noisy tensor completion with m observations $\rightarrow$ Refute random 3-SAT with m clauses

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Noisy tensor completion with m observations → Refute random 3-SAT with m clauses

Believed to be hard,
If $m = n^{3/2-\delta}$

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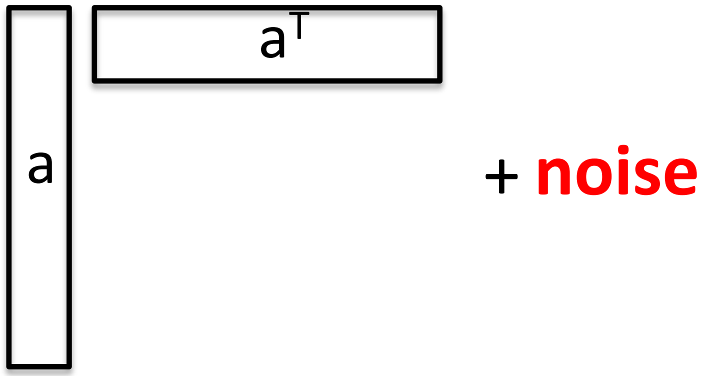
Case #1: Approximately low-rank



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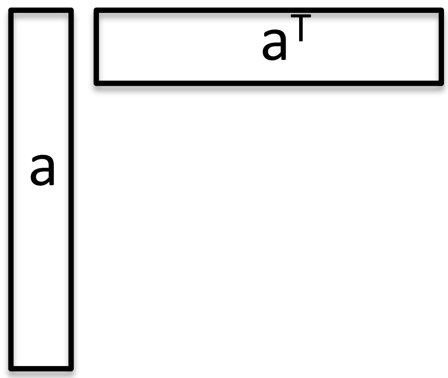

+ **noise**

For each $(i,j) \in \Omega$

$$M_{i,j} = \begin{cases} a_i a_j & \text{w/ probability } 3/4 \\ \text{random } \pm 1 & \text{w/ probability } 1/4 \end{cases}$$

where each $a_i = \pm 1$

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random

For each $(i,j) \in \Omega$, $M_{i,j}$ = random ± 1

In **Case #1** the entries are (somewhat) predictable, but in **Case #2** they are completely **unpredictable**

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The community working on **refuting random CSPs**

AN INTERPRETATION

We can interpret:

$$(i_1, j_1; \boldsymbol{\sigma_1}), (i_2, j_2; \boldsymbol{\sigma_2}), \ldots, (i_m, j_m; \boldsymbol{\sigma_m})$$

±1 r.v.

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In particular each observation/fctn value maps to a clause:

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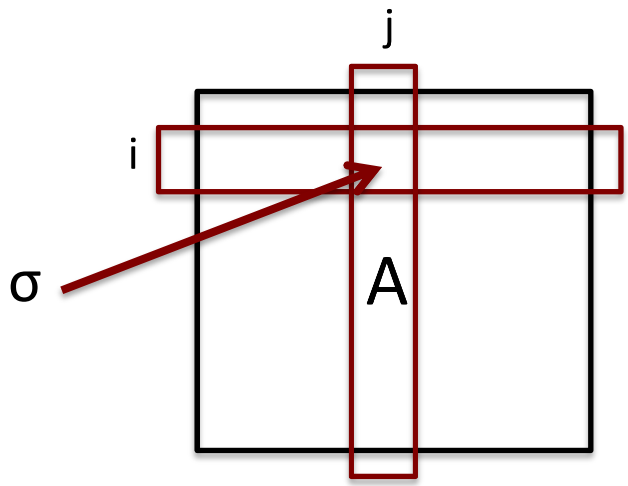
$$\mathrm{OPT}(\psi) \leq \mathrm{val}(\psi)$$

(2) With high probability (for random ψ with m clauses):

$$\mathrm{val}(\psi) = \frac{1}{2} + o(1)$$

Lemma: If $(i_1, j_1; \boldsymbol{\sigma_1}), \ldots, (i_m, j_m; \boldsymbol{\sigma_m}) \leftrightarrow \psi$ then



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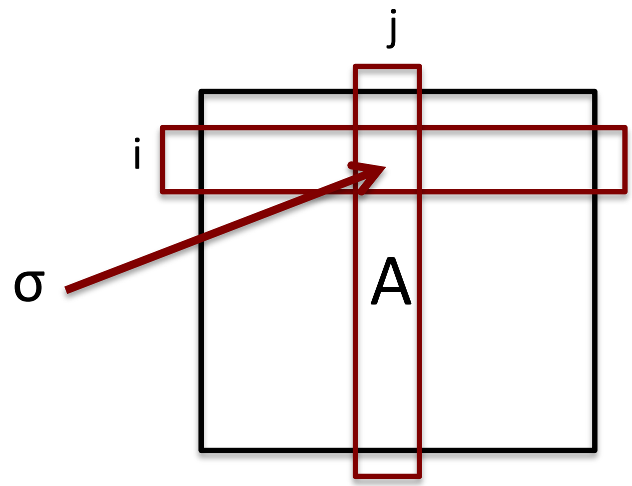

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Proof: Map the assignment to a unit vector so that $x_i = \pm 1/\sqrt{n}$ and take the quadratic form on A ■

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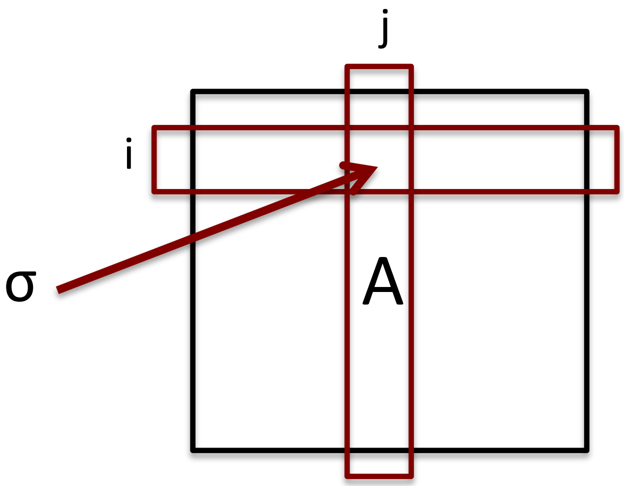

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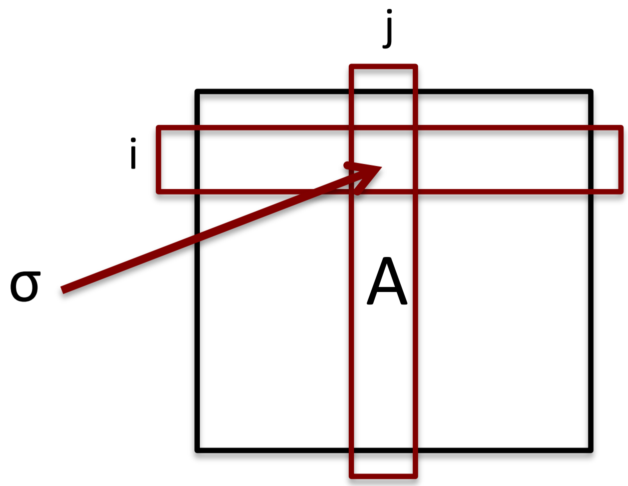

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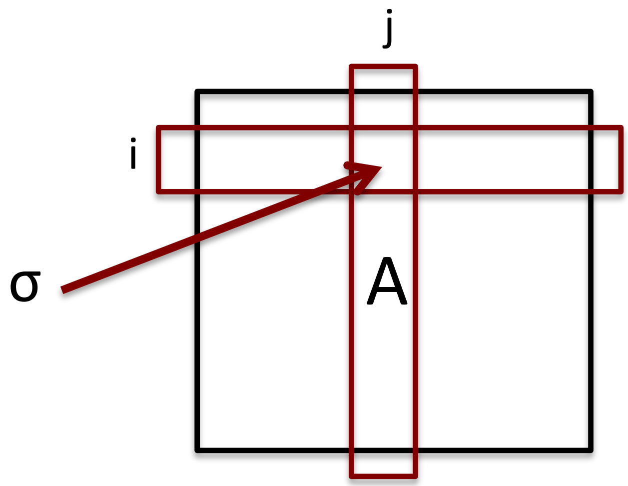

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This solves the strong refutation problem…

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Part III: Resolution

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It also yields bounds on how well the solution to the convex program generalizes **[Srebro, Shraibman]** …

$$\min \|X\|_* \text{ s.t. } \frac{1}{|\Omega|} \sum_{(i,j)\in\Omega} |X_{i,j} - M_{i,j}| \leq \eta$$

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It also yields bounds on how well the solution to the convex program generalizes **[Srebro, Shraibman]** …

More precisely:

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[Coja-Ohglan, Goerdt, Lanka]: Reduce 3-XOR to 4-XOR

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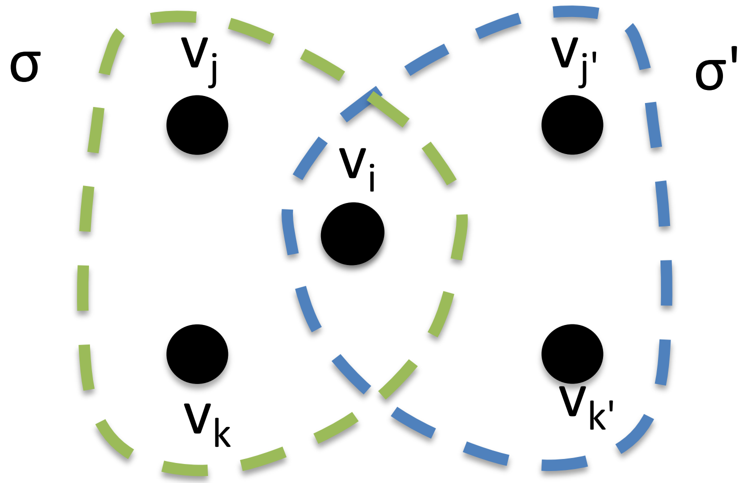

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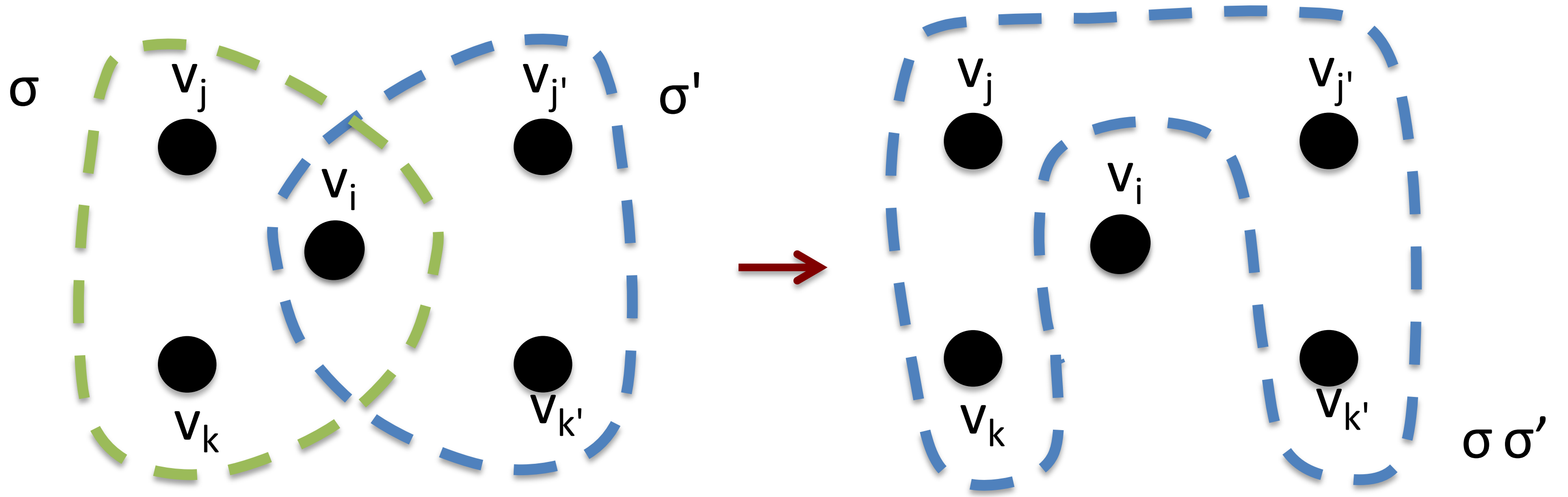

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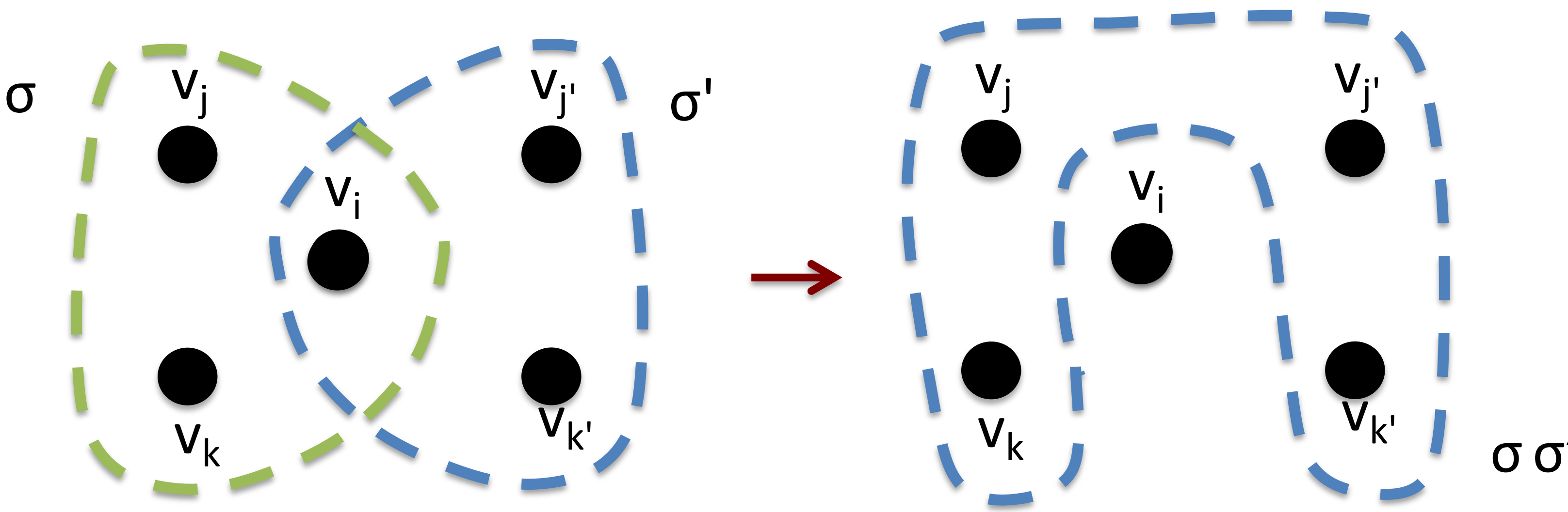

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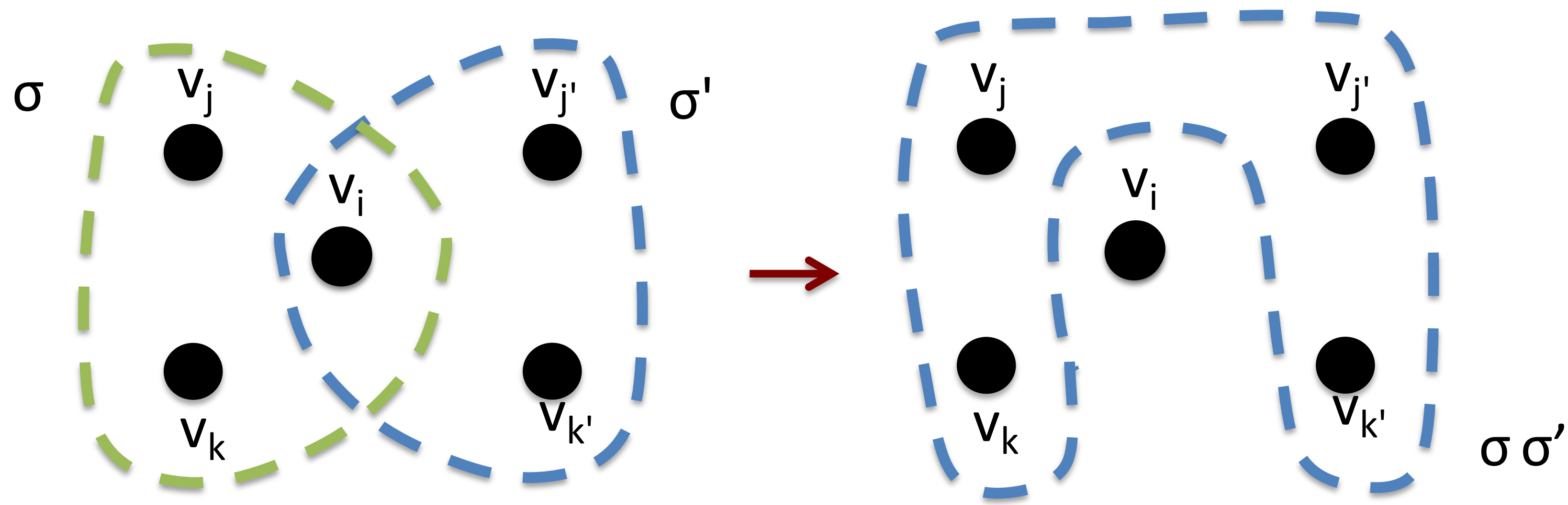

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Warning: The 4-XOR clauses are not independent!

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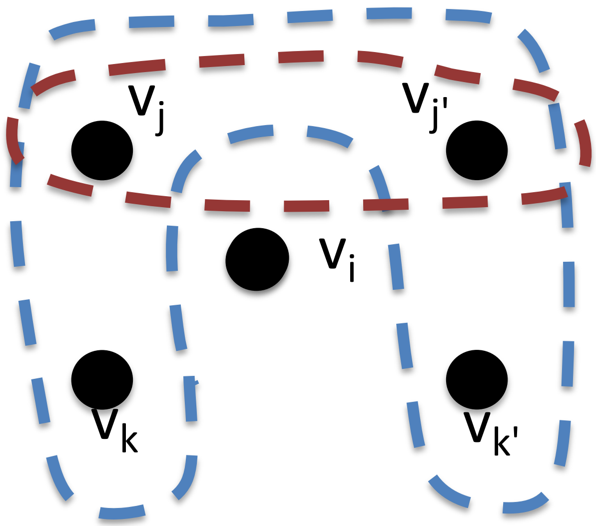

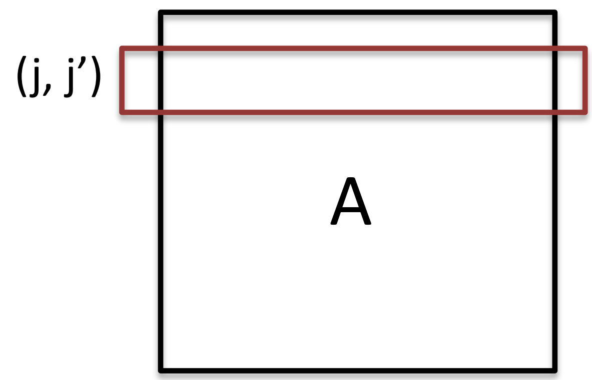

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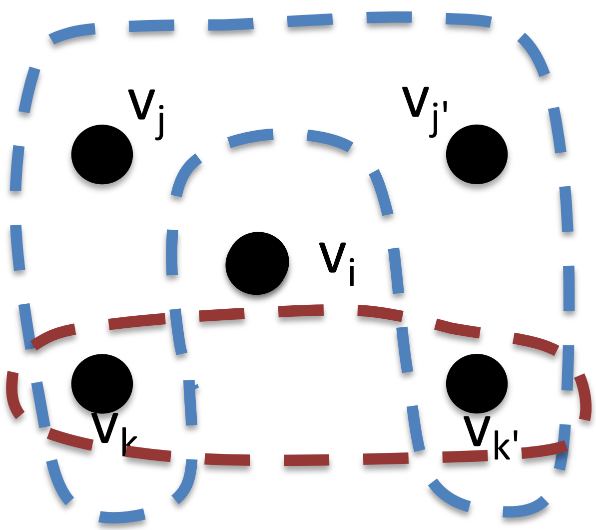

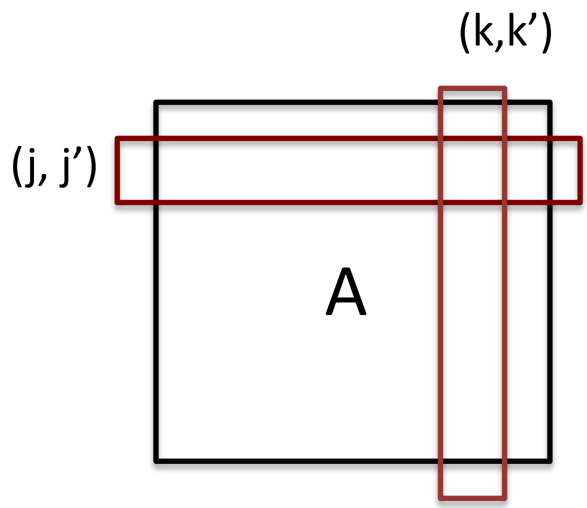

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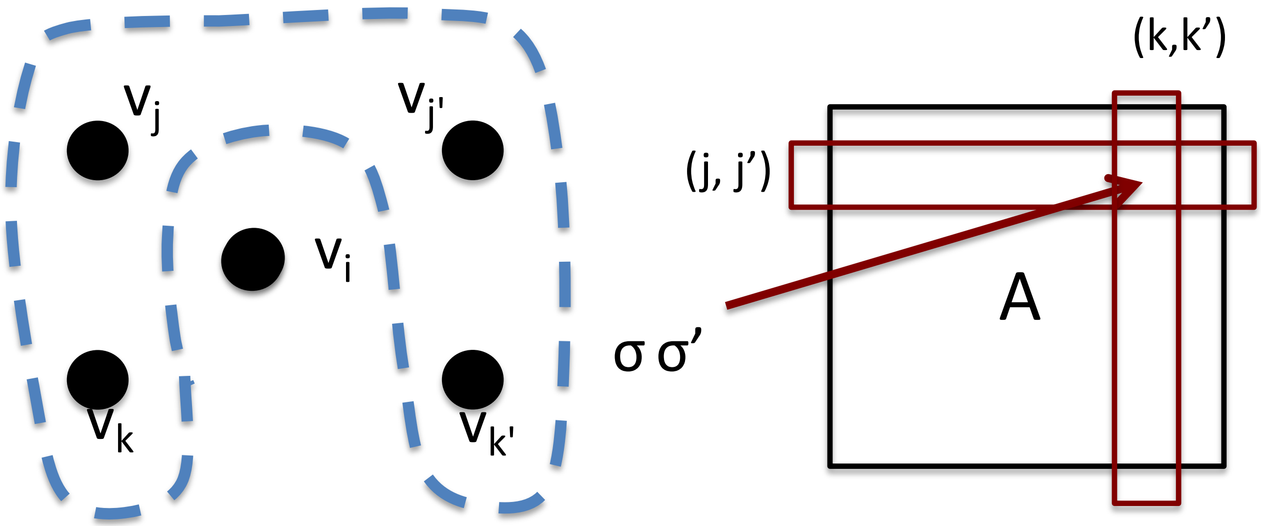

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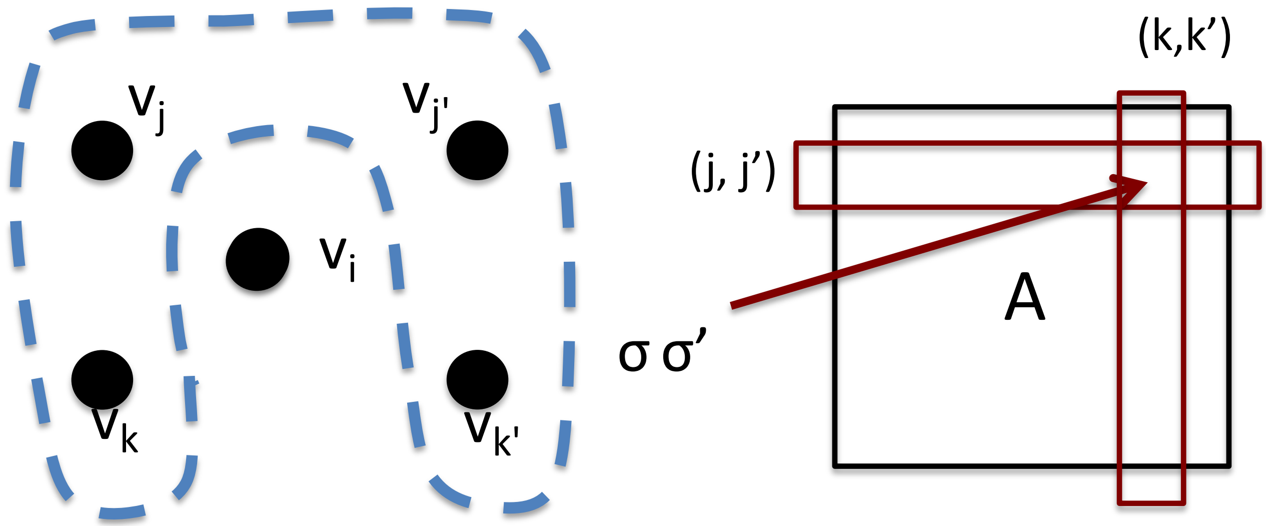

Hence the paired variables for the rows (and colns) come from different clauses!

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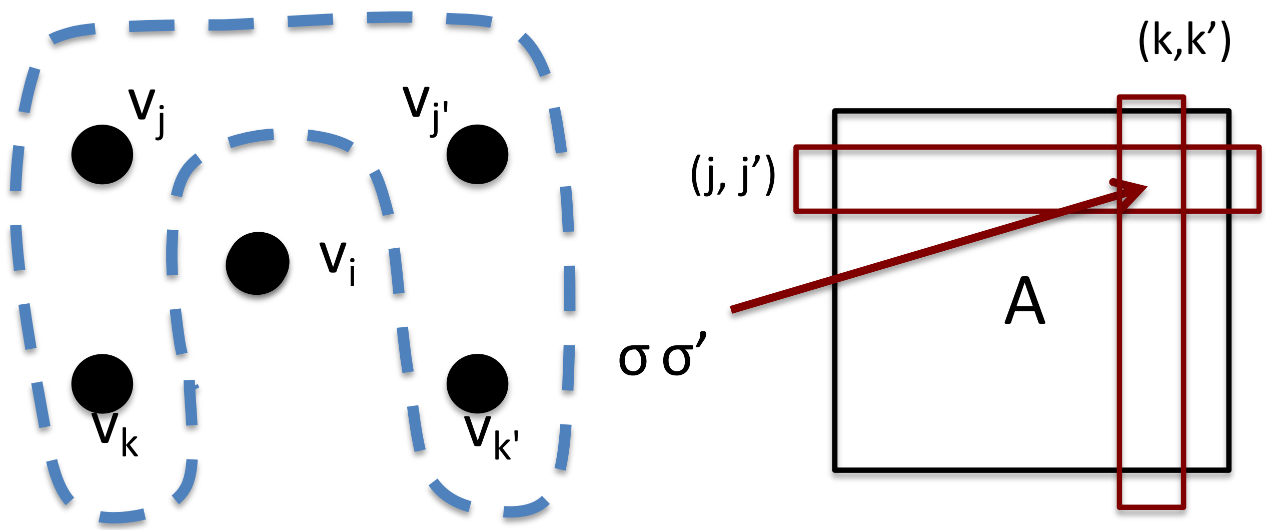

+ trace method

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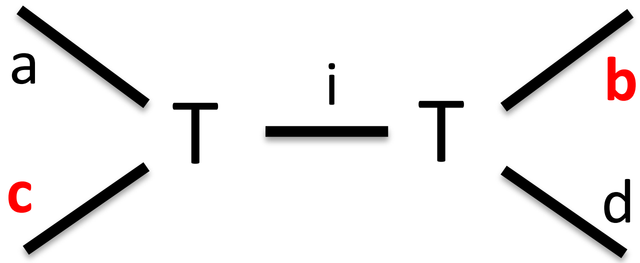

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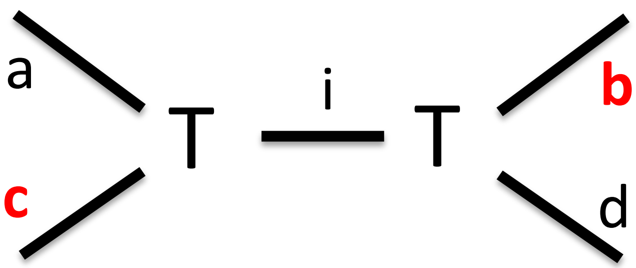

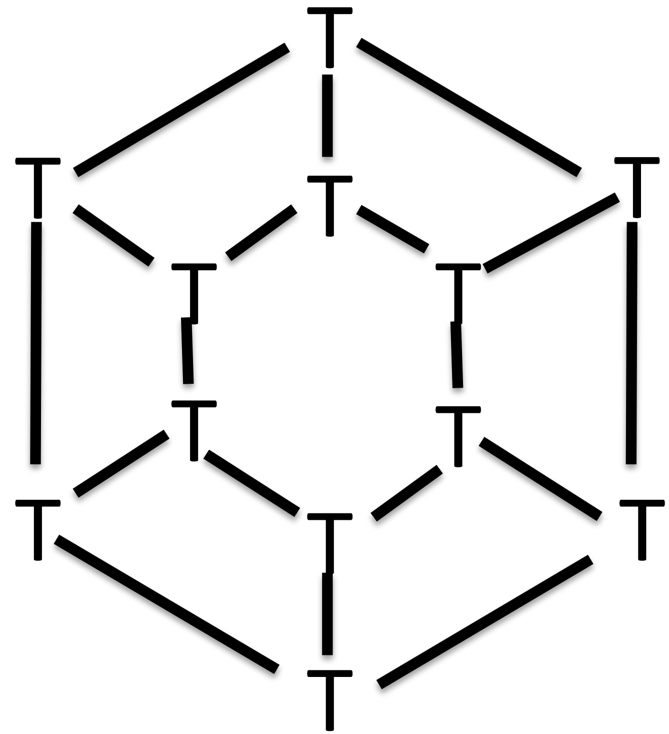

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Counts certainly labelings over this graph

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Noisy **tensor** completion with m observations ← **Embedding in SOS** — **Strongly refute*** random **3-XOR/3-SAT** with m clauses

[Coja-Oghlan, Goerdt, Lanka]

We then embed this algorithm into the **sixth** level of the sum-of-squares hierarchy, to get a relaxation for tensor prediction

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[Kivva, Potechin]: Yes, for random overcomplete tensors

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Scales to thousand-dimensional tensors!

Summary:

- Connections between tensors and random CSPs
- New algorithms for completing third-order tensors that beat flattening
- **Is practical tensor completion within reach?**

Thanks! Any Questions?