# Nearly Complete Graphs Decomposable Into Large Induced Matchings

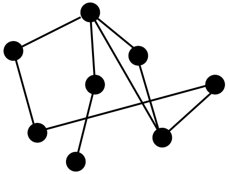
Ankur Moitra, IAS

joint with Noga Alon (IAS, TAU) and Benny Sudakov (UCLA)

May 22, 2012

# Rusza-Szemeredi Graphs

**Induced Matching:** an induced subgraph whose edges form a matching



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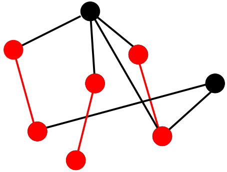

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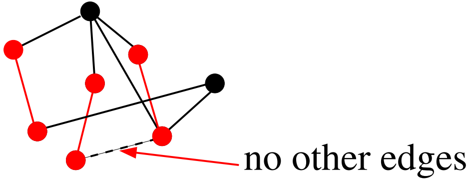

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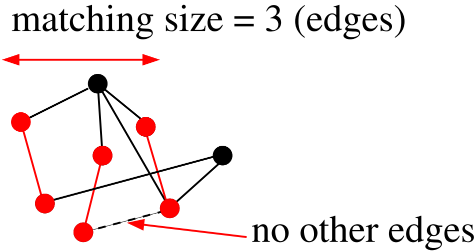

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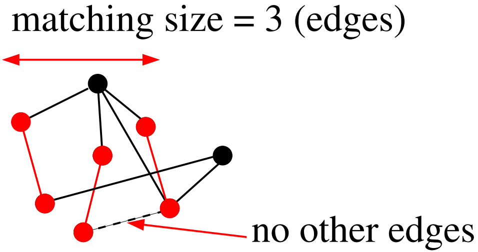


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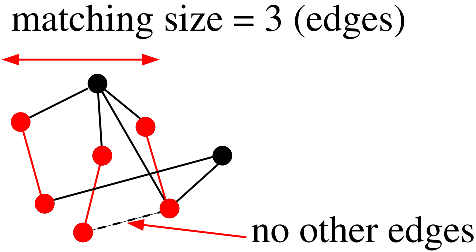


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**Note:** An induced matching of size $r \Rightarrow$ missing $r(r-1)$ edges

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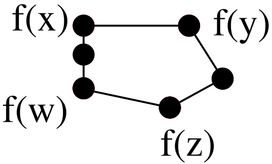

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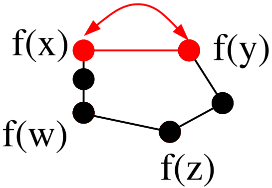

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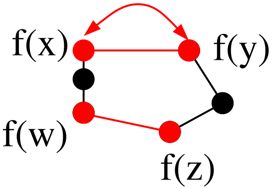


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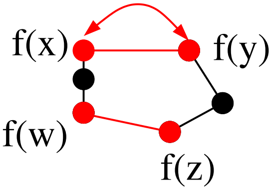


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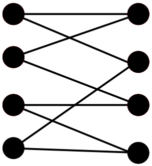

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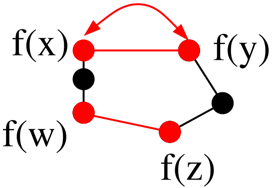


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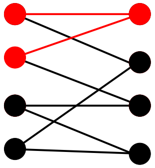

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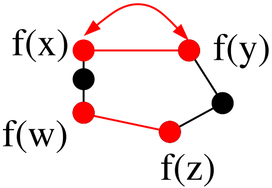


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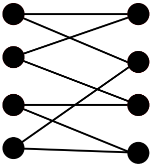

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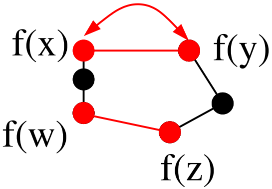


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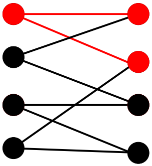

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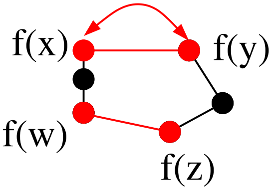


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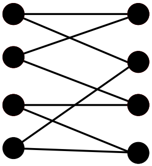

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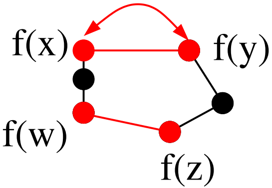


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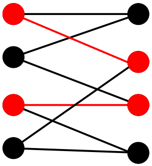

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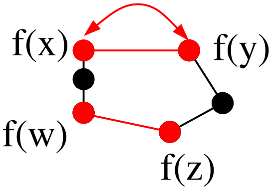


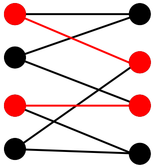

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**Additive Combinatorics:** arithmetic progressions

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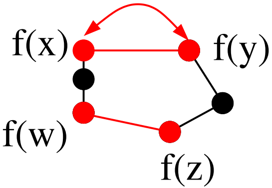


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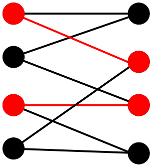

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**Question**

*Can Rusza-Szemeredi graphs be dense (for $r = N^{\Omega(1)}$)?*

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We give applications to linearity testing, routing in broadcast networks, and disprove conjectures of Meshulam and Vempala

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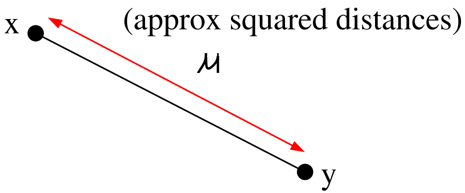

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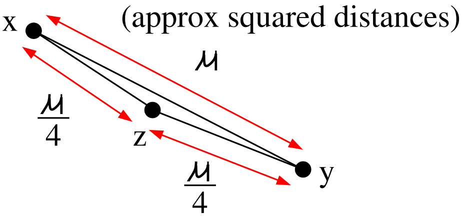

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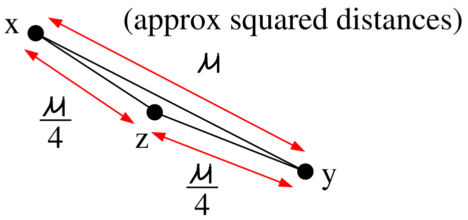

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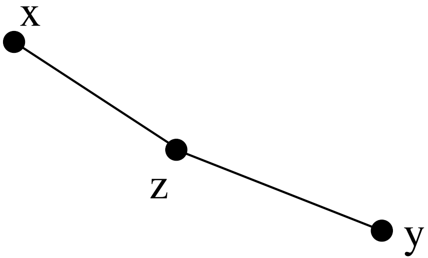

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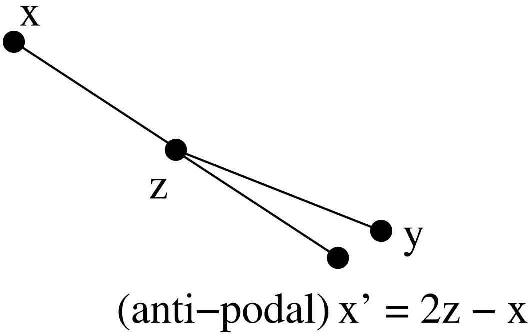

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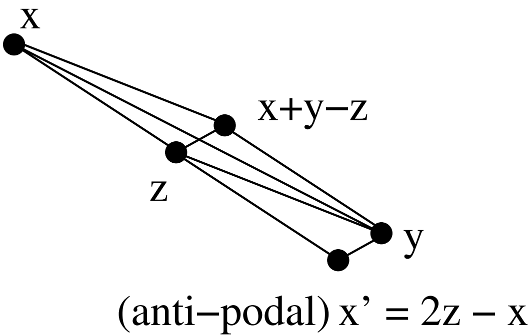

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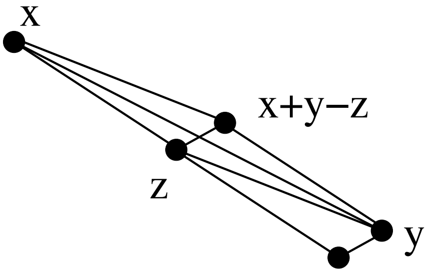


(anti–podal) x’ = 2z − x

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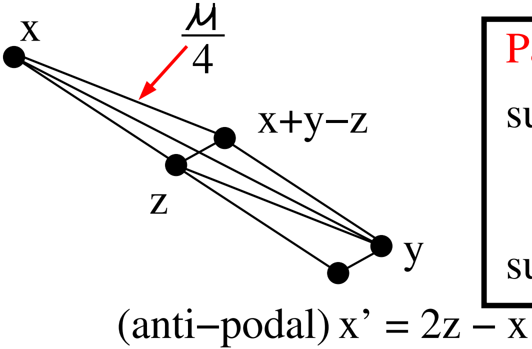


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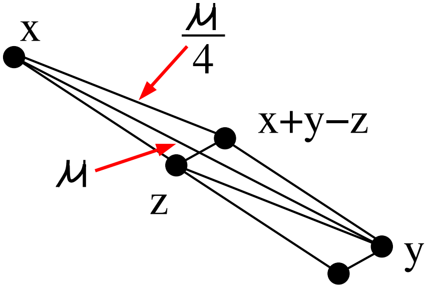


(anti–podal) $x' = 2z - x$

**Parallelogram Law:**

sum of squared sides

=

sum of squared diags

# Goal: Bound the Max Degree

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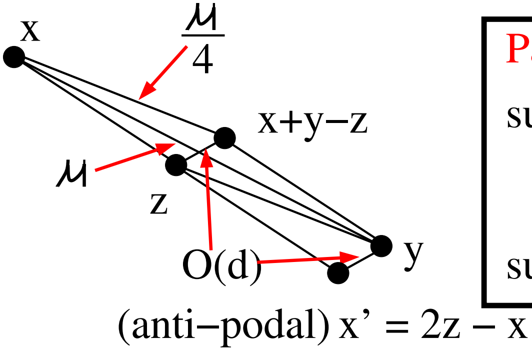


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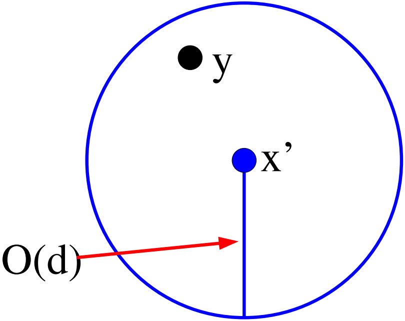
Degree of $x$ (in $G_z$) is bounded by number of $y$ close to $x'$:



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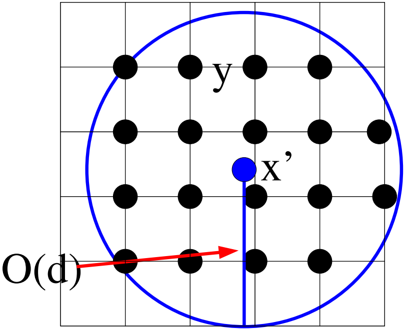

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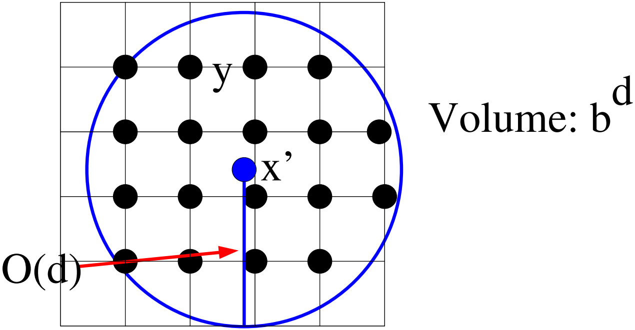

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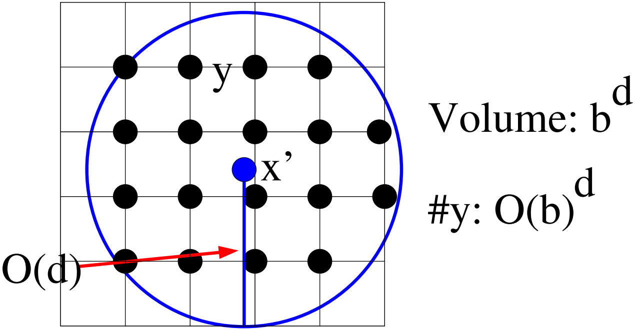
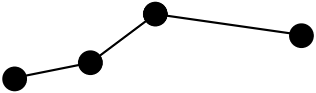


# Partitioning $G_z$ into Induced Matchings



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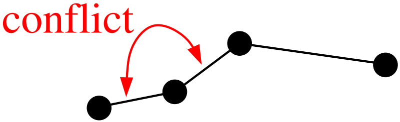

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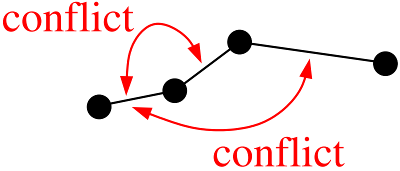

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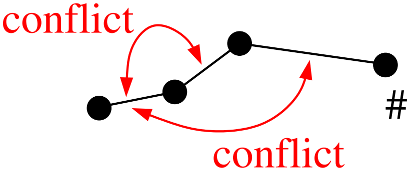


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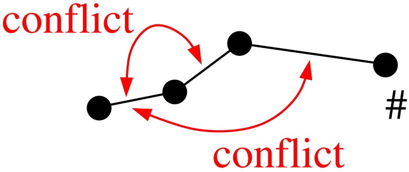


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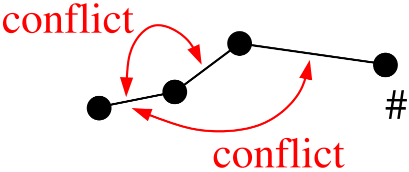


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Place each edge in first matching w/o conflicts

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- **Known:** The triangle removal lemma is **equivalent** to no $(r, t)$-RS graphs in certain ranges (e.g. no dense graphs with $t = N \log^* N$)

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Can our graphs be used to give an integrality gap for Directed Steiner Tree that is polynomial in the number of nodes?
(best so far is poly-logarithmic)

Questions?

Thanks!