An Information Complexity Approach to Extended Formulations

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joint work with Mark Braverman

The Permutahedron

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Yet Q has only $O(n^2)$ facets

Extended Formulations

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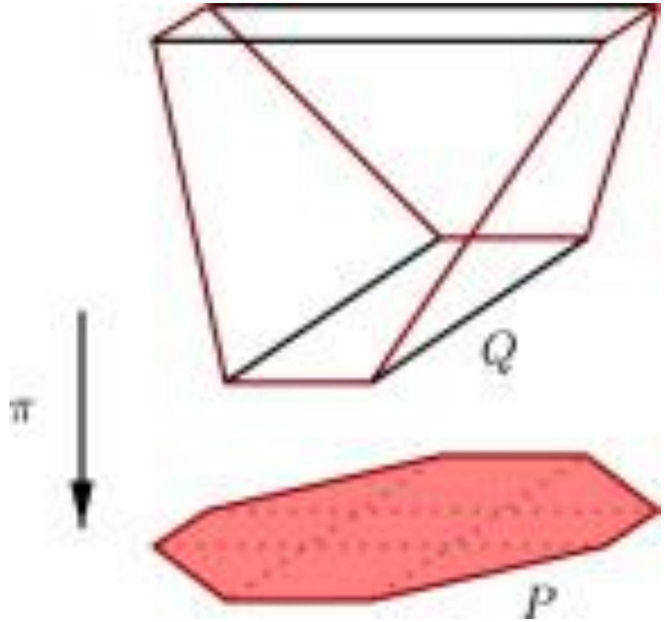

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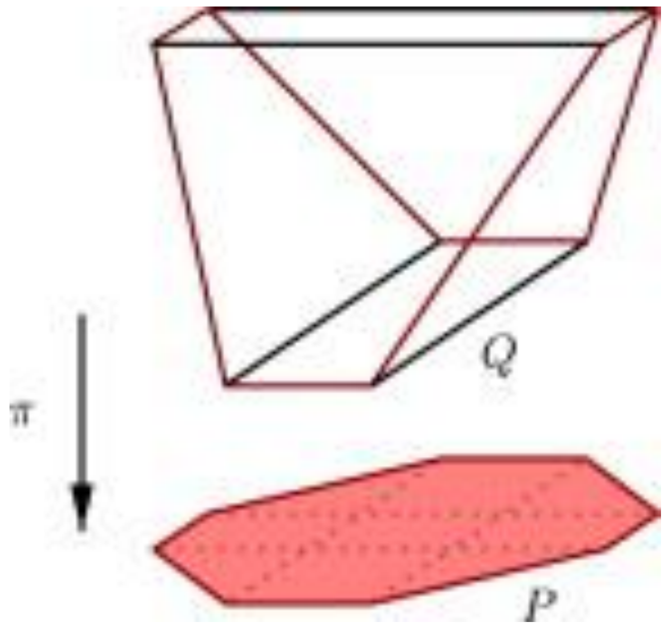

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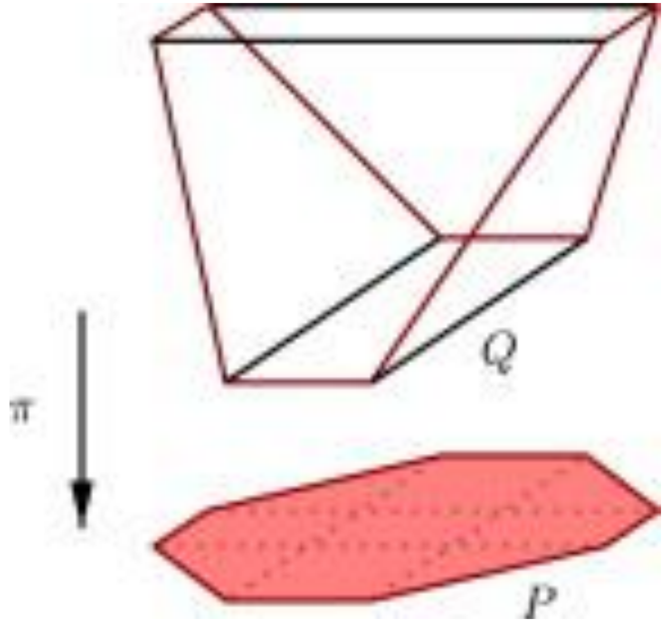

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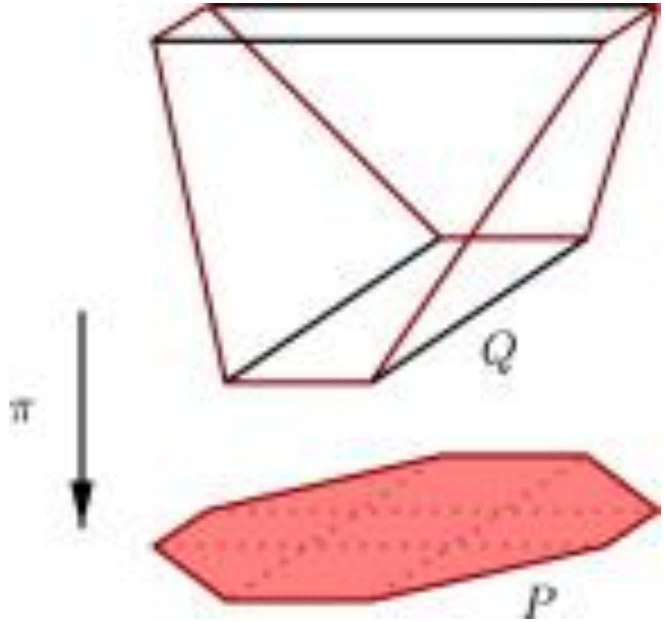

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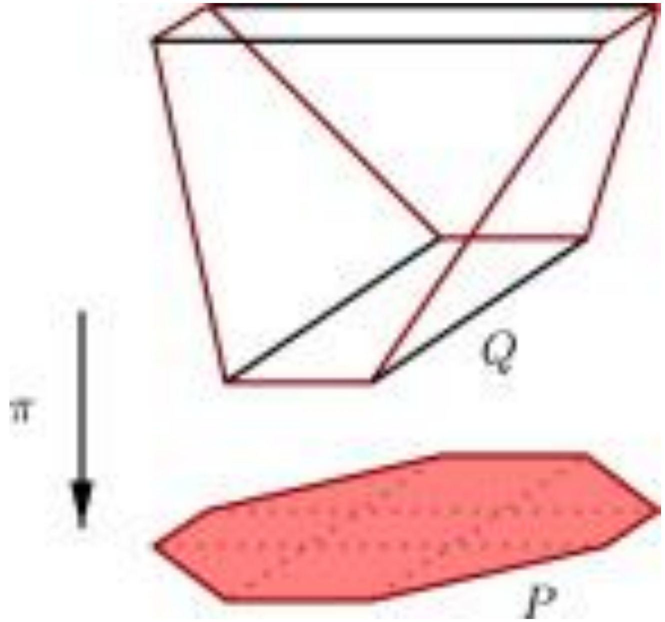

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…analogy with **quantifiers** in Boolean formulae

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e.g. prove there is low-cost object, through its polytope

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[Yannakakis ’90]: Yes, through the **nonnegative rank**

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Definition of the **slack matrix**…

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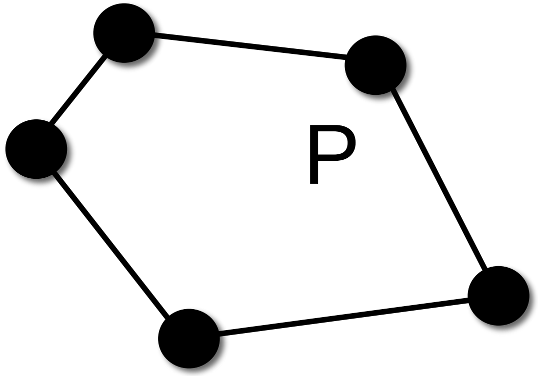

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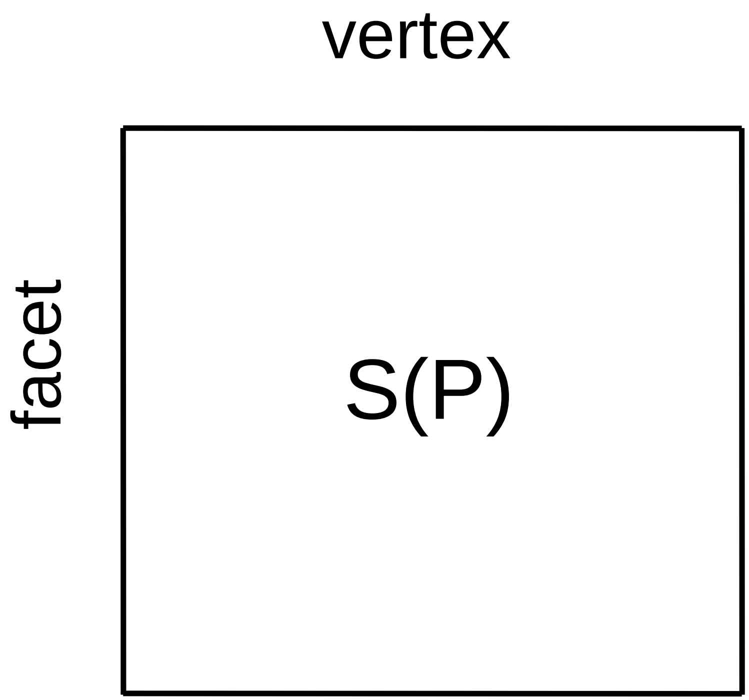

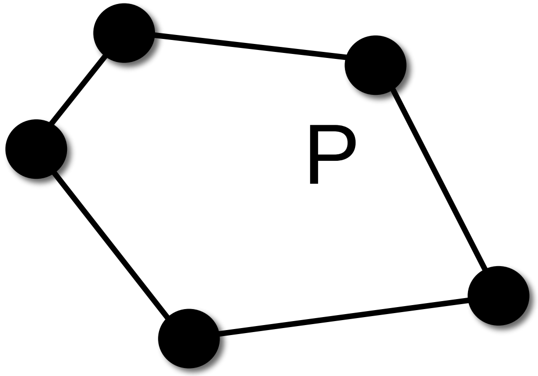

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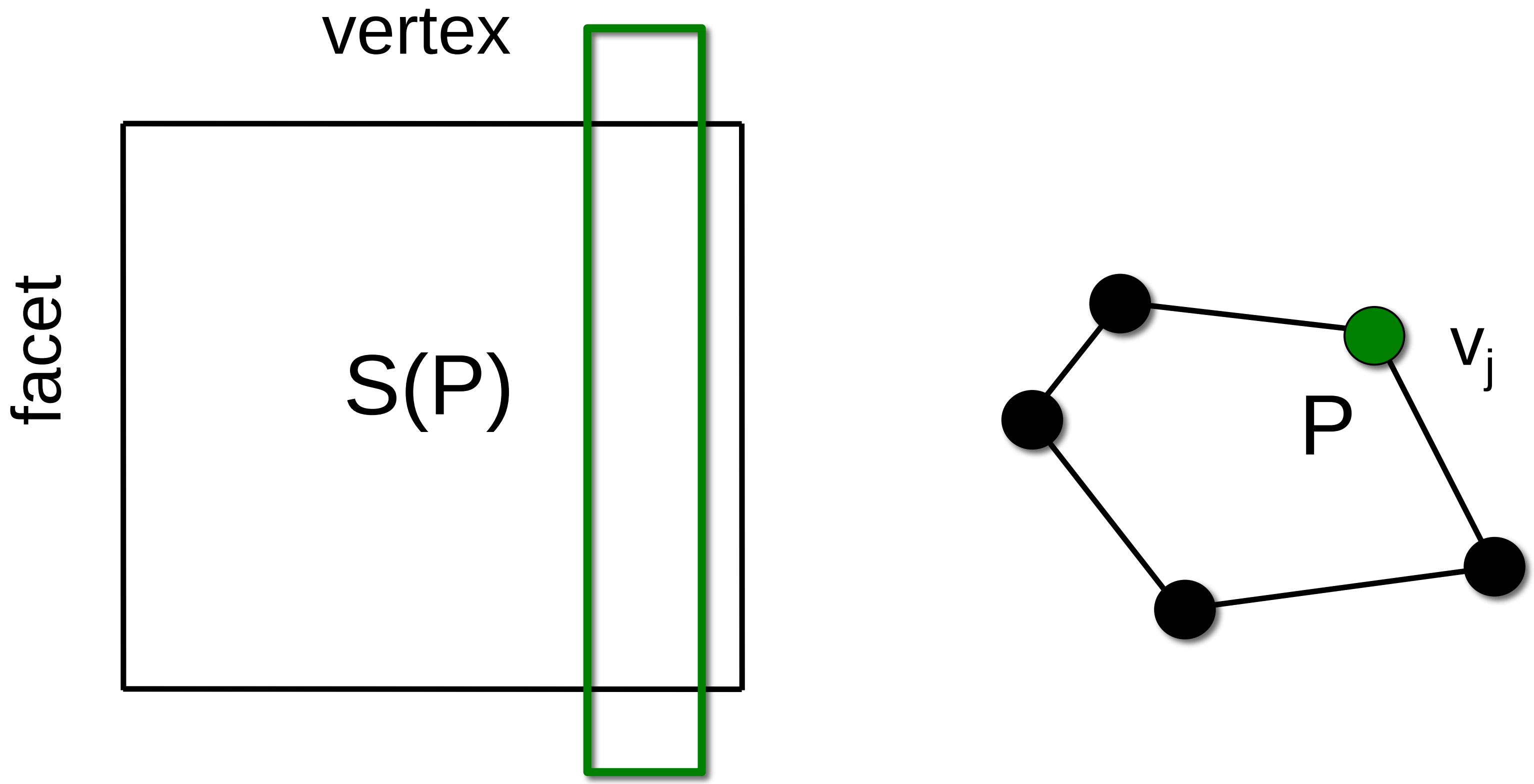

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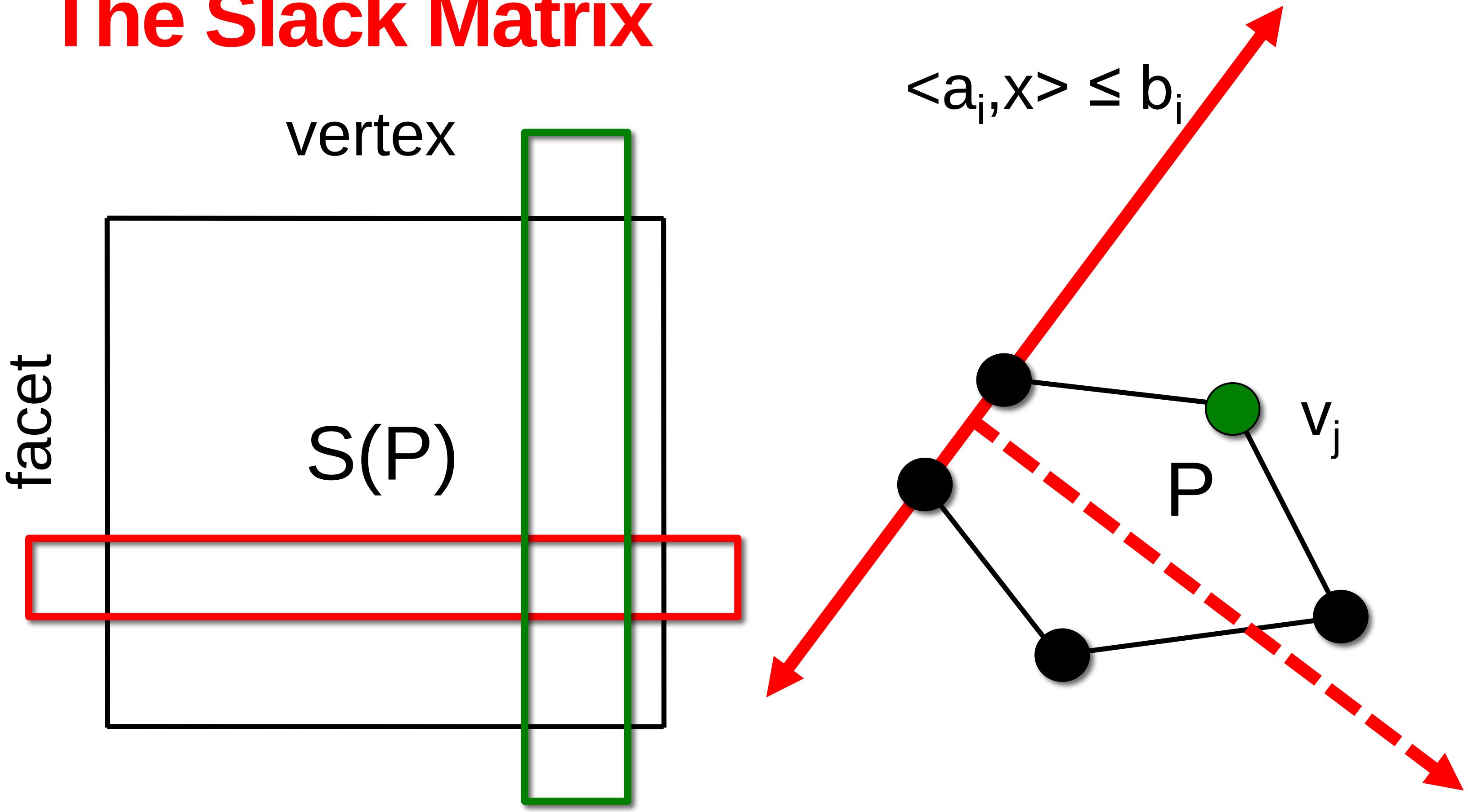

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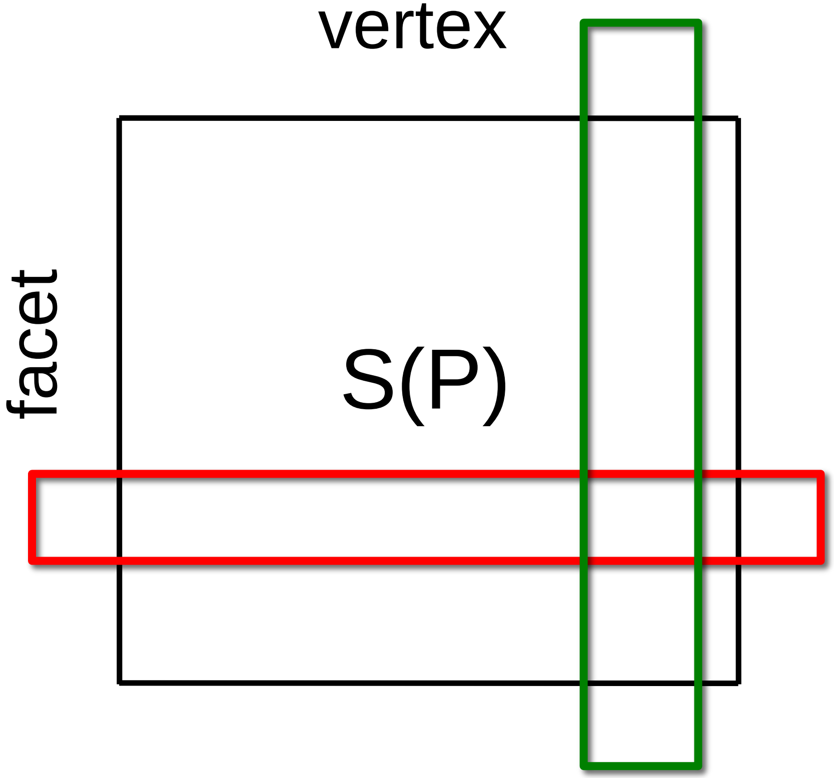

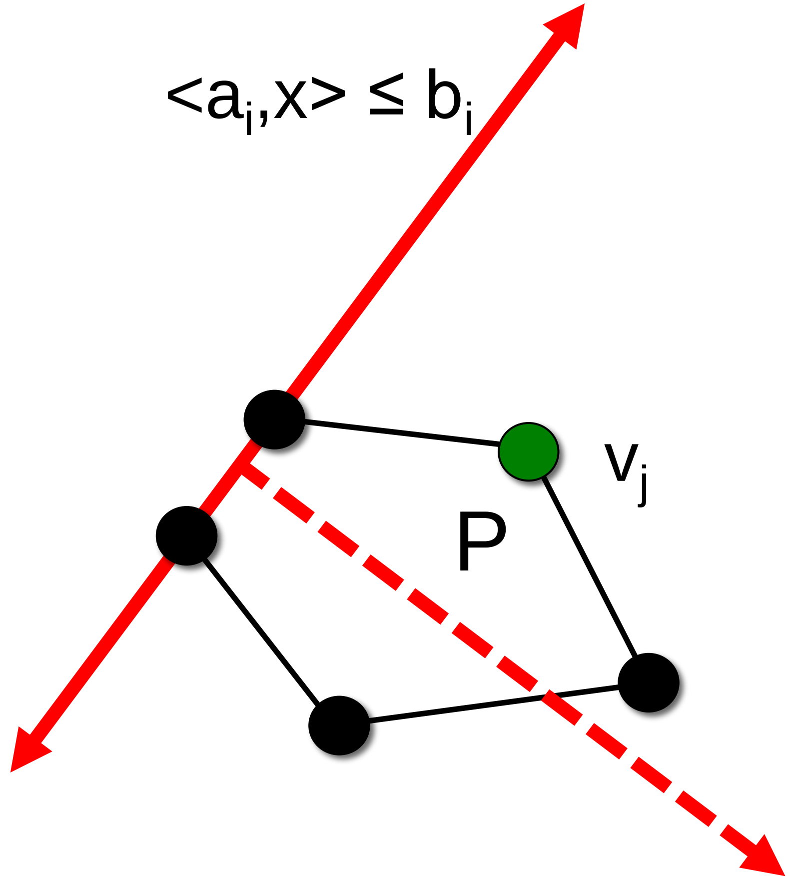

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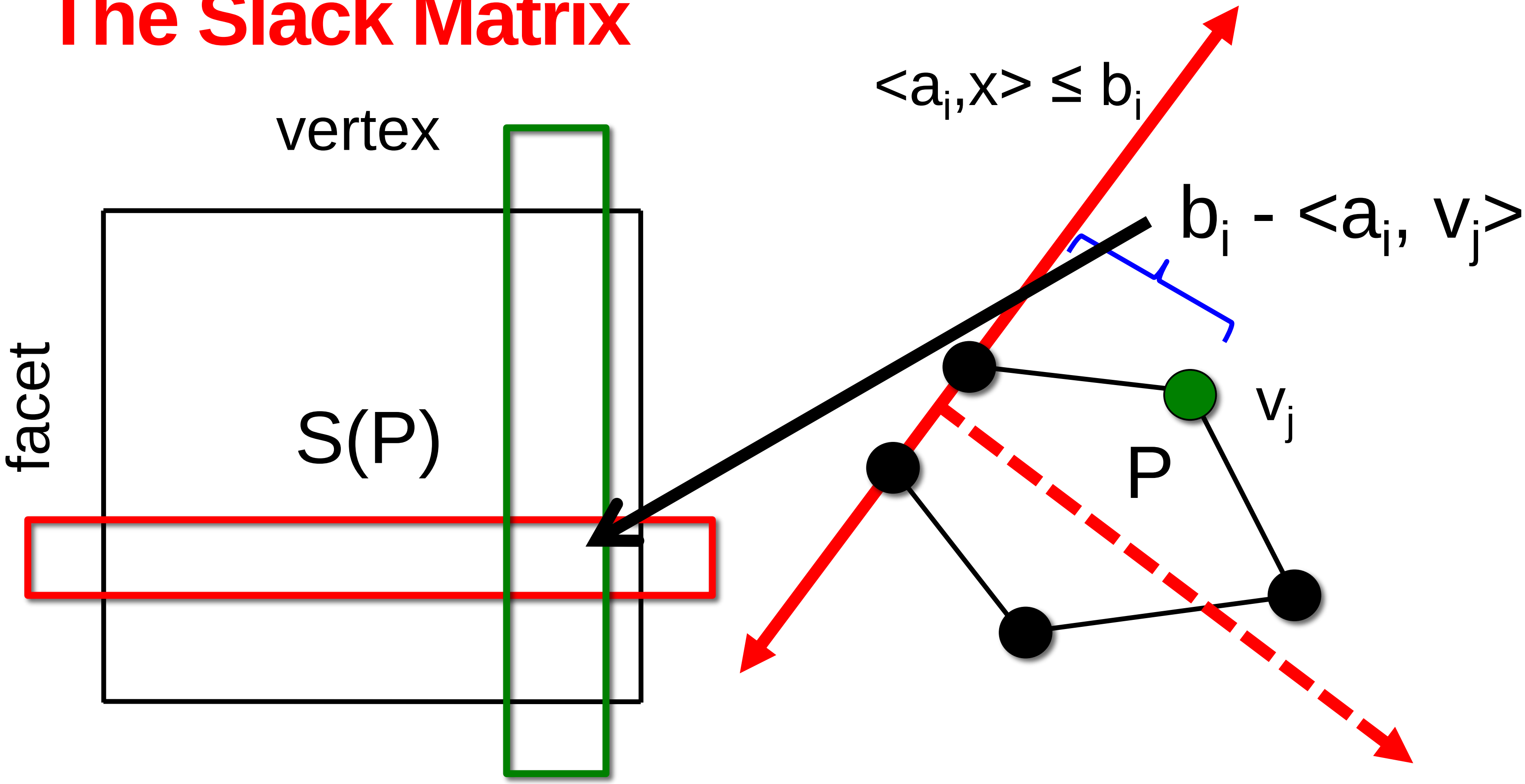

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Note: $\text{rank}^+(S) \geq \text{rank}(S)$, but can be much larger too!

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We will give a new way to lower bound nonnegative rank via **information theory**…

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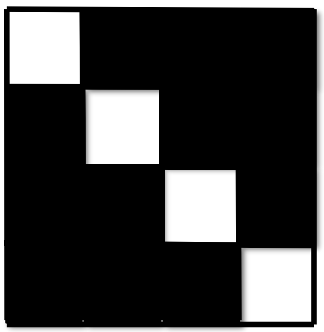

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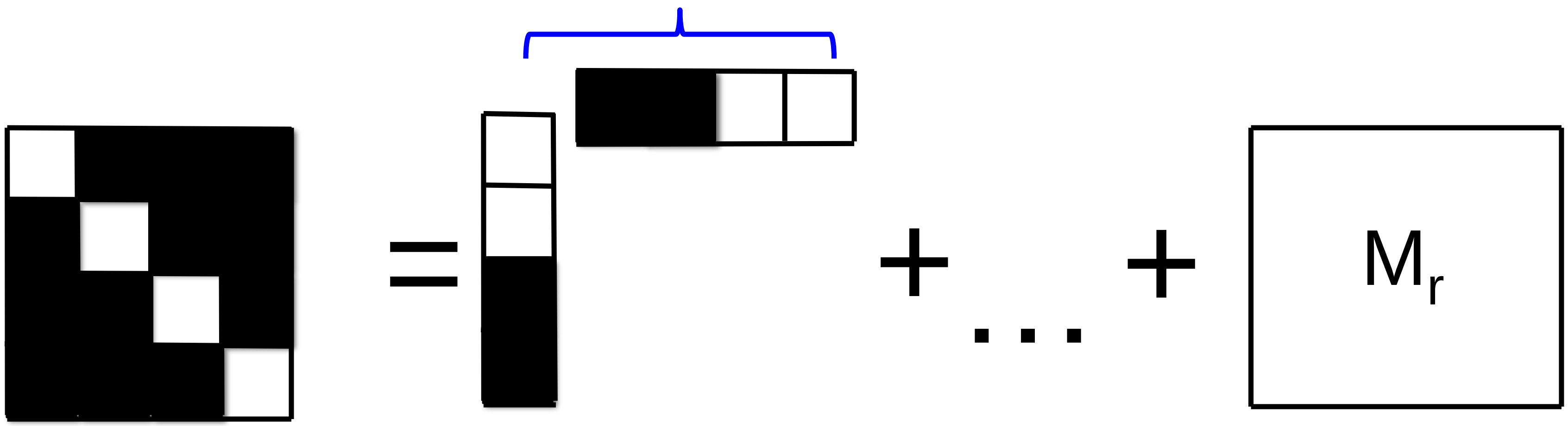

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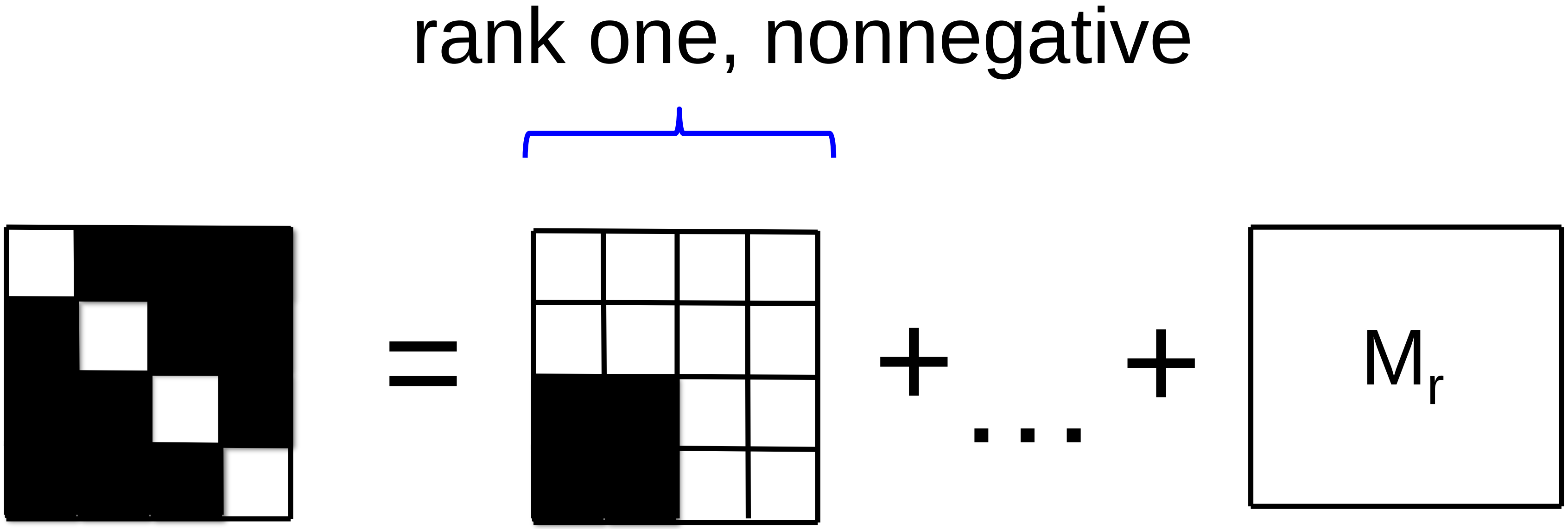

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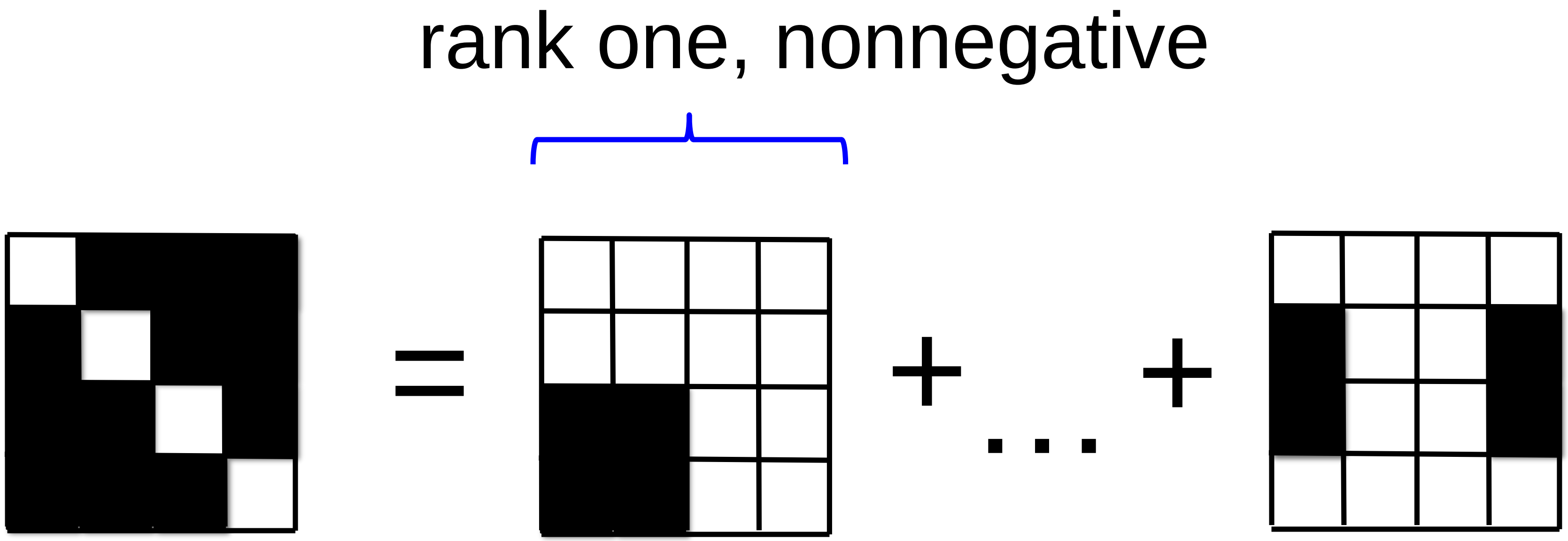

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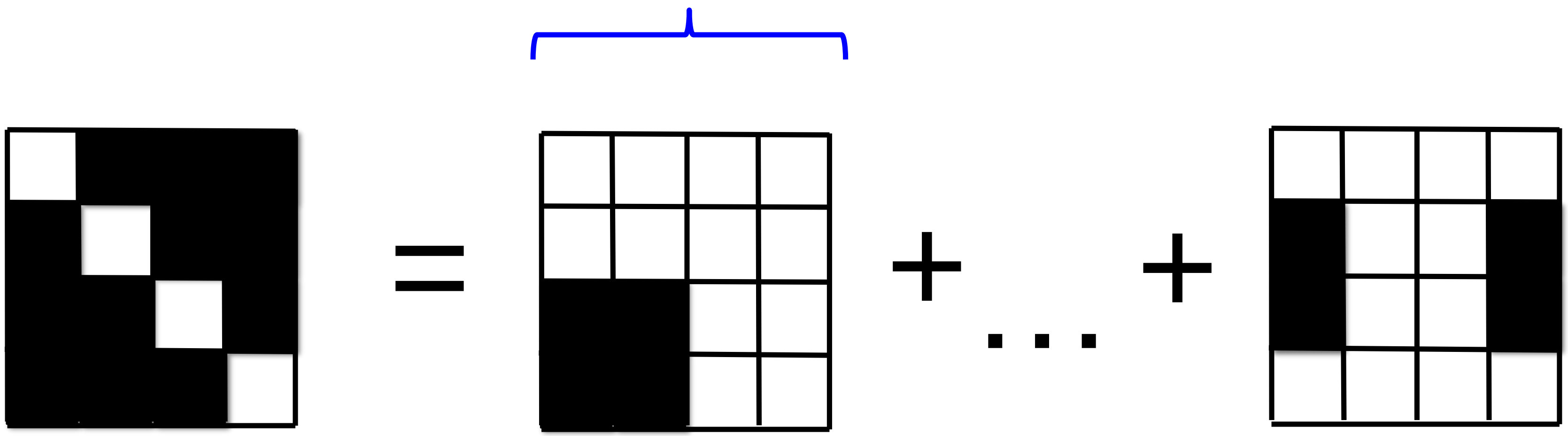

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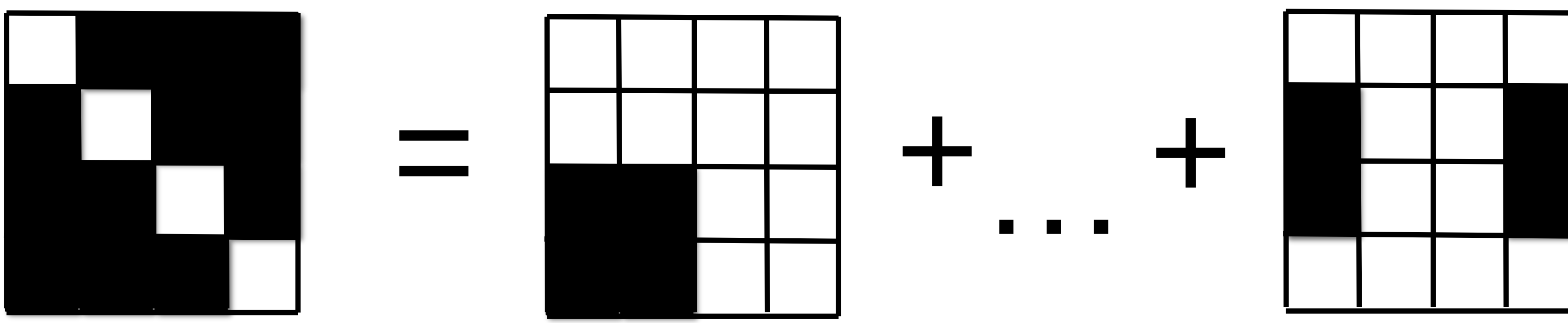

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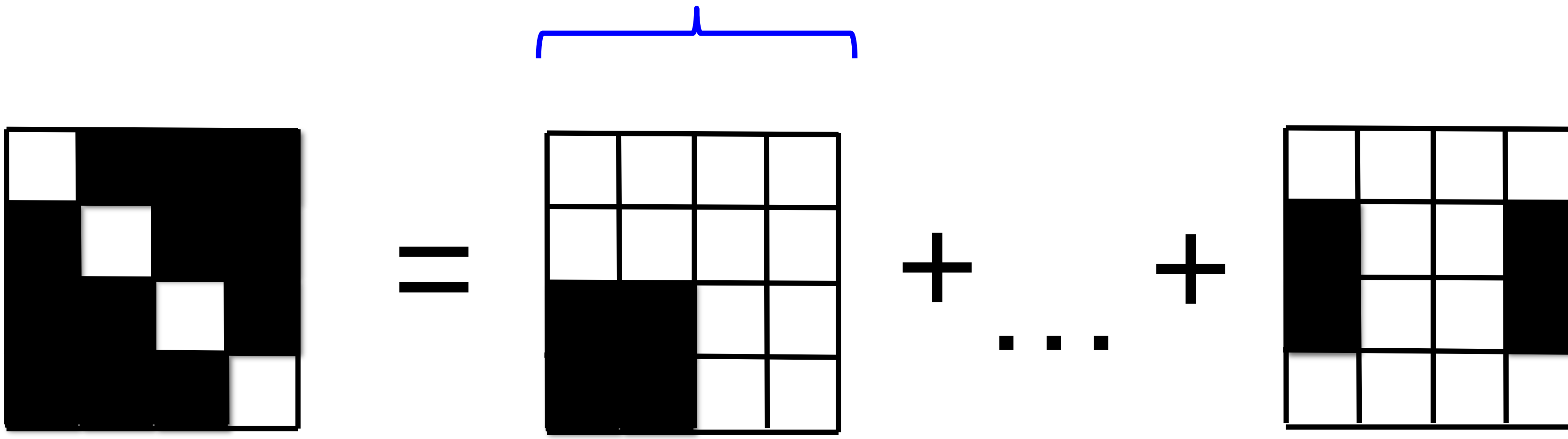

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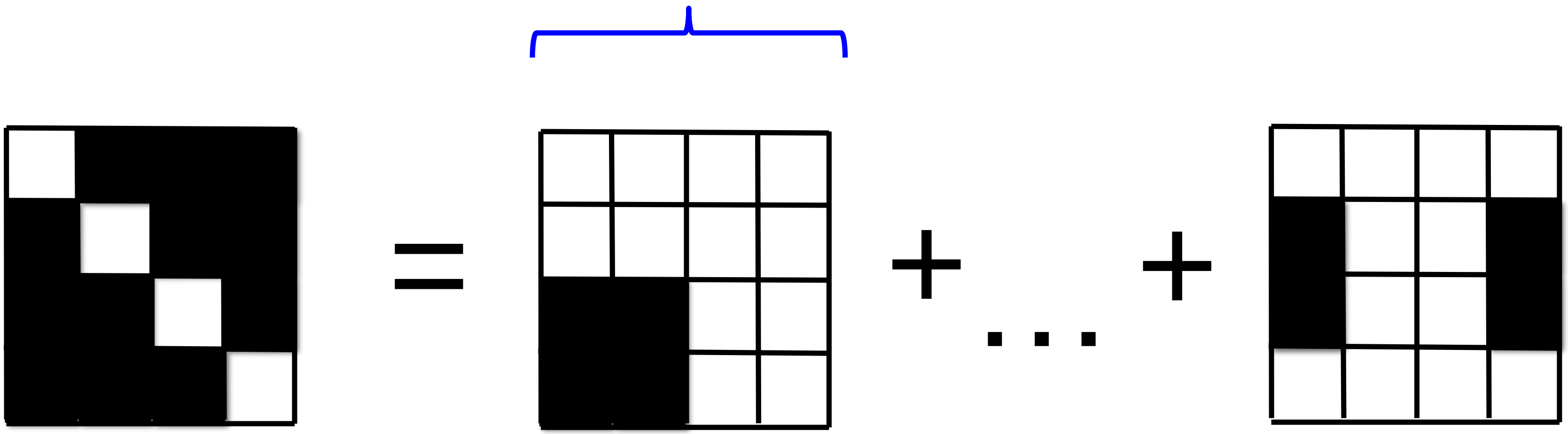

Non-deterministic Comm. Complexity

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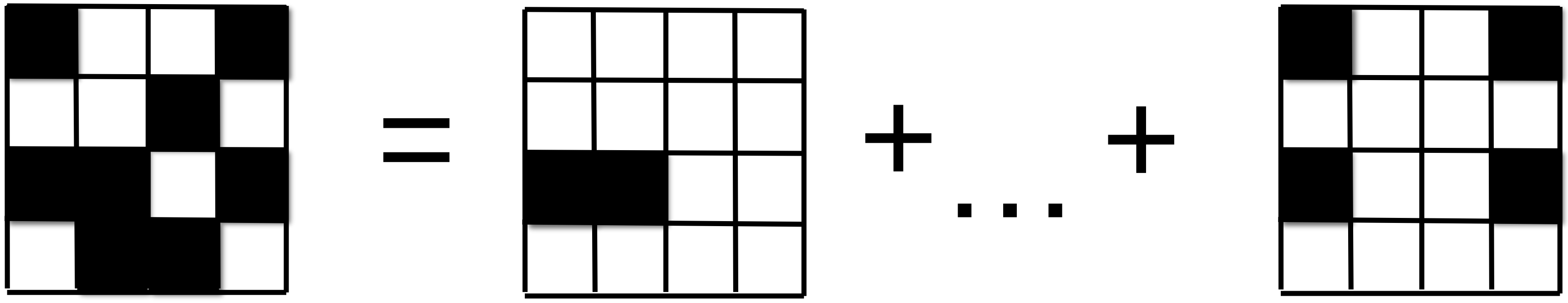

A Sampling Argument

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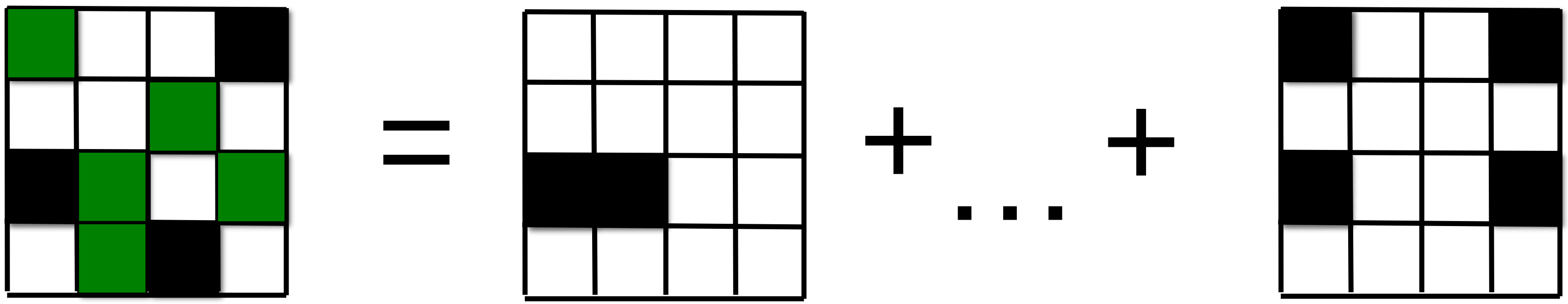

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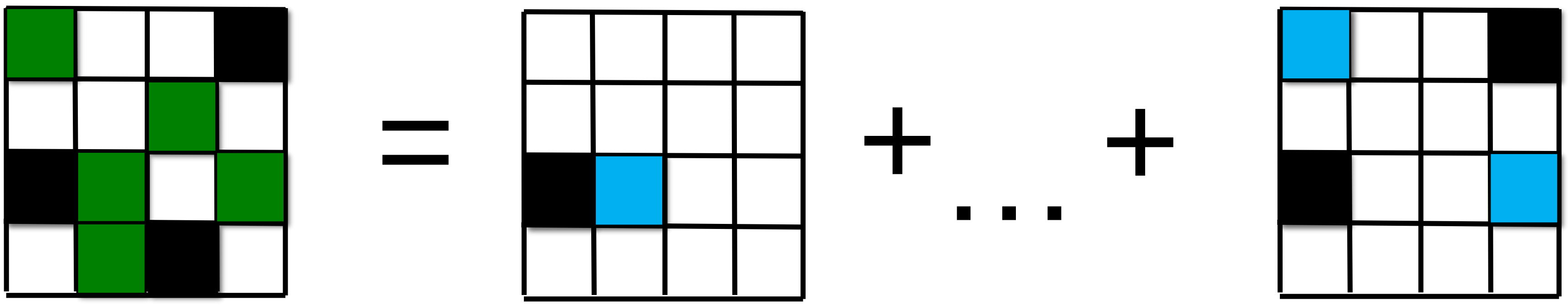

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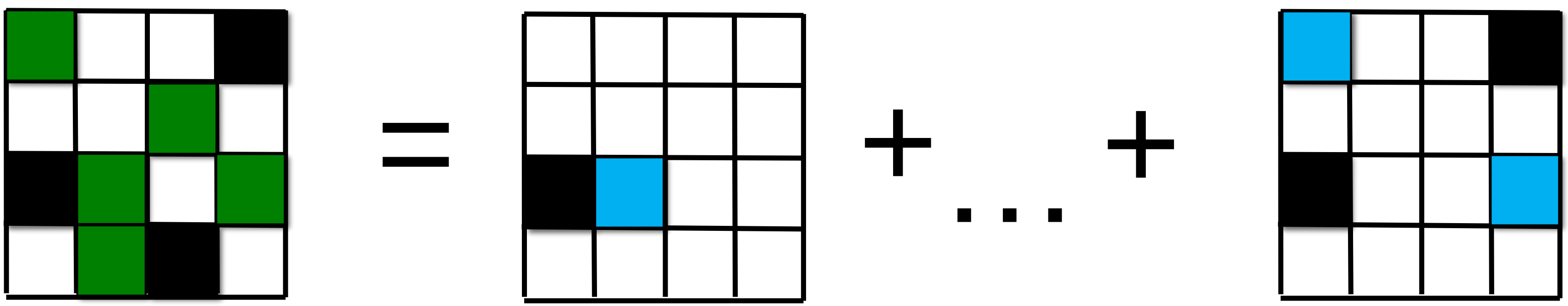

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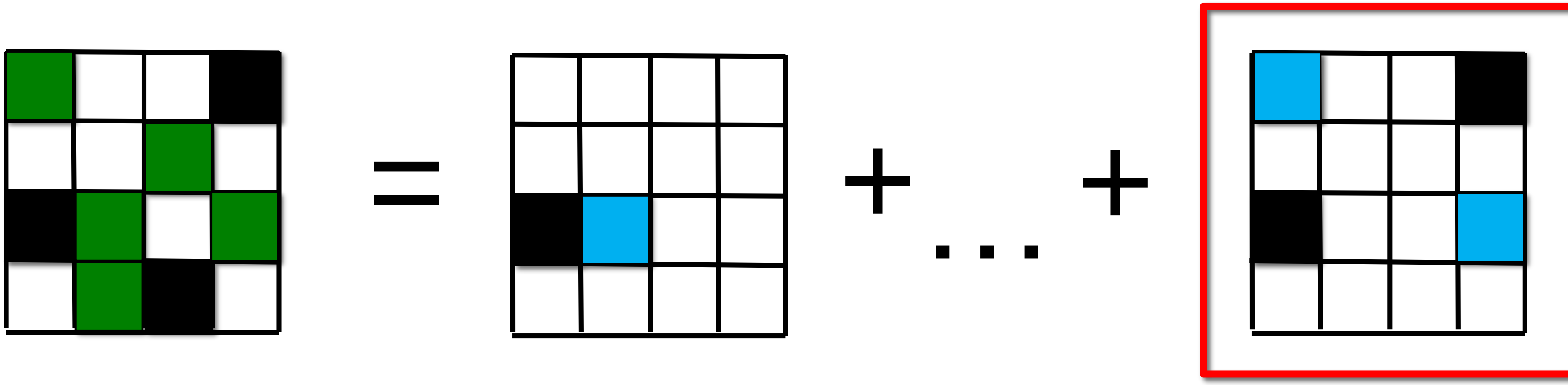

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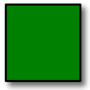
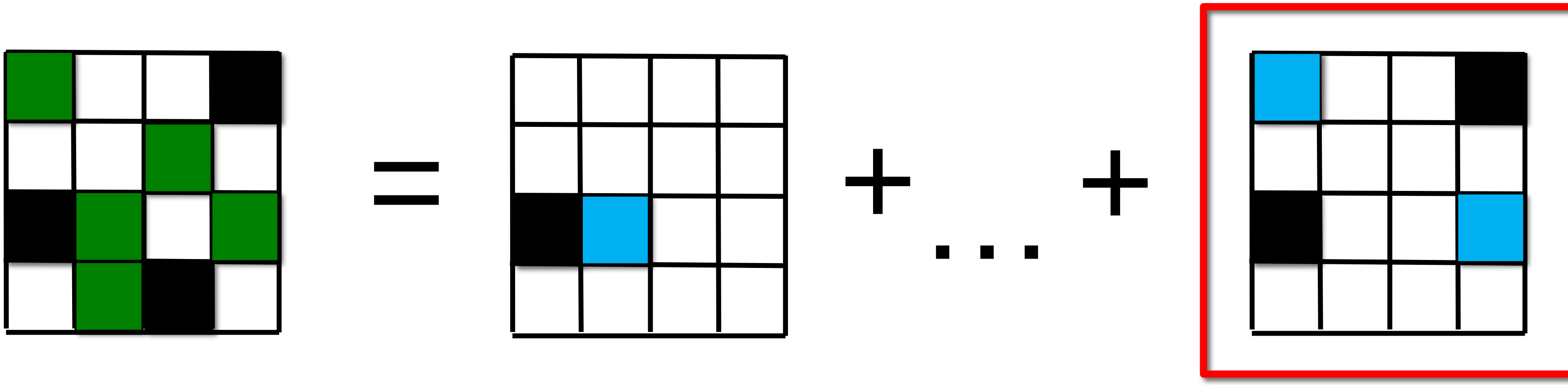
T = {}, set of entries in S with same value



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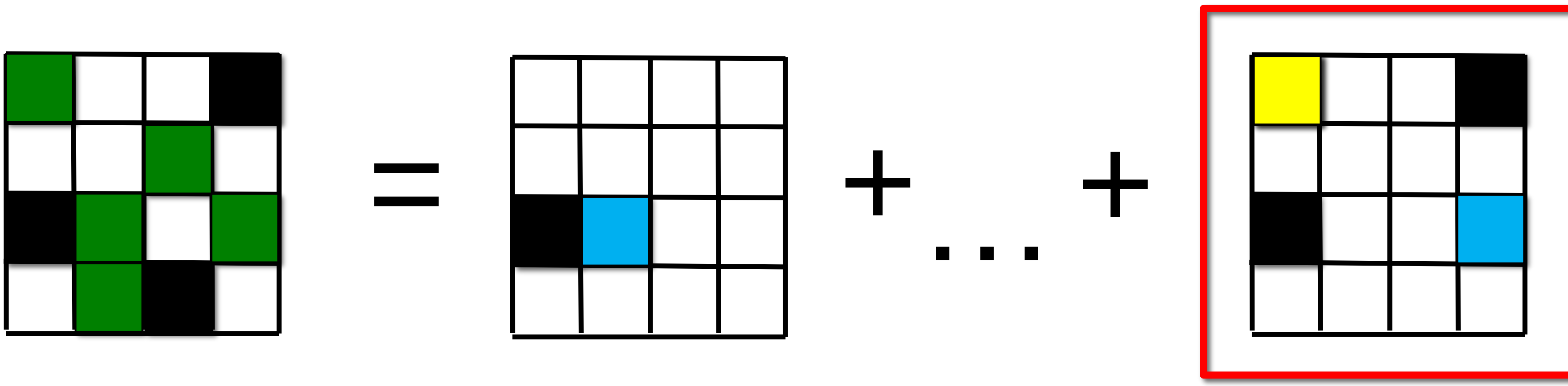

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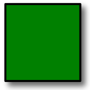
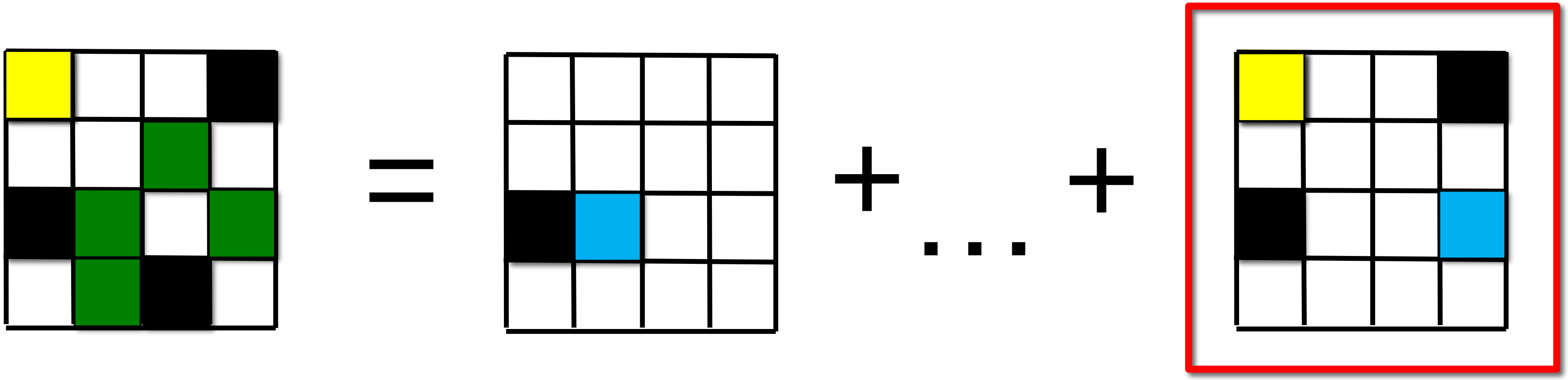
T = {}, set of entries in S with same value



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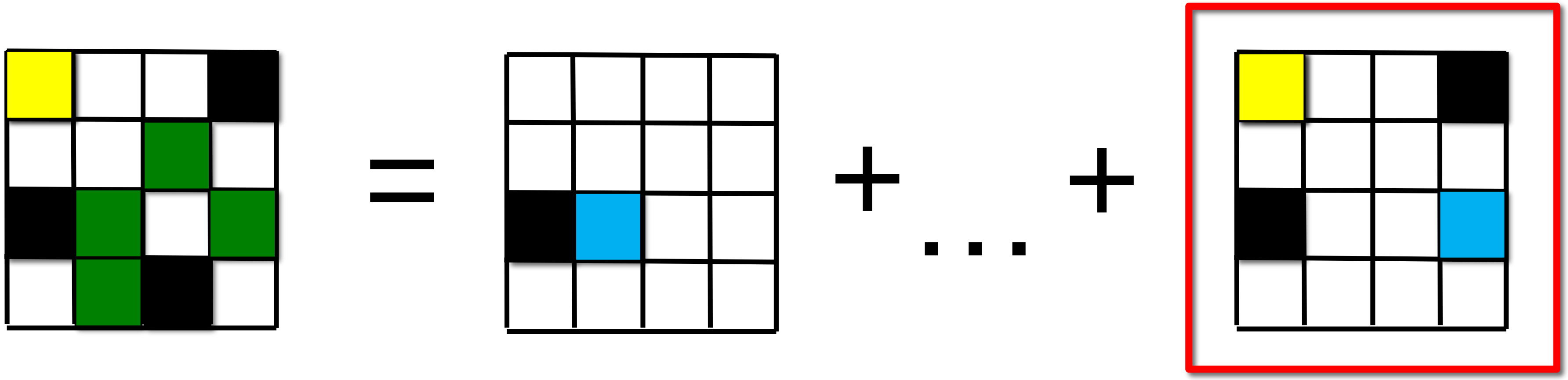

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If r is too small, this procedure uses too little entropy!

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The Construction of [Fiorini et al]

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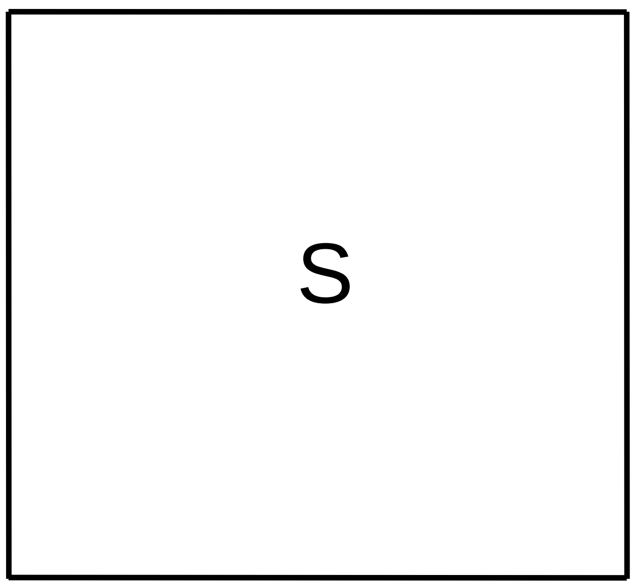
vertices:

constraints:



The Construction of [Fiorini et al]

correlation polytope: $P_{corr} = \text{conv}\{aa^T | a \text{ in } \{0,1\}^n\}$

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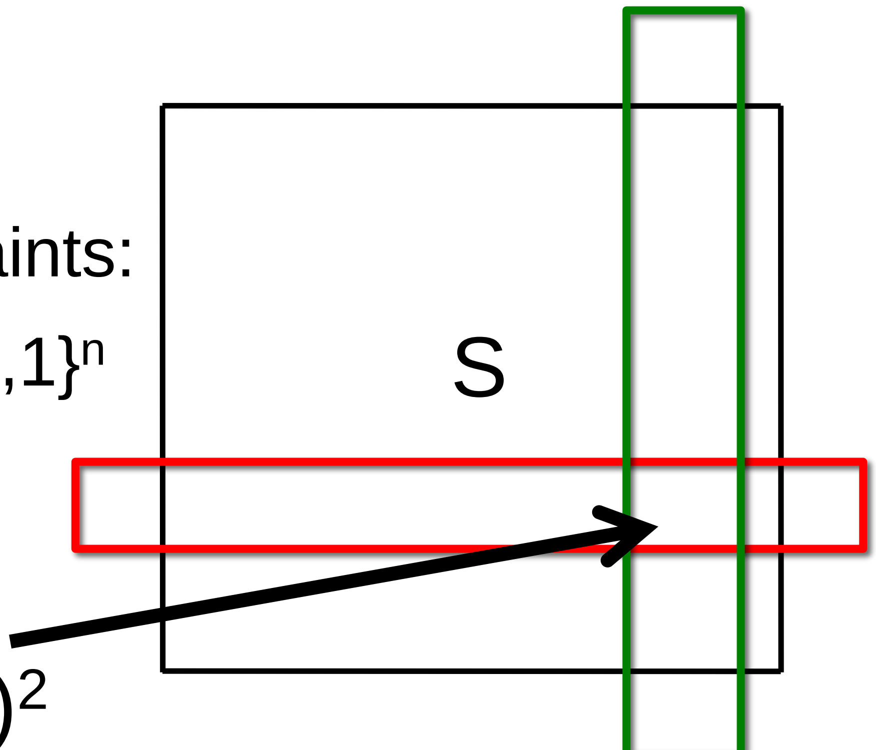

$(1-a^Tb)^2$

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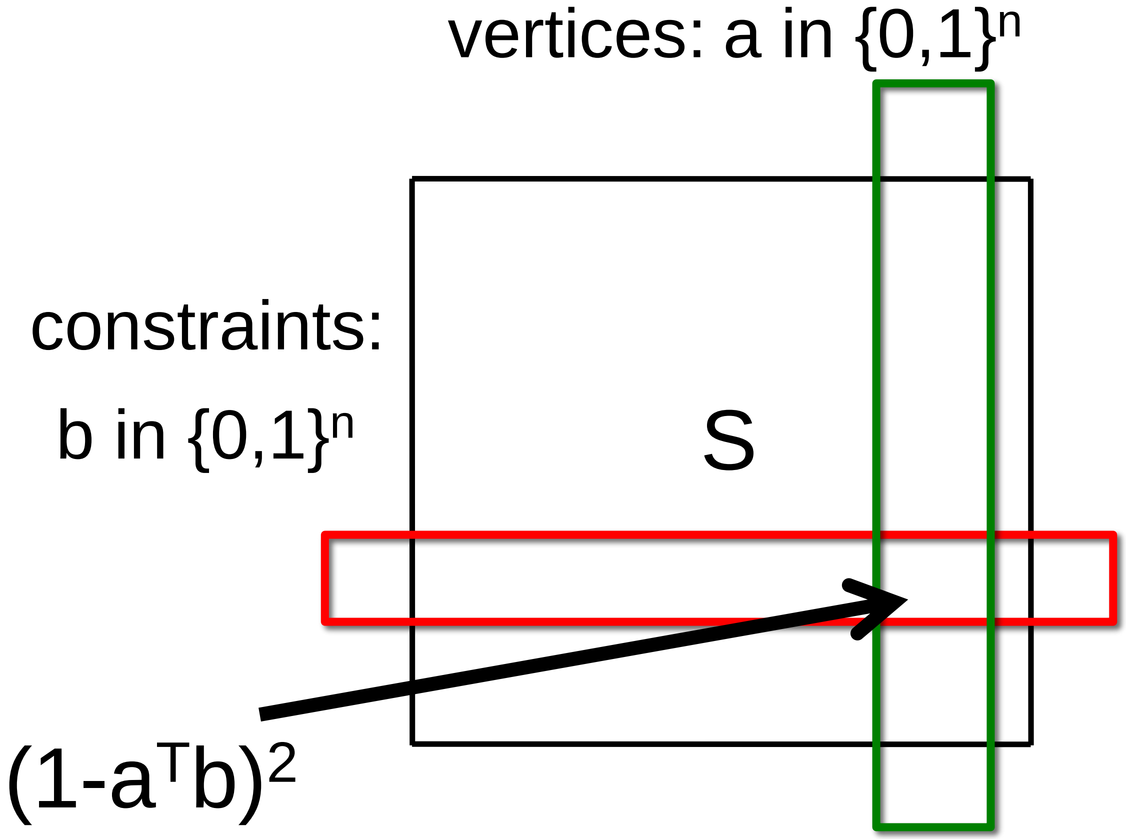

UNIQUE DISJ.
Output 'YES' if a and b as sets are disjoint, and 'NO' if a and b have one index in common

The Reconstruction Principle

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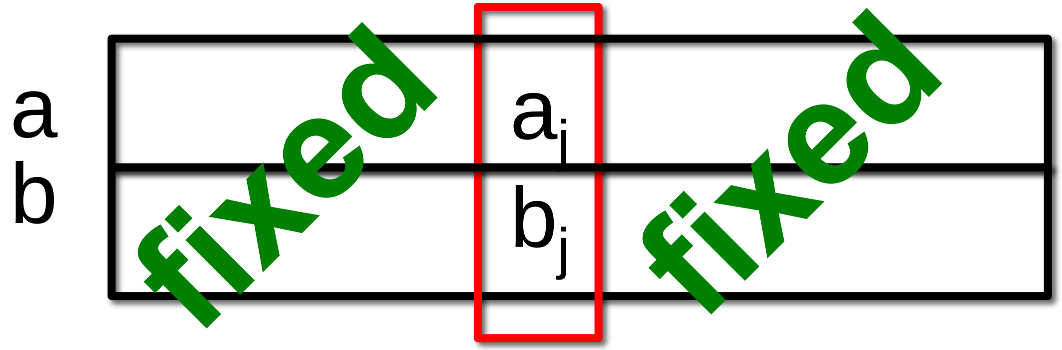
Suppose that a_{-j} and b_{-j} are **fixed**



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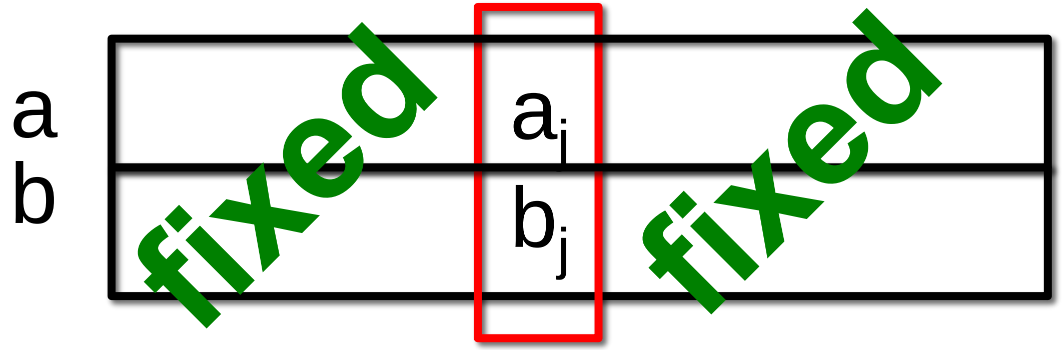

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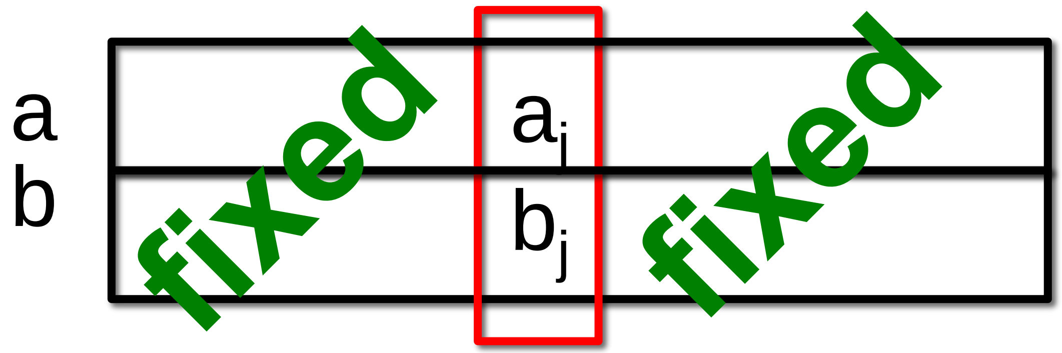

M_i restricted to (a_{-j}, b_{-j})

	$(\ldots b_j=0\ldots)$	$(\ldots b_j=1\ldots)$
$(a_{1..j-1}, a_j=0, a_{j+1\ldots n})$	$M_i(a,b)$	$M_i(a,b)$
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$$
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\hline
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$H(a_j,b_j \mid i, a_{-j}, b_{-j}) \leq 1 < \log_2 3$

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- Yannakakis's Factorization Theorem
- The Rectangle Bound
- A Sampling Argument

Part II: Applications

- Correlation Polytope
- Approximating the Correlation Polytope
- A Better Lower Bound for Disjointness

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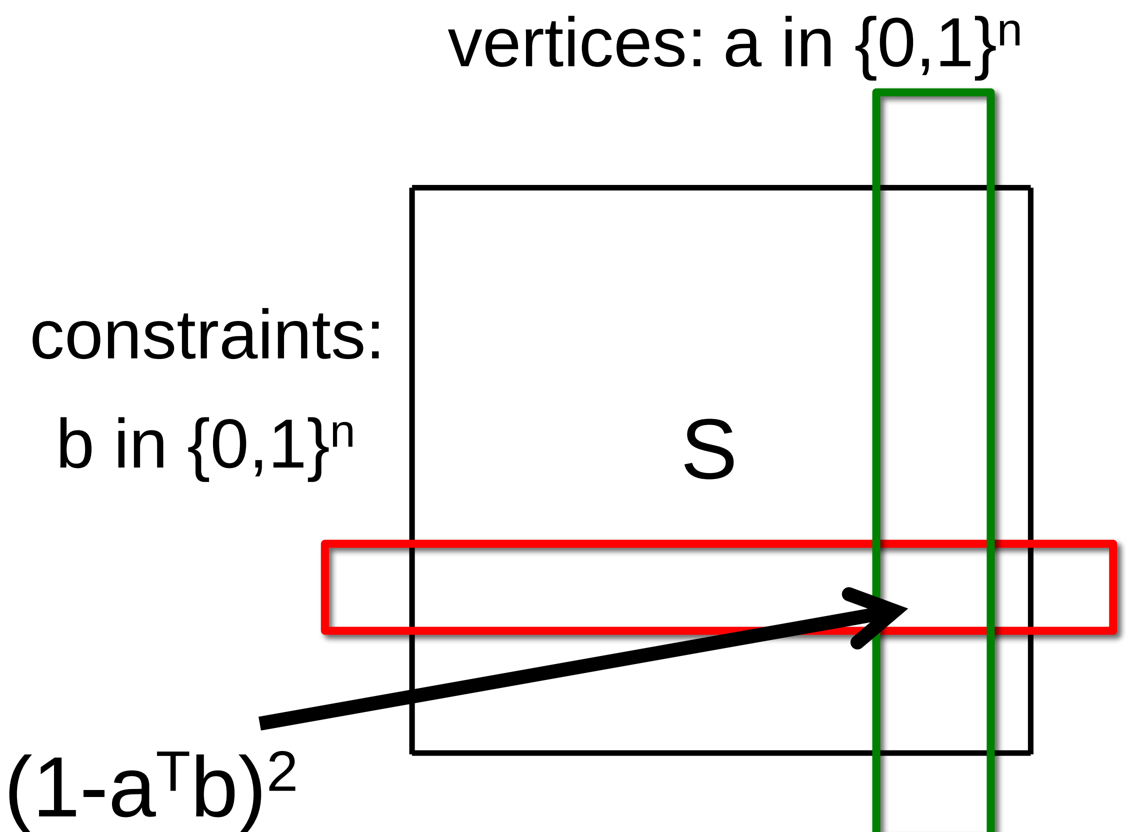
Approximate EFs [Braun et al]



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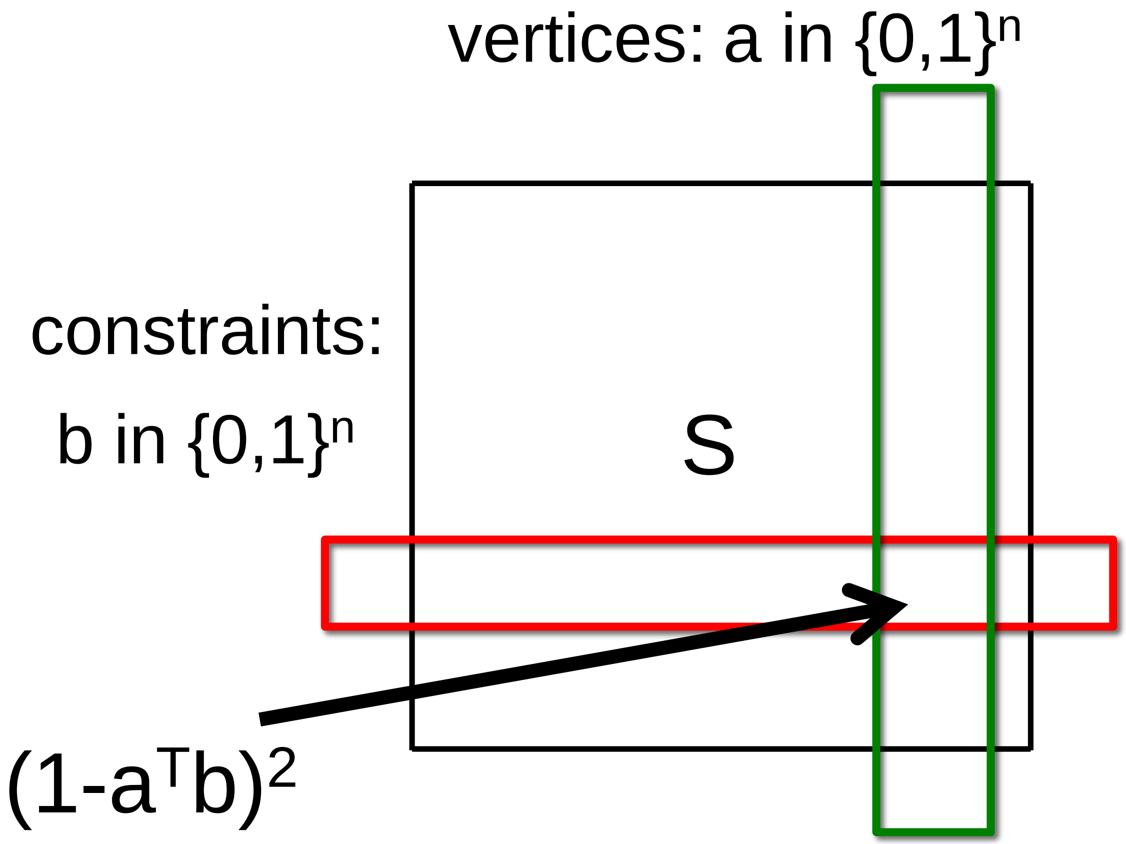

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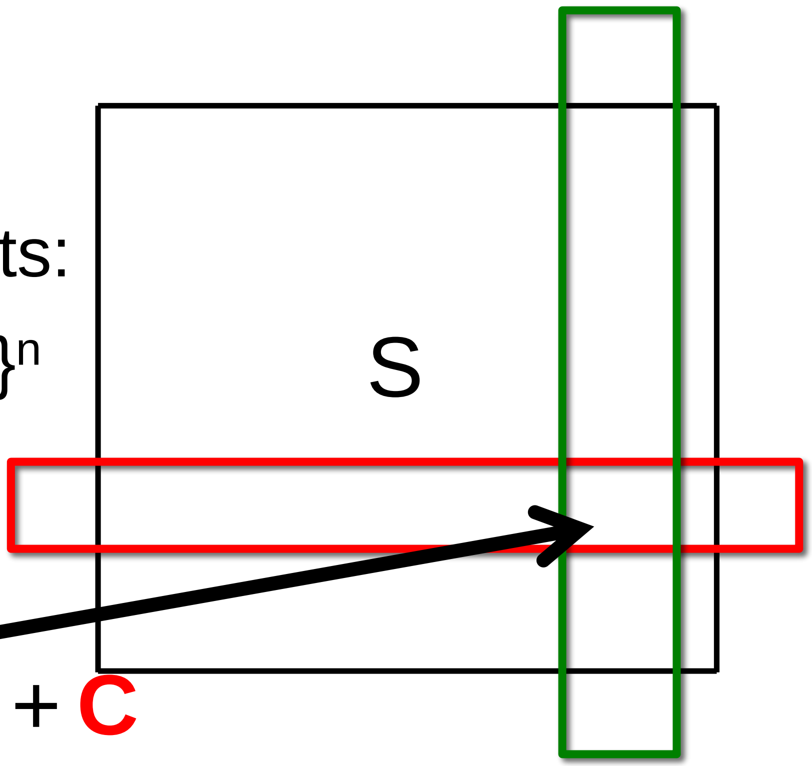

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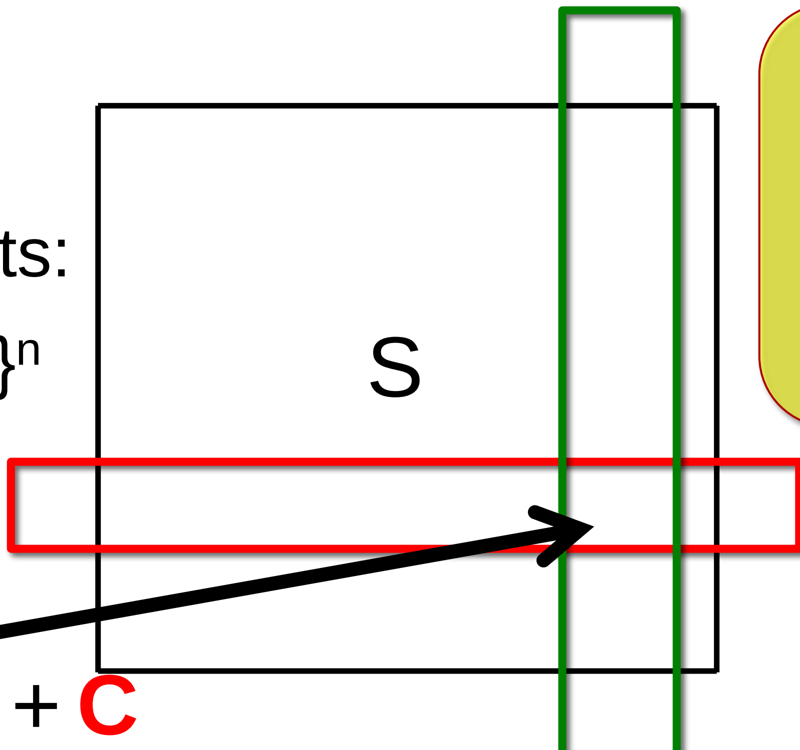

$(1-a^Tb)^2$ + **C**

New Goal:
Output the answer to UDISJ with prob. at least ½ + 1/2(C+1)

Is the correlation polytope hard to approximate for large values of C?

Analogy: Is UDISJ hard to compute with prob. $\frac{1}{2}+1/2(C+1)$ for large values of C?

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Claim: If UDISJ can be computed with prob. ½+1/2(C+1) using $o(n/C^2)$ bits, then UDISJ can be computed with prob. ¾ using $o(n)$ bits

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Proof: Run the protocol $O(C^2)$ times and take the majority vote

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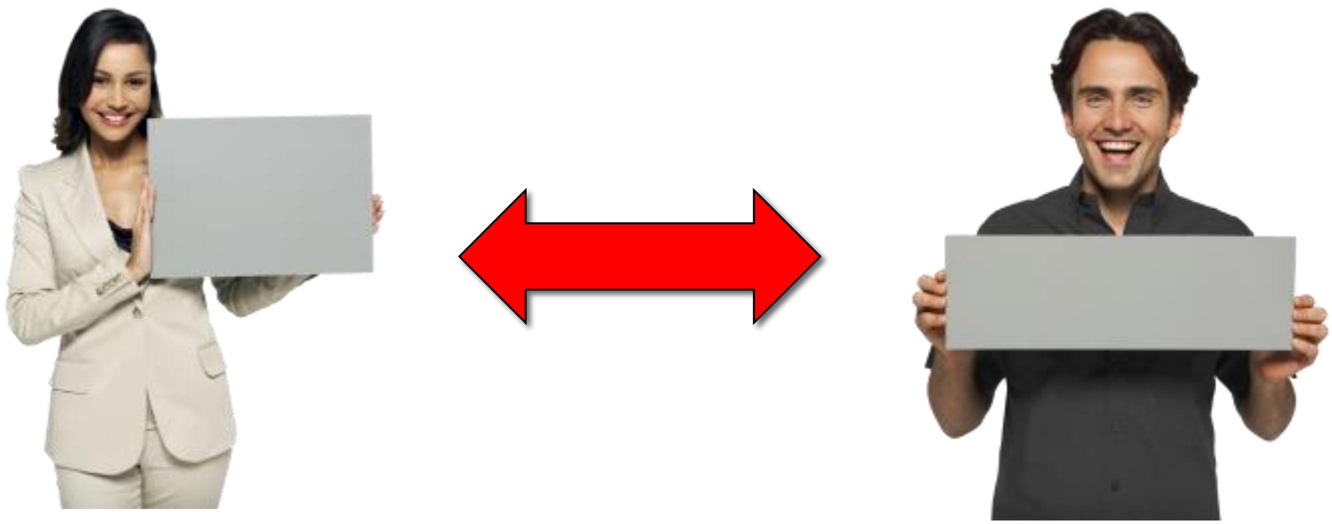

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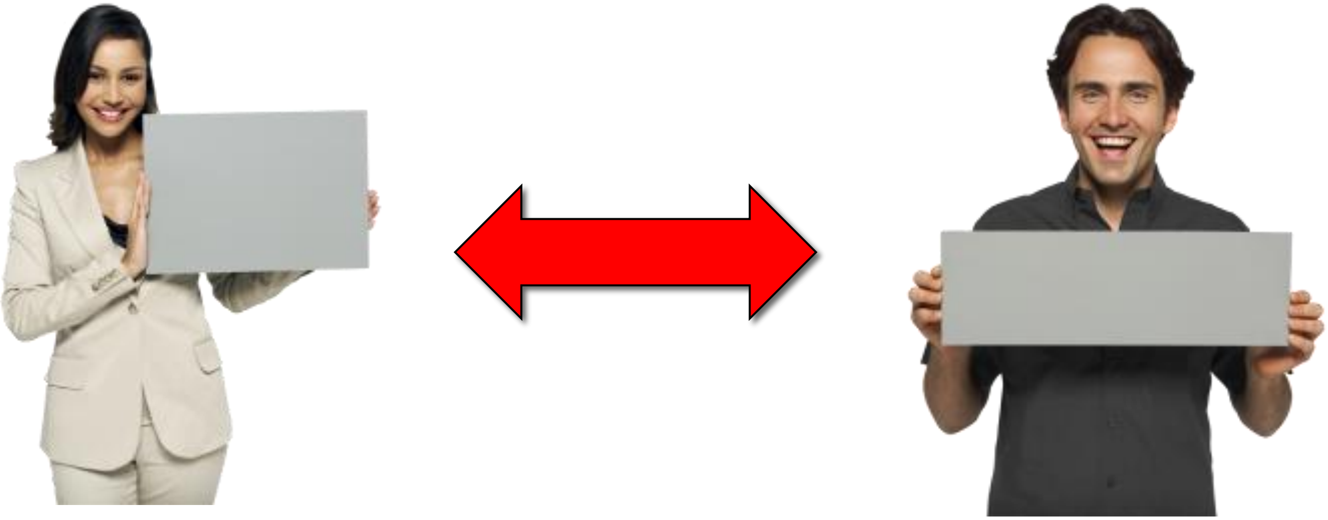

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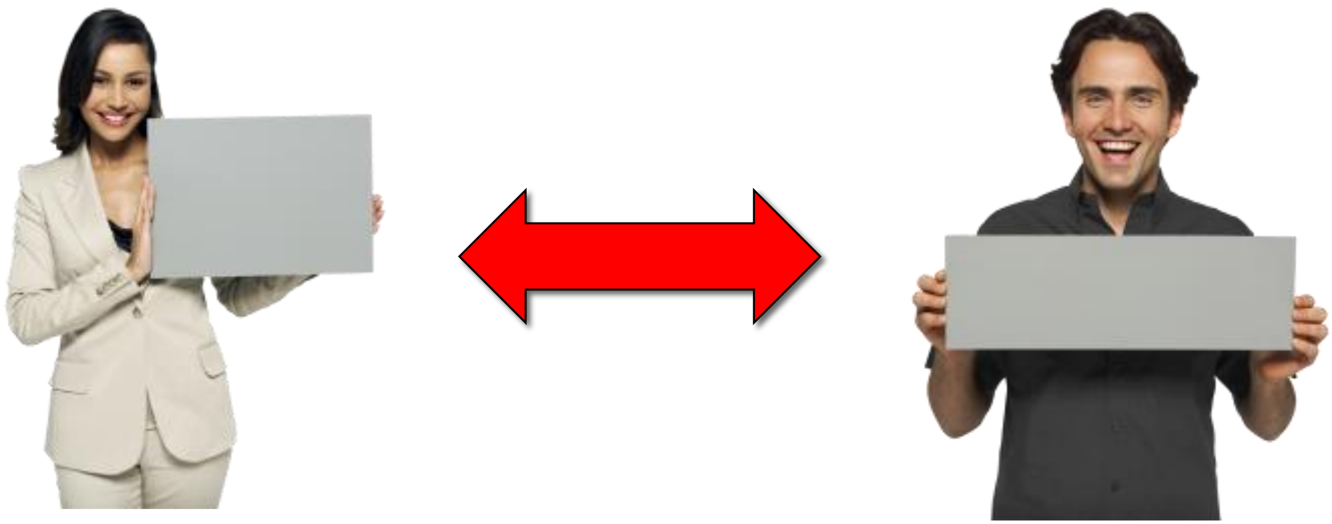

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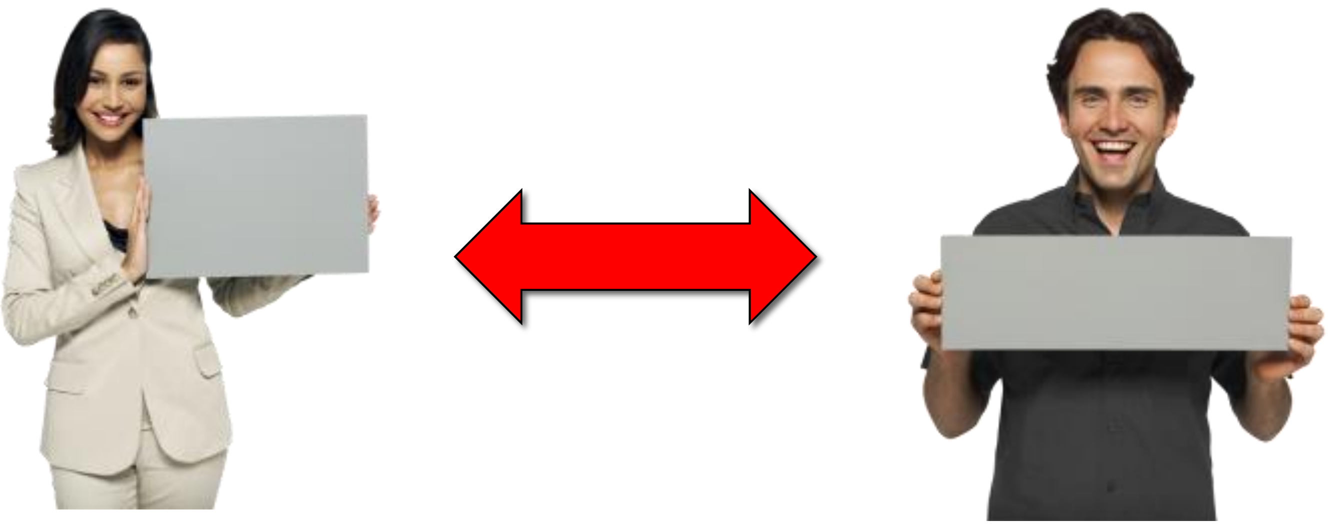
Is there a **stricter** one bit problem that we could reduce to instead?



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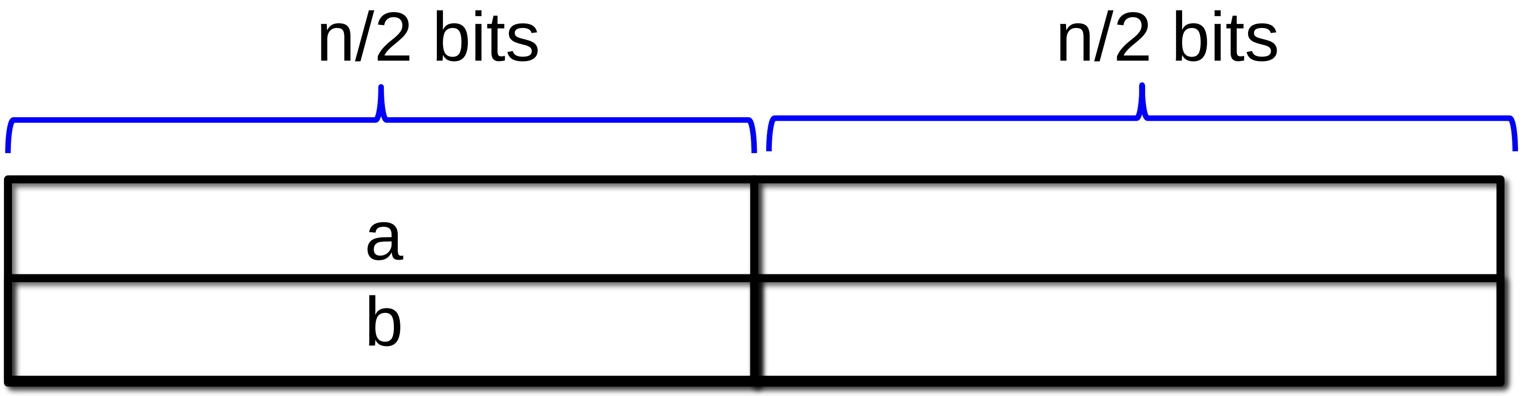
How can we reduce to such a one-bit problem?



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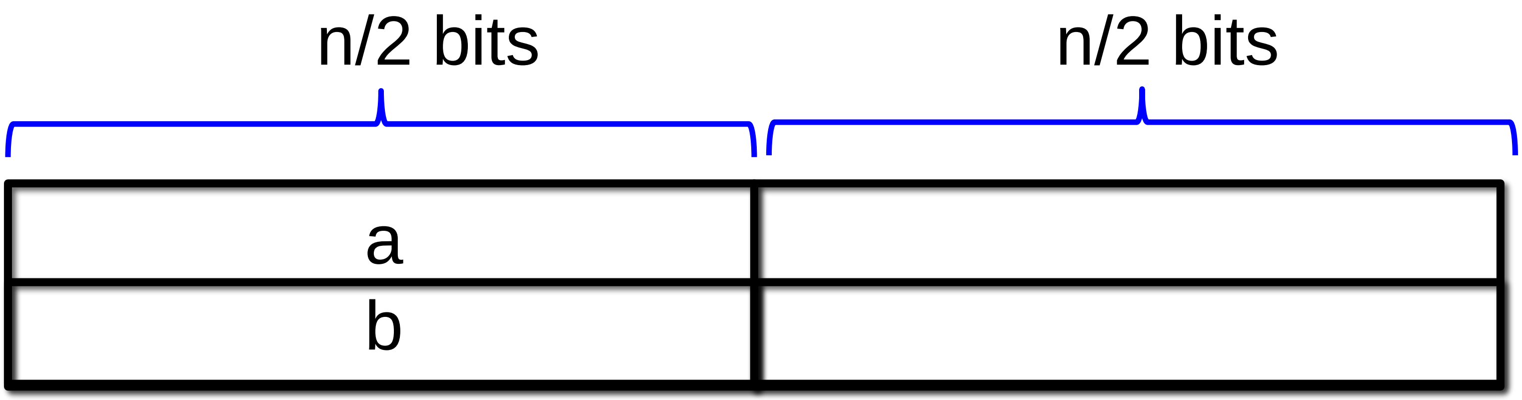

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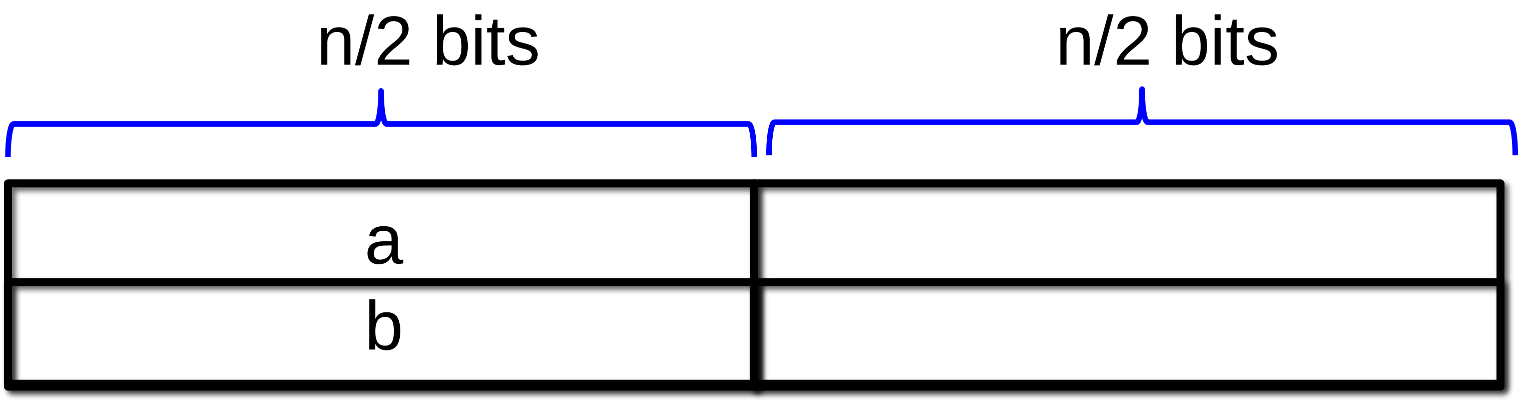

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- Generate a random partition (x, y, z) of n/2 and fill in x (0,0), y (0,1) and z (1,0) pairs

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n/2 bits	n/2 bits
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Claim: The protocol T' has bias 1/C for DISJ.

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- Run T on the n bit string, return the output

Claim: The protocol T' has bias 1/C for DISJ.

Claim: Yet flipping a pair (a_i, b_i) between (0,0), (0,1) or (1,0) results in two distributions p,q (on inputs to T) with $|p-q|_1 \leq 1/n$

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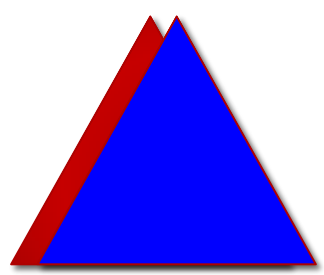
Proof:



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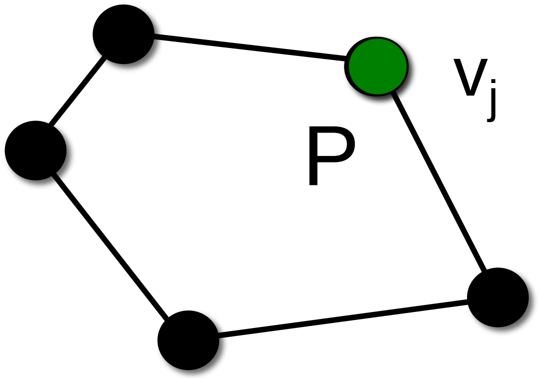
Thanks!



Any Questions?