Sum-of-Squares, with a View Towards Average-case Complexity

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KITP Tutorial, January 11, 2019
A CLASSIC HARD PROBLEM: MAXCUT

Goal: given a graph $G = (V, E)$:

find a cut $U \subseteq V$ that maximizes the number of crossing edges
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Simple $\frac{1}{2}$-approximation algorithm: Choose $U$ randomly.
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Simple $\frac{1}{2}$-approximation algorithm: Choose U randomly. But can we do better?
MAXCUT AS A QUADRATIC PROGRAM

We can also formulate MAXCUT as optimizing a polynomial, subject polynomial constraints:

$$\max \sum_{(i,j) \in E} (x_i - x_j)^2$$

s.t. $x_i^2 = x_i$ for all $i$
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Now we can leverage the **Sum-of-Squares (SOS) Hierarchy**...
# SUM-OF-SQUARES HIERARCHY

Introduced by [Parrilo ‘00], [Lasserre ‘01]

- strengthens **Sherali-Adams, Lovasz-Schrijver, LS+**
- breaks integrality gaps for other hierarchies [Barak et al, ‘12]
- highly successful convex relaxation
  - sparsest cut [ARV ‘04]
  - unique games [ABS ‘10], [BRS ‘12], [GS ‘12]
- optimal among all poly. sized SDPs for random CSPs [LRS ‘15]
- best known algorithm for several **average-case** problems
  - planted sparse vector, dictionary learning [BKS ‘14, ‘15]
  - noisy tensor completion [BM ‘15], tensor PCA [HSS ‘15]
OUTLINE

Part I: Introduction
  • MAXCUT and the Sum-of-Squares Hierarchy
  • A Dual View via Pseudo-expectation

Part II: Rounding SOS

Part III: Fooling SOS
  • Planted Clique and its Applications
  • The MPW Moments and Corrections
  • Pseudo-calibration and Fourier Analysis

Part IV: Sparse PCA and Computational vs. Statistical Tradeoffs

Part V: Equivalence with Spectral Methods
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A DUAL VIEW

(Usually introduced as proof system related to Hilbert’s 17th prob.)
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**Goal:** Find operator that behaves like the expectation over a distribution on solutions

\[ \mathcal{E} : \mathcal{P}_{n, \leq d} \rightarrow \mathbb{R} \]

degree \( \leq d \) polynomials in \( n \) variables
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(Usually introduced as proof system related to Hilbert’s 17\textsuperscript{th} prob.)

\textbf{Goal:} Find operator that behaves like the expectation over a distribution on solutions

\[
\tilde{E} : \mathcal{P}_n^{\leq d} \rightarrow \mathbb{R}
\]

degree \leq d polynomials in n variables

Called a \textbf{Pseudo-expectation}
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Called a **Pseudo-expectation**

Let’s see what it looks like for MAXCUT...
Degree d relaxation for MAXCUT:

\[ \max \tilde{E}[\sum_{(i,j) \in E} (x_i - x_j)^2] \]

such that:

1. \( \tilde{E} \) is linear
2. \( \tilde{E}[1] = 1 \)
3. \( \tilde{E}[p^2] \geq 0 \) for all \( \deg(p) \leq d/2 \)
4. \( \tilde{E}[x_i^2p] = \tilde{E}[x_ip] \) for all \( \deg(p) \leq d-2 \)
Degree $d$ relaxation for MAXCUT:

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(1) – (3) are the usual constraints that say $\tilde{E}$ behaves like it is taking the expectation under some distribution on assignments to the variables.
Degree d relaxation for MAXCUT:

$$\max \quad \mathbb{E}\left[\sum_{(i,j) \in E} (x_i - x_j)^2\right]$$

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(1) – (3) are the usual constraints that say $\tilde{\mathbb{E}}$ behaves like it is taking the expectation under some distribution on assignments to the variables

(4) is because we want the distribution to be supported on 0/1 valued assignments
Degree d relaxation for MAXCUT:

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such that:

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But why is this a relaxation for MAXCUT?
Degree d relaxation for MAXCUT:

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**Claim:** If there is a cut that has at least $k$ edges crossing, there is a feasible solution to (1) – (4) with objective value $\geq k$
Degree d relaxation for MAXCUT:

\[
\max \quad \tilde{\mathbb{E}}\left[ \sum_{(i,j) \in E} (x_i - x_j)^2 \right]
\]

such that:

1. \( \tilde{\mathbb{E}} \) is linear
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**Claim:** If there is a cut that has at least \( k \) edges crossing, there is a feasible solution to (1) – (4) with objective value \( \geq k \)

**Proof:** if \( a_1, a_2, \ldots, a_n \) is the indicator vector of the cut \( U \), set

\[
\tilde{\mathbb{E}}[p(x_1, x_2, \cdots, x_n)] = p(a_1, a_2, \cdots, a_n)
\]
Can we efficiently solve this relaxation?
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**Theorem:** There is an $n^{O(d)}$-time algorithm for finding such an operator, if it exists.
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It is a semidefinite program on a $n^{O(d)} \times n^{O(d)}$ matrix whose entries are the pseudo-expectation applied to monomials.
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How well does SOS approximate MAXCUT?
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APPORXIMATION ALGORITHMS FOR MAXCUT

Revolutionary work of [Goemans, Williamson]:

**Theorem:** There is a $\alpha_{GW}$-approximation algorithm for MAXCUT

$$\alpha_{GW} = \min_{-1 \leq \rho \leq 1} \frac{2 \arccos \rho}{(1-\rho)\pi} \geq 0.878$$

for MAXCUT
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We will give an alternate proof by rounding the degree two Sum-of-Squares relaxation
Main Question: How do you round a pseudo-expectation to find a cut?

I.e. if I give you \( \tilde{E} \) how do you find a cut with at least

\[
\alpha_{GW} \tilde{E} \left[ \sum_{(i,j) \in E} (x_i - x_j)^2 \right]
\]

edges crossing (in expectation)?
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Main Idea: Use a sample from a Gaussian distribution whose moments match the pseudo-moments
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edges crossing (in expectation)?

**Main Idea:** Use a sample from a Gaussian distribution whose moments match the pseudo-moments

**Aside:** Rounding higher degree relaxations is much harder b/c you cannot necc. find a r.v. whose moments match the pseudo-moments
Claim: Without loss of generality, can assume for all $i$

$$\mathbb{E}[x_i] = \frac{1}{2}$$
Claim: Without loss of generality, can assume for all $i$

$$\tilde{\mathbb{E}}[x_i] = \frac{1}{2}$$

Intuition: You can always change $U$ to $V \setminus U$ without changing the value of the cut, so WLOG $x_i$ has probability $1/2$ of being in $U$
GAUSSIAN ROUNDING

Let $y$ be a Gaussian vector with mean $\mu$ and covariance $\Sigma$ for

$$\mu = \mathbb{E}[x] \quad \text{and} \quad \Sigma = \mathbb{E}[(x - \mu)(x - \mu)^T]$$
GAUSSIAN ROUNding

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Now set $a_i = 0$ if $y_i \leq \frac{1}{2}$ and otherwise $a_i = 1$
GAUSSIAN ROUNING

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We will show that for each $(i, j)$ we have

$$\mathbb{E}[(a_i - a_j)^2] \geq \alpha_{GW} \mathbb{E}[(x_i - x_j)^2]$$

which, by linearity of expectation, will complete the proof
For each edge (i,j), calculate contribution to **objective value**:

\[
\tilde{E}[(x_i - x_j)^2] = \tilde{E}[x_i^2] - 2\tilde{E}[x_i x_j] + \tilde{E}[x_j^2]
\]
For each edge \((i,j)\), calculate contribution to **objective value**:

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For each edge \((i,j)\), calculate contribution to \textbf{objective value}:

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\tilde{\mathbb{E}}[(x_i - x_j)^2] = \tilde{\mathbb{E}}[x_i^2] - 2\tilde{\mathbb{E}}[x_i x_j] + \tilde{\mathbb{E}}[x_j^2]
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\[
= \tilde{\mathbb{E}}[x_i] - 2\tilde{\mathbb{E}}[x_i x_j] + \tilde{\mathbb{E}}[x_j]
\]

\[
= \frac{1}{2}(1 - \rho) \quad \text{for} \quad \rho = 4\tilde{\mathbb{E}}[x_i x_j] - 1
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And its contribution to the \textbf{expected number of edges crossing}:
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And its contribution to the expected number of edges crossing:

\[
\text{Var}(y_i) = \tilde{E}[(x_i - \frac{1}{2})^2] = \tilde{E}[x_i] - \frac{1}{4} = \frac{1}{4}
\]
For each edge \((i,j)\), calculate contribution to **objective value**:

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\]

Now we can compute:

\[
\mathbb{P}[a_i \neq a_j] = \mathbb{P}[\text{sgn}(s) \neq \text{sgn}(\rho s + \sqrt{1 - \rho^2} t)]
\]
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\]

\[
= \frac{\text{arccos} \rho}{\pi} \quad \text{independent std Gaussians}
\]
Putting it all together, we have for every edge \((i, j)\):

\[
\mathbb{P}[a_i \neq a_j] \geq \frac{2 \arccos \rho}{(1-\rho)\pi} \bar{\mathbb{E}}[(x_i - x_j)^2] \geq \alpha_{GW} \bar{\mathbb{E}}[(x_i - x_j)^2]
\]

which completes the proof
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PLANTED CLIQUE

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Can we find the planted clique?
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Can we find the planted clique?

And how large does $\omega$ need to be?
Quasi-polynomial time:

**Fact:** There is an $n^{O(\log n)}$-time algorithm (brute-force) that can find planted cliques of size $\omega \geq C \log n$, for any $C > 2$
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Polynomial time:

**Fact:** There is a polynomial time algorithm that succeeds (whp) for $\omega \geq C \sqrt{n \log n}$ (degree counting)
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**Polynomial time:**

**Fact:** There is a polynomial time algorithm that succeeds (whp) for $\omega \geq C \sqrt{n \log n}$ (degree counting)

**Theorem [Alon, Krivelevich, Sudakov]:** There is a polynomial time algorithm that succeeds (whp) for $\omega \geq C \sqrt{n}$ (spectral)
**Quasi-polynomial time:**

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**Theorem [Alon, Krivelevich, Sudakov]:** There is a polynomial time algorithm that succeeds (whp) for \( \omega \geq C \sqrt{n} \) (spectral)

**Theorem [Deshpande, Montanari]:** There is a nearly linear time algorithm that succeeds (whp) for \( \omega \geq \sqrt{n/e} \)
APPLICATIONS OF PLANTED CLIQUE

Planted Clique (and variants) are basic problems in average-case complexity, imply many other hardness results:
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Planted Clique (and variants) are basic problems in average-case complexity, imply many other hardness results:

- Discovering motifs in biological networks [Milo et al ‘02]
- Computing the best Nash Equilibrium [HK ‘11], [ABC ‘13]
- Property testing [Alon et al ‘07]
- Sparse PCA [Berthet, Rigollet ‘13]
- Compressed sensing [Koiran, Zouzias ‘14]
- Cryptography [Juels, Peinado ‘00], [Applebaum et al ‘10]
- Mathematical finance [Arora et al ‘10]
LOWER BOUNDS?

Is it *actually* hard to find $n^{1/2-\epsilon}$-sized planted cliques?
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Complexity-theoretic reasons lower bound are unlikely to be based on P vs. NP

  e.g. [Feigenbaum, Fortnow ’93], [Bogdanov, Trevisan ’06]
LOWER BOUNDS?

Is it actually hard to find $n^{1/2-\epsilon}$-sized planted cliques?

Complexity-theoretic reasons lower bound are unlikely to be based on **P vs. NP**

- e.g. [Feigenbaum, Fortnow ’93], [Bogdanov, Trevisan ’06]

Our best evidence seems to Sum-of-Squares lower bounds
Sum-of-Squares for planted clique:

1. $\tilde{E}$ is linear
2. $\tilde{E}[1] = 1$
3. $\tilde{E}[p^2] \geq 0$
   for all $\text{deg}(p) \leq d/2$

*general*
Sum-of-Squares for planted clique:

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2. $\tilde{\mathbb{E}}[1] = 1$
3. $\tilde{\mathbb{E}}[p^2] \geq 0$
   for all $\deg(p) \leq d/2$
4. $\tilde{\mathbb{E}}[x_i^2 p] = \tilde{\mathbb{E}}[x_i p]$ (booleanity)
Sum-of-Squares for planted clique:

(1) $\tilde{\mathbf{E}}$ is linear

(2) $\tilde{\mathbf{E}}[1] = 1$

(3) $\tilde{\mathbf{E}}[p^2] \geq 0$

for all $\text{deg}(p) \leq d/2$

(4) $\tilde{\mathbf{E}}[x_i^2p] = \tilde{\mathbf{E}}[x_ip]$}

(5) $\tilde{\mathbf{E}}[\sum x_i] = \omega$

(clique size)
Constraints on the pseudo-expectation:

1. \( \tilde{E} \) is linear
2. \( \tilde{E}[1] = 1 \)
3. \( \tilde{E}[p^2] \geq 0 \)
   for all \( \text{deg}(p) \leq d/2 \)
4. \( \tilde{E}[x_i^2 p] = \tilde{E}[x_i p] \)
5. \( \tilde{E}[\sum x_i] = \omega \)
6. \( \tilde{E}[x_i x_j p] = 0 \)
   for all \((i,j)\) not an edge

(clique constraints)
Constraints on the pseudo-expectation:

1. $\tilde{\mathbb{E}}$ is linear
2. $\tilde{\mathbb{E}}[1] = 1$
3. $\tilde{\mathbb{E}}[p^2] \geq 0$
   - for all $\deg(p) \leq d/2$
4. $\tilde{\mathbb{E}}[x_i^2 p] = \tilde{\mathbb{E}}[x_ip]$
5. $\tilde{\mathbb{E}}[\sum x_i] = \omega$
6. $\tilde{\mathbb{E}}[x_ix_j p] = 0$
   - for all $(i,j)$ not an edge

---

**general**

**specific to planted clique**
Constraints on the pseudo-expectation:

(1) \( \tilde{E} \) is linear

(2) \( \tilde{E}[1] = 1 \)

(3) \( \tilde{E}[p^2] \geq 0 \)

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---

Can SOS find \(n^\epsilon\)-sized planted cliques in polynomial time?
A STRONG LOWER BOUND

Nearly optimal lower bound against SOS, for the planted clique problem (via pseudo-Bayesian techniques):

**Theorem [Barak, Hopkins, Kelner, Kothari, Moitra, Potechin]:**
The integrality gap of the level $d$ Sum-of-Squares hierarchy is

$$n^{\frac{1}{2} - c \sqrt{d / \log n}}$$

for some constant $c > 0$

For any $d = o(\log n)$, the integrality gap is $n^{1/2 - o(1)}$
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For any $d = o(\log n)$, the integrality gap is $n^{1/2-o(1)}$

Builds on [Meka, Potechin, Wigderson ‘14], [Deshpande Montanari ‘15], [Hopkins, Kothari, Potechin, Raghavendra, Scrhamm ‘16]
A STRONG LOWER BOUND

Our Approach: **Pseudo-calibration**

New insights into what makes SOS powerful, and how to fool it
A STRONG LOWER BOUND

Our Approach: **Pseudo-calibration**

New insights into what makes SOS powerful, and how to fool it

When our *recipe* fails, it often yields spectral algorithms
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PSEUDO-MOMENTS

How can we fool the SOS algorithm into thinking there is a $n^{1/2-o(1)}$ sized clique in $G(n, 1/2)$?
PSEUDO-MOMENTS

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**Usual Approach:** Adapt integrality gaps from weaker hierarchies
PSEUDO-MOMENTS

How can we fool the SOS algorithm into thinking there is a $n^{1/2-o(1)}$ sized clique in $G(n, ½)$?

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This works for random CSPs
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**Usual Approach:** Adapt integrality gaps from weaker hierarchies

This works for random CSPs

**Theorem [Feige, Krauthgamer]:** The integrality gap of the level $d$ LS+ hierarchy is

$$\sqrt{\frac{n}{2^d}}$$
Theorem [Meka, Potechin, Wigderson]: The integrality gap of the level \(d\) Sum-of-Squares hierarchy is

\[ n^{1/d-o(1)} \]
Theorem [Meka, Potechin, Wigderson]: The integrality gap of the level $d$ Sum-of-Squares hierarchy is

$$n^{1/d-o(1)}$$

In particular, set:

$$\tilde{E}_{MPW}\left[ \prod_{i \in A} x_i \right] = 2^{\left( \frac{|A|}{2} \right)} \left( \frac{\omega}{n} \right)^{|A|}$$

if $A$ is clique, zero otherwise.
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In particular, set:

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if $A$ is clique, zero otherwise. Extend by linearity to all $\deg(p) \leq d$

**Approach:** Spectral bounds on *locally random matrices*
Theorem [Meka, Potechin, Wigderson]: The integrality gap of the level $d$ Sum-of-Squares hierarchy is

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Improved analysis due to [Deshpande, Montanari], for $d = 4$

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\[ n^{1/([d/2]+1)-o(1)} \]

But these bounds are tight (for these moments)
KELNER’S POLYNOMIAL

Do the MPW moments work beyond $n^{1/\left(\lceil a/2 \rceil + 1\right)}$?
KELNER’S POLYNOMIAL

Do the MPW moments work beyond $n^{1/(\lceil a/2 \rceil + 1)}$?

Set

$$G_{i,j} = \begin{cases} 
+1 & \text{if (i,j) an edge} \\
-1 & \text{else} 
\end{cases}$$

$$P_{G,i} = \left( \sum_j G_{i,j} x_j \right)^{\ell}$$
KELNER’S POLYNOMIAL

Do the MPW moments work beyond $n^{1/\left\lceil \frac{a}{2} \right\rceil + 1}$?

Set $G_{i,j} = \begin{cases} +1 & \text{if } (i,j) \text{ an edge} \\ -1 & \text{else} \end{cases}$

$$P_{G,i} = \left( \sum_j G_{i,j} x_j \right)^\ell$$

If there is an $\omega$-sized planted clique:

$$\mathbb{E}[P_{G,i}^2] \geq \left( \frac{\omega}{n} \right) \omega^{2\ell}$$

$$(G, x) \leftarrow G(n, 1/2, \omega)$$
**KELNER’S POLYNOMIAL**

Do the MPW moments work beyond $n^{1/([\alpha/2] + 1)}$?

Set $G_{i,j} = \begin{cases} +1 & \text{if } (i,j) \text{ an edge} \\ -1 & \text{else} \end{cases}$

$$P_{G,i} = \left( \sum_j G_{i,j} x_j \right)^\ell$$

If there is an $\omega$-sized planted clique:

$$\mathbb{E}[P_{G,i}^2] \geq \left( \frac{\omega}{n} \right) \omega^{2\ell}$$

But if $G$ is sampled from $G(n, \frac{1}{2})$:

$$\mathbb{E}[\mathbb{E}_{MPW}[P_{G,i}^2]] \leq n^\ell \left( \frac{\omega}{n} \right)^\ell = \omega^\ell$$
KELNER’S POLYNOMIAL

Do the MPW moments work beyond \( n^{1/([\alpha/2]+1)} \)?

Set \( G_{i,j} = \begin{cases} +1 & \text{if (i,j) an edge} \\ -1 & \text{else} \end{cases} \)

\[
PG_i = \left( \sum_j G_{i,j} x_j \right) \ell
\]

If there is an \( \omega \)-sized planted clique:

\[
\mathbb{E}[P^2_{G,i}] \geq \left( \frac{\omega}{n} \right) \omega^{2\ell}
\]

But if G is sampled from \( G(n, \frac{1}{2}) \):

\[
\mathbb{E}[\mathbb{E}_{\text{MPW}}[P^2_{G,i}]] \leq (n^{\ell}) \left( \frac{\omega}{n} \right)^\ell = \omega^{\ell}
\]

**Need:** \( \omega \leq n^{1/((\ell+1)} = n^{1/(d/2+1)} \) otherwise something is wrong
KELNER’S POLYNOMIAL

Do the MPW moments work beyond $n^{1/(\lceil a/\sigma \rceil + 1)}$?
KELNER’S POLYNOMIAL

Do the MPW moments work beyond $n^{1/(\lceil d/2 \rceil + 1)}$?

This example can be used to find a squared polynomial whose pseudo-expectation is negative for $\omega > n^{1/(\lceil d/2 \rceil + 1)}$

$$\widehat{\mathbb{E}}_{MPW}[P^2] < 0$$
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**Intuition:** A good pseudo-expectation attempts to hide info about what vertices participate in the planted clique
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$$\tilde{\mathbb{E}}_{MPW}[P^2] < 0$$

**Intuition:** A good pseudo-expectation attempts to hide info about what vertices participate in the planted clique

But vertices with a **standard deviation higher degree**, should be a constant factor more likely to be in the p.c. (soft constraint)
FIXING THE MPW-MOMENTS

This family of polynomials is essentially the only thing that goes wrong at $d = 4$

**Theorem [Hopkins et al.], [Raghavendra, Schramm]:** The integrality gap of the level 4 Sum-of-Squares hierarchy is

$$n^{1/2-o(1)}$$
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**Approach:** Add an explicit correction term of fix all $P_{G,i}$’s, even more dependent random matrix theory
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**Is there a fix for higher degrees?**
FIXING THE MPW-MOMENTS

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It turns out for \( d = 6 \), even the fixes need fixes, and on and on...
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PART V: EQUIVALENCE WITH SPECTRAL METHODS
Can we find pseudo-moments that satisfy the following:

$$
\mathbb{E}[\mathbb{E}[f(G, x)]] = \mathbb{E}[f(G, x)]
$$

for all *simple* functions $f$?
Can we find pseudo-moments that satisfy the following:

\[ \mathbb{E}[\mathbb{E}[f(G, x)]] = \mathbb{E}[f(G, x)] \]

for all polynomials \( f \) that are low-degree in \( G_{i,j} \)'s and \( x_i \)'s?
Consider the pseudo-expectation of some monomial:

$$\hat{\mathbb{E}}[x_A] : G \rightarrow \mathbb{R}, \text{ and let } \chi_T(G) = \prod_{(i,j) \in T} G_{i,j}$$
Consider the pseudo-expectation of some monomial:

\[ \tilde{\mathbb{E}}[x_A] : G \to \mathbb{R}, \text{ and let } \chi_T(G) = \prod_{(i,j) \in T} G_{i,j} \]

We can write any such function in terms of its Fourier expansion

\[ \tilde{\mathbb{E}}[x_A](G) = \sum_{T \subseteq ([n]/2)} \tilde{\mathbb{E}}[x_A](T) \chi_T(G) \]
Consider the pseudo-expectation of some monomial:

\[ \tilde{E}[x_A] : G \to \mathbb{R}, \text{ and let } \chi_T(G) = \prod_{(i,j) \in T} G_{i,j} \]

We can write any such function in terms of its **Fourier expansion**

\[ \tilde{E}[x_A](G) = \sum_{T \subseteq \left[ \frac{n}{2} \right]} \tilde{E}[x_A](T) \chi_T(G) \]

**How should we set the Fourier coefficients?**
The Fourier coefficients are chosen for us, by pseudo-calibration.
Utilizing the expression

\[
\hat{E}[x_A](G) = \sum_{T \subseteq ([n]/2)} \hat{E}[x_A](T) \chi_T(G)
\]

we can calculate:

\[
\mathbb{E}[\mathbb{E}[x_A \chi_T(G)]]
\]

\( G \leftarrow G(n, 1/2) \)
Utilizing the expression

$$\widetilde{\mathbb{E}}[x_A](G) = \sum_{T \subseteq \binom{[n]}{2}} \widetilde{\mathbb{E}}[x_A](T) \chi_T(G)$$

we can calculate:

$$\mathbb{E}[\widetilde{\mathbb{E}}[x_A] \chi_T(G)] \quad \text{(by linearity)}$$

$$G \leftarrow G(n, 1/2)$$
The Fourier coefficients are chosen for us, by pseudo-calibration.

Utilizing the expression

$$\hat{E}[x_A](G) = \sum_{T \subseteq \binom{n}{2}} \hat{E}[x_A](T) \chi_T(G)$$

we can calculate:

$$\mathbb{E}[\hat{E}[x_A]\chi_T(G)] = \sum_{T' \subseteq \binom{n}{2}} \hat{E}[x_A](T') \mathbb{E}[\chi_T(G)\chi_{T'}(G)]$$
Utilizing the expression

$$\tilde{\mathbb{E}}[x_A](G) = \sum_{T \subseteq ([n]/2)} \tilde{\mathbb{E}}[x_A](T)\chi_T(G)$$

we can calculate:

$$\mathbb{E}[\tilde{\mathbb{E}}[x_A]\chi_T(G)] = \sum_{T' \subseteq ([n]/2)} \tilde{\mathbb{E}}[x_A](T')\mathbb{E}[\chi_T(G)\chi_{T'}(G)]$$

$$= \begin{cases} 
+1 & \text{if } T = T' \\
0 & \text{else}
\end{cases}$$
Utilizing the expression

$$\tilde{E}[x_A](G) = \sum_{T \subseteq \binom{[n]}{2}} \tilde{E}[x_A](T) \chi_T(G)$$

we can calculate:

$$\mathbb{E}[\tilde{E}[x_A x_T(G)]] = \tilde{E}[x_A](T)$$
The Fourier coefficients are chosen for us, by pseudo-calibration

Utilizing the expression

\[ \tilde{\mathbb{E}}[x_A](G) = \sum_{T \subseteq ([n]/2)} \tilde{\mathbb{E}}[x_A](T) \chi_T(G) \]

we can calculate:

\[ \mathbb{E}[\tilde{\mathbb{E}}[x_A \chi_T(G)]] = \tilde{\mathbb{E}}[x_A](T) \]

\[ \triangleq \mathbb{E}[x_A \chi_T(G)] \]
Utilizing the expression

$$\tilde{\mathbb{E}}[x_A](G) = \sum_{T \subseteq ([n]/2)} \tilde{\mathbb{E}}[x_A](T) \chi_T(G)$$

we can calculate:

$$\mathbb{E}[\tilde{\mathbb{E}}[x_A \chi_T(G)]] = \tilde{\mathbb{E}}[x_A](T)$$

vertices of $T$

pseudo-calibration

$$\Delta \mathbb{E}[x_A \chi_T(G)] = \left(\frac{\omega}{n}\right)|V(T) \cup A|$$
Utilizing the expression

$$\tilde{\mathbb{E}}[x_A](G) = \sum_{T \subseteq \binom{n}{2}} \tilde{\mathbb{E}}[x_A](T) \chi_T(G)$$

we can calculate:

$$\mathbb{E}[\tilde{\mathbb{E}}[x_A \chi_T(G)]] = \tilde{\mathbb{E}}[x_A](T)$$

$$\Delta \mathbb{E}[x_A \chi_T(G)] = \left( \frac{\omega}{n} \right)^{|V(T) \cup A|}$$

It turns out, we need to **truncate** but at what degree?
Our pseudo-moments are:

\[
\tilde{\mathbb{E}}[x_A] = \sum_{T \subseteq [n]} \left( \frac{n}{\omega} \right)^{|V(T) \cup A|} |V(T) \cup A| \chi_T(G)
\]

where \(|V(T) \cup A| \leq \tau\).
Our pseudo-moments are:

$$\tilde{E}[x_A] = \sum_{T \subseteq ([n]/2)} \left( \frac{\omega}{n} \right)^{|V(T) \cup A|} \chi_T(G)$$

with

$$|V(T) \cup A| \leq \tau$$

**Lemma:** With high probability,

$$|\tilde{E}[1] - 1| \leq \tau \max_{t \leq \tau} 2^{t^2} \left( \frac{\omega}{\sqrt{n}} \right)^t$$
Our pseudo-moments are:

$$\tilde{\mathbb{E}}[x_A] = \sum_{T \subseteq \binom{[n]}{2}} \binom{\omega}{n} |V(T) \cup A| \chi_T(G)$$

$$|V(T) \cup A| \leq \tau$$

**Lemma:** With high probability,

$$|\tilde{\mathbb{E}}[1] - 1| \leq \tau \max_{t \leq \tau} 2^{t^2} \left( \frac{\omega}{\sqrt{n}} \right)^t$$

(1) This is why we need to truncate
Our pseudo-moments are:

\[ \tilde{\mathbb{E}}[x_A] = \sum_{T \subseteq \binom{[n]}{2}} (\frac{\omega}{n})^{|V(T) \cup A|} \chi_T(G) \]

\[ |V(T) \cup A| \leq \tau \]

**Lemma:** With high probability,

\[ |\tilde{\mathbb{E}}[1] - 1| \leq \tau \max_{t \leq \tau} 2^{t^2} \left( \frac{\omega}{\sqrt{n}} \right)^t \]

\[ n^{-\Omega(\epsilon)} \]

(2) is small enough for any \( \omega \leq n^{1/2-\epsilon} \) for \( \tau \leq \frac{\epsilon}{2} \log n \)
Our pseudo-moments are:

\[
\tilde{E}[x_A] = \sum_{T \subseteq \binom{[n]}{2}} \left( \frac{\omega}{n} \right)^{|V(T) \cup A|} \chi_T(G)
\]

\[|V(T) \cup A| \leq \tau\]

**Lemma:** With high probability,

\[
|\tilde{E}[1] - 1| \leq \tau \max_{t \leq \tau} 2^{t^2} \left( \frac{\omega}{\sqrt{n}} \right)^t
\]

(3) Can always renormalize pseudo-expectation so \(\tilde{E}[1] = 1\)
Our pseudo-moments are:

\[
\tilde{\mathbb{E}}[x_A] = \sum_{T \subseteq \binom{[n]}{2}} \left( \frac{\omega}{n} \right)^{|V(T) \cup A|} \chi_T(G)
\]

where \(|V(T) \cup A| \leq \tau\).

**Lemma**: With high probability,

\[
|\tilde{\mathbb{E}}[1] - 1| \leq \tau \max_{t \leq \tau} 2^{t^2} \left( \frac{\omega}{\sqrt{n}} \right)^t
\]

(4) Similar bound holds (again by standard concentration) for

\[
\tilde{\mathbb{E}}\left[ \sum_i x_i \right] = \omega \left( 1 \pm n^{-\Omega(\epsilon)} \right)
\]
Our pseudo-moments are:

\[ \tilde{E}[x_A] = \sum_{T \subseteq \binom{[n]}{2}} \left( \frac{\omega}{n} \right)^{|V(T) \cup A|} |V(T) \cup A| \chi_T(G) \]

\[ |V(T) \cup A| \leq \tau \]
Our pseudo-moments are:

$$\tilde{\mathbb{E}}[x_A] = \sum_{T \subseteq \binom{[n]}{2}, |V(T) \cup A| \leq \tau} \left( \frac{\omega}{n} \right)^{|V(T) \cup A|} \chi_T(G)$$

**Lemma**: If $A$ is not a clique then

$$\tilde{\mathbb{E}}[x_A] = 0$$
Our pseudo-moments are:

\[ \tilde{\mathbb{E}}[x_A] = \sum_{T \subseteq \binom{[n]}{2}} \left( \frac{\omega}{n} \right)^{|V(T) \cup A|} \chi_T(G) \]

\[ |V(T) \cup A| \leq \tau \]

**Lemma:** If A is not a clique then

\[ \tilde{\mathbb{E}}[x_A] = 0 \]

Follows from Fourier expansion of AND, and grouping terms
Our pseudo-moments are:

\[
\widetilde{E}[X_A] = \sum_{T \subseteq \binom{[n]}{2}} \left( \frac{n}{\omega} \right)^{|V(T) \cup A|} \chi_T(G)
\]

where \(|V(T) \cup A| \leq \tau\)

**Lemma:** If A is not a clique then

\[
\widetilde{E}[X_A] = 0
\]

Follows from Fourier expansion of AND, and grouping terms

This is why we use \(|V(T) \cup A| \leq \tau\) for truncation
Our pseudo-moments are:

$$\tilde{\mathbb{E}}[x_A] = \sum_{\substack{T \subseteq [n] \backslash 2 \leq N \leq \tau}} \binom{\omega}{n} |V(T) \cup A| \chi_T(G)$$
Our pseudo-moments are:

\[ \widetilde{\mathbb{E}}[x_A] = \sum_{T \subseteq \{n\} \atop |V(T) \cup A| \leq \tau} \left( \frac{\omega}{n} \right)^{|V(T) \cup A|} \chi_T(G) \]

**Lemma:** Let \( f_G(x) = \sum_{|S| \leq 2d} c_A(G') x_A \) where \( \text{deg}(c_A) \leq \tau \), then

\[
\mathbb{E}[\widetilde{\mathbb{E}}[f_G(x)]] = \mathbb{E}[f_G(x)]
\]
What about proving positivity? e.g. $\mathbb{E}[p^2] \geq 0$
What about proving positivity? e.g. $\tilde{\mathbb{E}}[p^2] \geq 0$

This step is *by far* the most challenging (*as usual*)
What about proving positivity?

e.g. $\widetilde{E}[p^2] \geq 0$

This step is by far the most challenging (as usual)

Interestingly it is much easier to show that

$$\mathbb{E}[\widetilde{E}[p^2]] \geq 0$$

$G \leftarrow G(n, 1/2)$
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Part V: Equivalence with Spectral Methods
SPARSE PRINCIPAL COMPONENT ANALYSIS

Goal: Given samples $X_1, X_2, \ldots, X_n \in \mathbb{R}^d$ from

$$\mathcal{N}(0, I + \theta vv^T)$$

spiked covariance model

where $v$ is $k$-sparse and its nonzero entries are $\pm 1/\sqrt{k}$
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where $v$ is $k$-sparse and its nonzero entries are $\pm 1/\sqrt{k}$

How large does the signal parameter $\theta$ need to be to detect the spike?
**Theorem:** There is a $d^{O(k)}$-time algorithm (brute-force) that can detect the spike (with failure probability $\delta$) when

$$\theta \geq \sqrt{\frac{k \log d}{\delta n}}$$
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**Theorem:** There is a polynomial time algorithm that can detect the spike (with failure probability $\delta$) when

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Select the $k$ largest entries along the diagonal of the empirical covariance matrix
In an influential paper, [Berthet, Rigollet] showed:
LOWER BOUNDS FROM PLANTED CLIQUE

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Theorem: Assuming that there is no polynomial time algorithm for finding a planted clique of size

\[ k = n^{1/2 - \epsilon} \]

for any \( \epsilon > 0 \) then there is no polynomial time algorithm for subgaussian sparse PCA with

\[ \sqrt{\frac{k^\alpha}{n}} \leq \theta \leq \sqrt{\frac{k^2 \log d}{n}} \]

for any \( 1 \leq \alpha < 2 \) that succeeds with constant probability.
DISCUSSION

Their reduction leaves open the following possibility:

Is there a quasi-polynomial time algorithm for detecting a spike in sparse PCA for much smaller values of $\theta$?
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Evidence for average-case complexity without reductions!
OUTLINE

Part I: Introduction

• MAXCUT and the Sum-of-Squares Hierarchy
• A Dual View via Pseudo-expectation

Part II: Rounding SOS

Part III: Fooling SOS

• Planted Clique and its Applications
• The MPW Moments and Corrections
• Pseudo-calibration and Fourier Analysis

Part IV: Sparse PCA and Computational vs. Statistical Tradeoffs

Part V: Equivalence with Spectral Methods
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A SPECTRAL CHARACTERIZATION

Is SOS only as powerful as low degree polynomials?
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**Theorem [Hopkins et al.]:** Suppose degree $d$ SOS can distinguish between planted and unplanted instances and that the problem is resilient to rerandomizing most coordinates.

Then there is an $n^{O(d)} \times n^{O(d)}$ matrix $Q$ whose entries are degree $O(d)$ polynomials in the instance variables where

\[(1) \quad \mathbb{E}_{\mathcal{I} \sim \text{unplanted}}[\lambda^+(Q(\mathcal{I}))] \leq 1\]

\[(2) \quad \mathbb{E}_{\mathcal{I} \sim \text{planted}}[\lambda^+(Q(\mathcal{I}))] \geq n^{10d}\]
OPEN QUESTIONS

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Can you prove SOS lower bounds for community detection beneath the Kesten-Stigum bound?

Can tools from random graph theory/statistics (e.g. small subgraph conditioning method, contiguity) be useful?
Summary:

• Sum-of-Squares hierarchy as a relaxation for polynomial optimization

• Upper bounds for MAXCUT and lower bounds for planted clique

• Lower bounds as a form of evidence for average-case hardness, computational vs. statistical gaps
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Thanks! Any Questions?