A Polynomial Time Algorithm for Lossy Population Recovery

Ankur Moitra
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joint work with Mike Saks

A Story…

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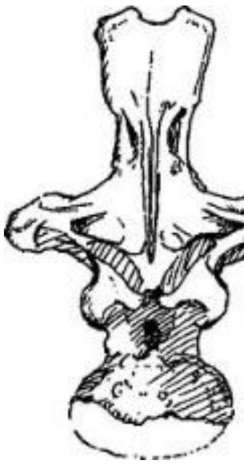

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features (n)

species (k)

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1	0	1	1	0	1	0	1	0	0
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	⋮				⋮			⋮	
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p_1	1	0	1	1	0	1	0	1	0	0
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$\vdots$		$\vdots$				$\vdots$			$\vdots$	
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samples:	?	1	?	?	?	0	?	?	?	?

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Is there a polynomial time algorithm for any fixed μ>0?

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Theorem [folk]: There is a quasi-polynomial time algorithm to PAC learn DNFs under the **uniform distribution**

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Is there a natural **grey-box** model? Can we design better algorithms?

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Population Recovery → Learning DNFs in Restriction Access

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Corollary: There is a polynomial time algorithm for learning DNFs in the Restriction Access Model for any $\mu > 0$.

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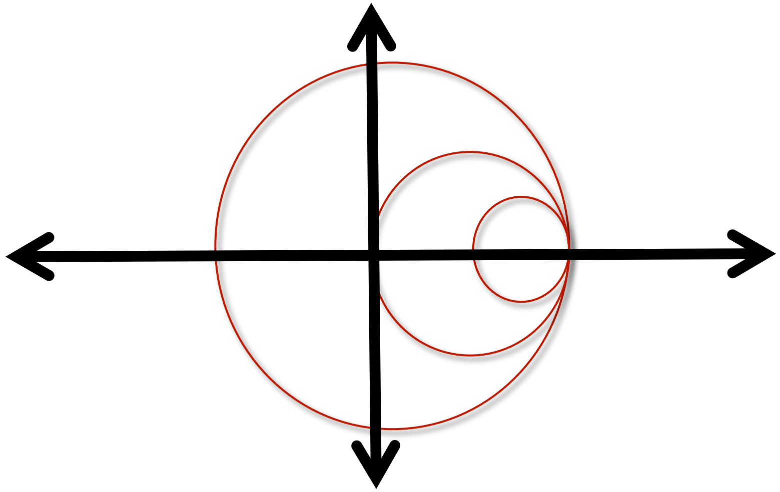

Complex Analysis:

uncertainty principles…

Two Reductions (Dvir et al)

Claim: We can assume we know the strings a_1, a_2, …, a_k (all we need is to find p_1, p_2, …, p_k)

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Suppose we had an algorithm for population recovery when the strings $a_1, a_2, \ldots, a_k$ are known:

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Suppose we had an algorithm for population recovery when the strings $a_1, a_2, \ldots, a_k$ are known:

p_1	$\boxed{1}$
p_2	$\boxed{0}$
$\vdots$	$\vdots$
p_k	$\boxed{1}$

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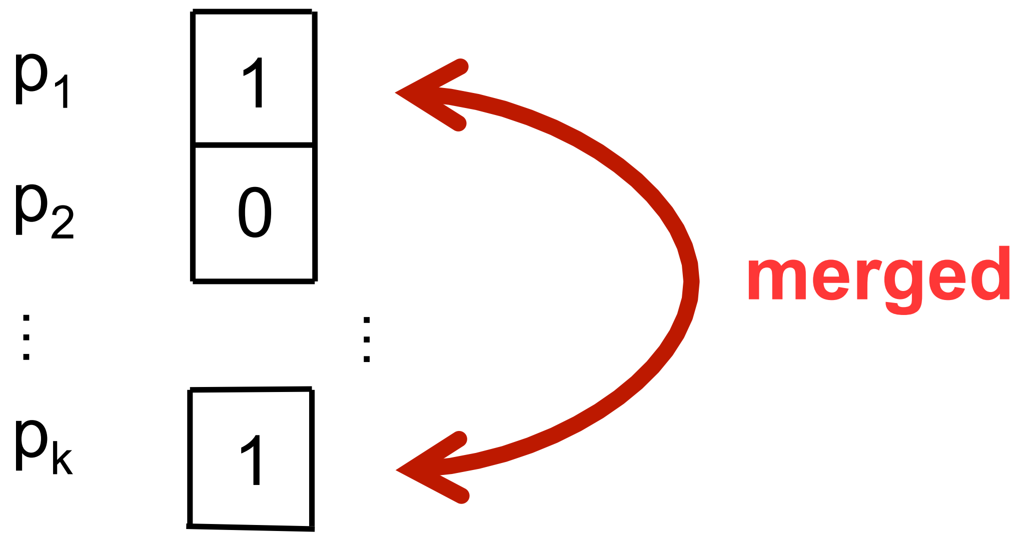

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p_2	0	1
$\vdots$		$\vdots$
p_k	1	1

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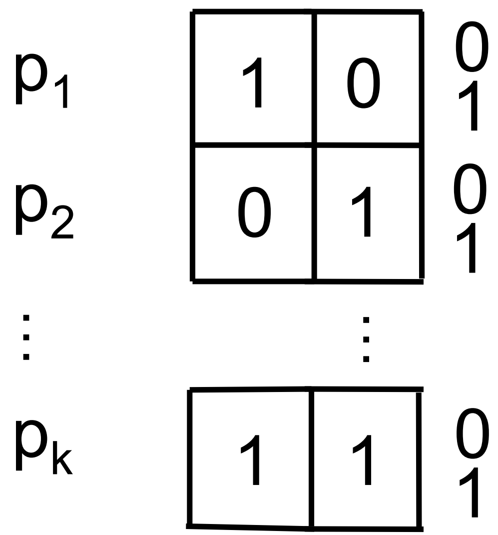

and so on…

Two Reductions (Dvir et al)

p_1	1	0	1	1	0	1	0	1	0	0
p_2	0	1	1	1	0	0	1	1	0	0
$\vdots$		$\vdots$				$\vdots$			$\vdots$	
p_k	1	1	1	1	0	0	0	1	0	0

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Claim: We just need to learn p_i for the all zero string

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E.g. we can **XOR** with a_1

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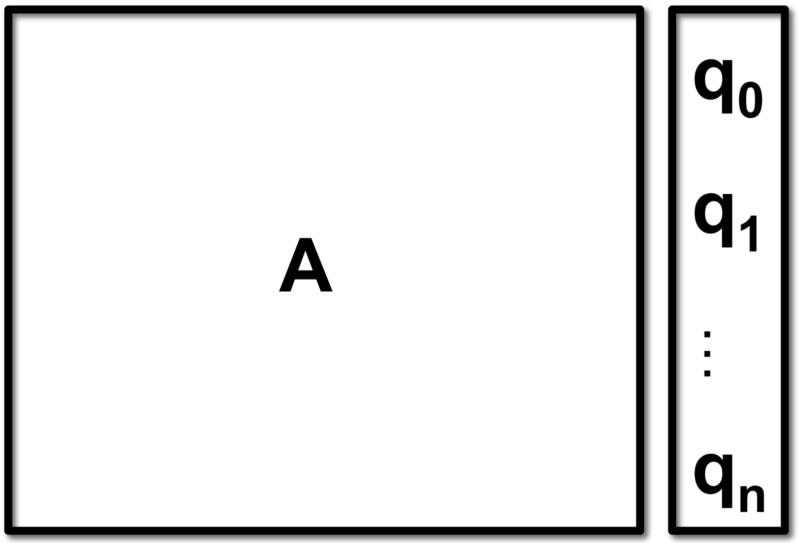

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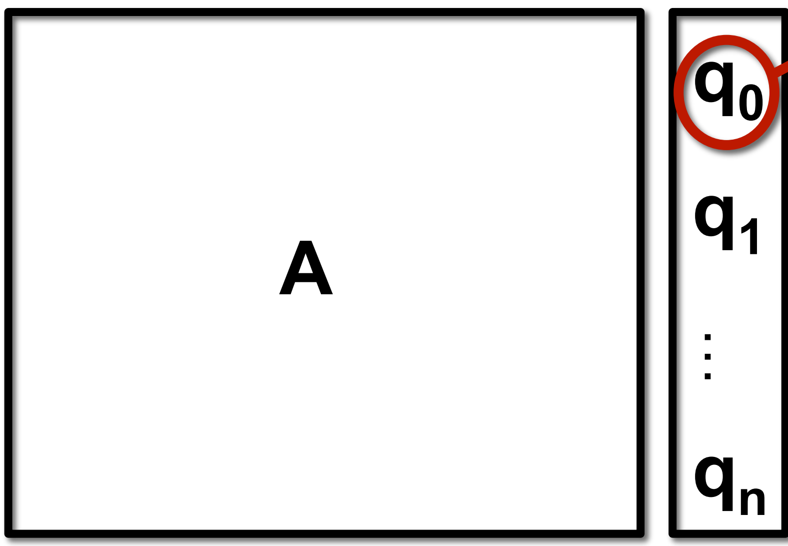

probability of all zero string

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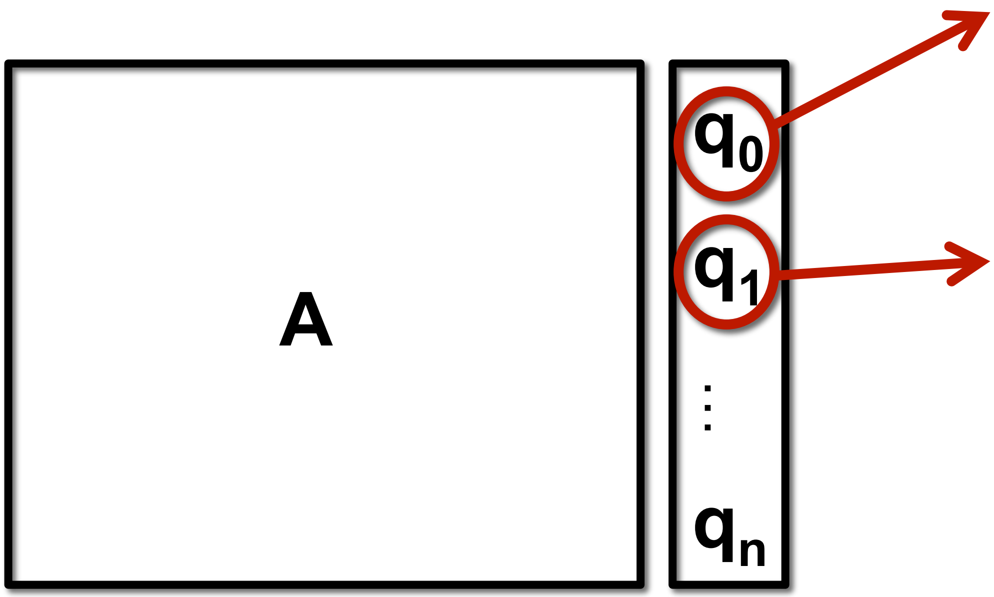

probability of all zero string

total probability of strings with one ‘1’

⋮

i.e. $q_1 = \Sigma_{i \text{ in } S}\, p_i$ for $S = \{i \mid a_i$ has one ‘1’$\}$

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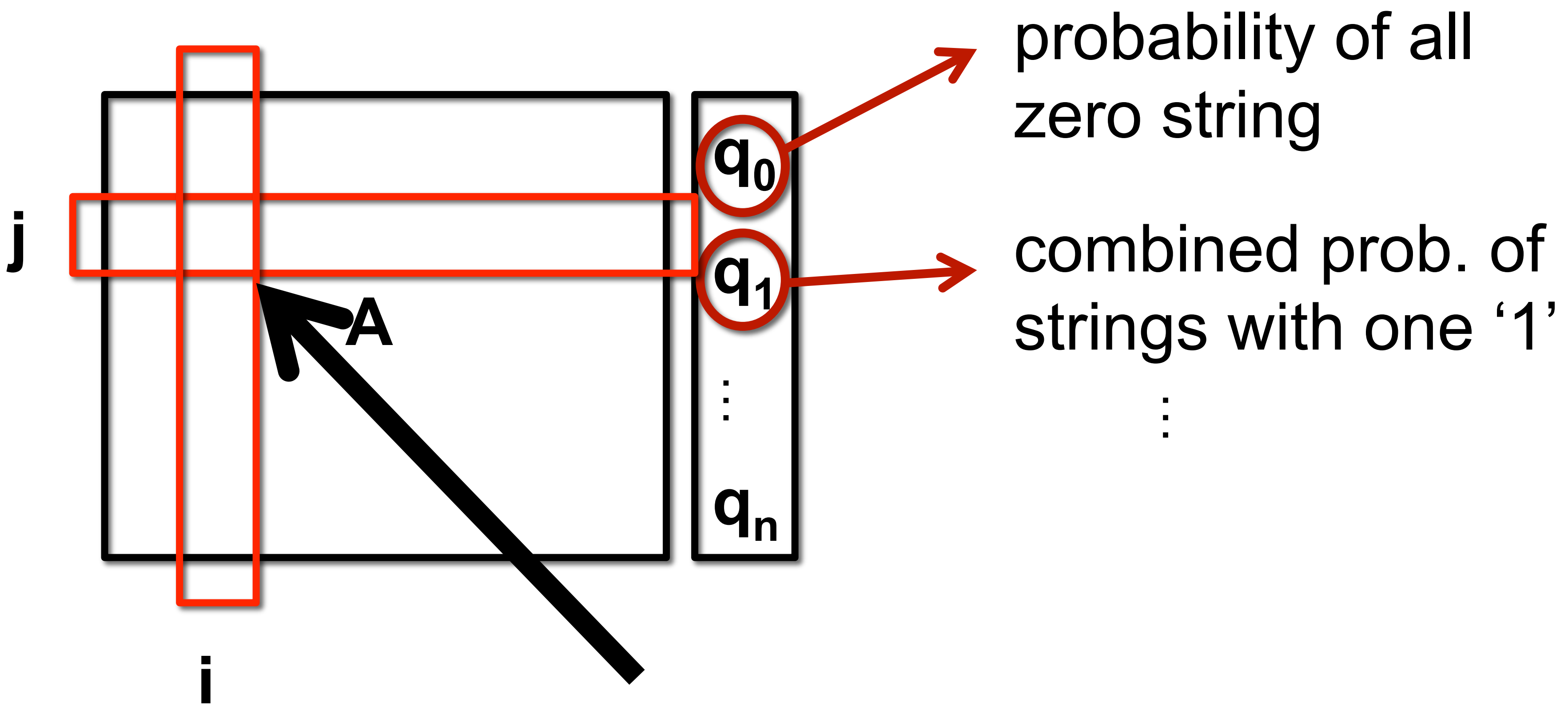

“probability that if there are **i** ones, **j** remain”

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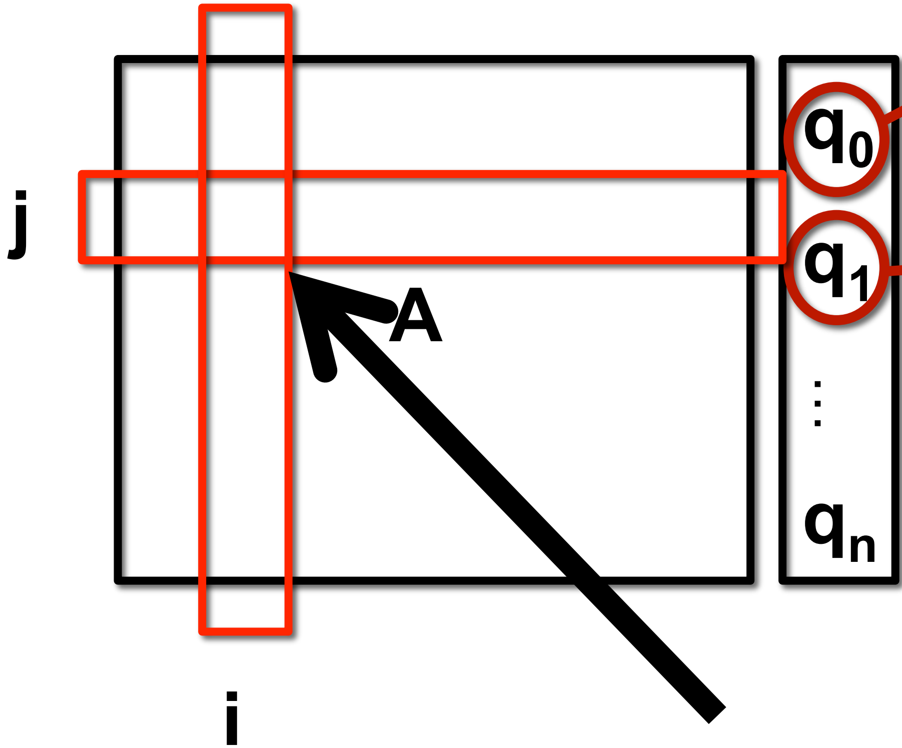

probability of all zero string

combined prob. of strings with one '1'

⋮

$$\binom{i}{j}\mu^j(1-\mu)^{i-j}$$

"probability that if there are **i** ones, **j** remain"

The Setup

$$
\begin{bmatrix} & & \\ & \mathbf{A} & \\ & & \end{bmatrix}
\begin{bmatrix} \mathbf{q_0} \\ \mathbf{q_1} \\ \vdots \\ \mathbf{q_n} \end{bmatrix}
=
\begin{bmatrix} \mathbf{b_0} \\ \mathbf{b_1} \\ \vdots \\ \mathbf{b_n} \end{bmatrix}
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$$\boxed{\quad A \quad} \begin{bmatrix} q_0 \\ q_1 \\ \vdots \\ q_n \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_n \end{bmatrix}$$

probability of all ‘0’s and ‘?’s in the sample

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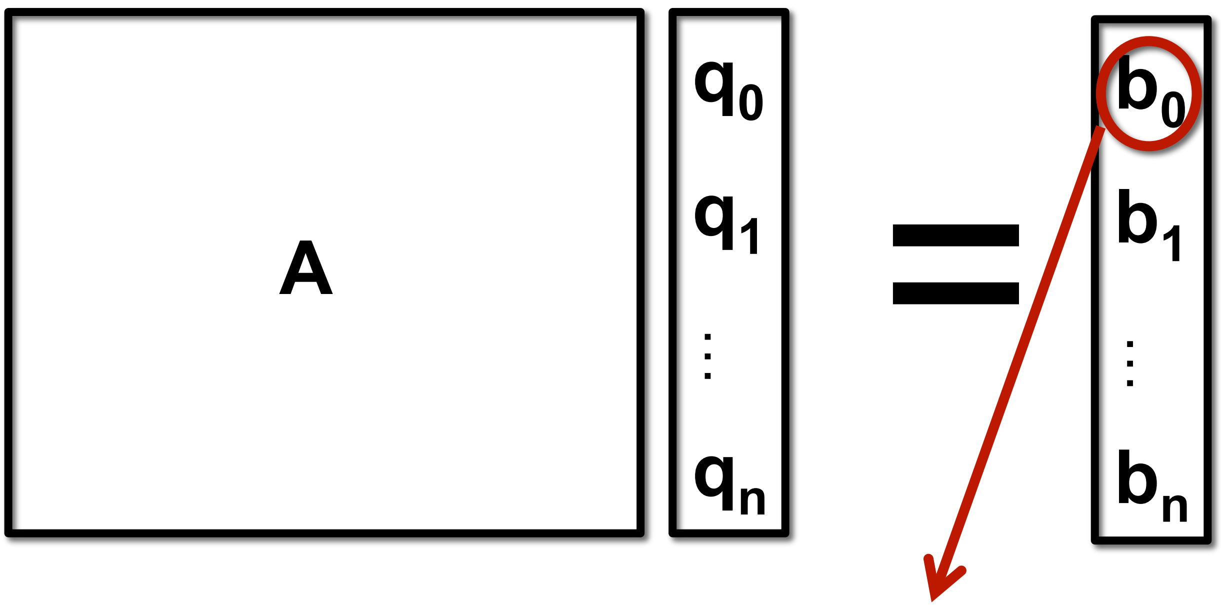

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e.g.

?	0	?	?	?	0	?	?	?	?

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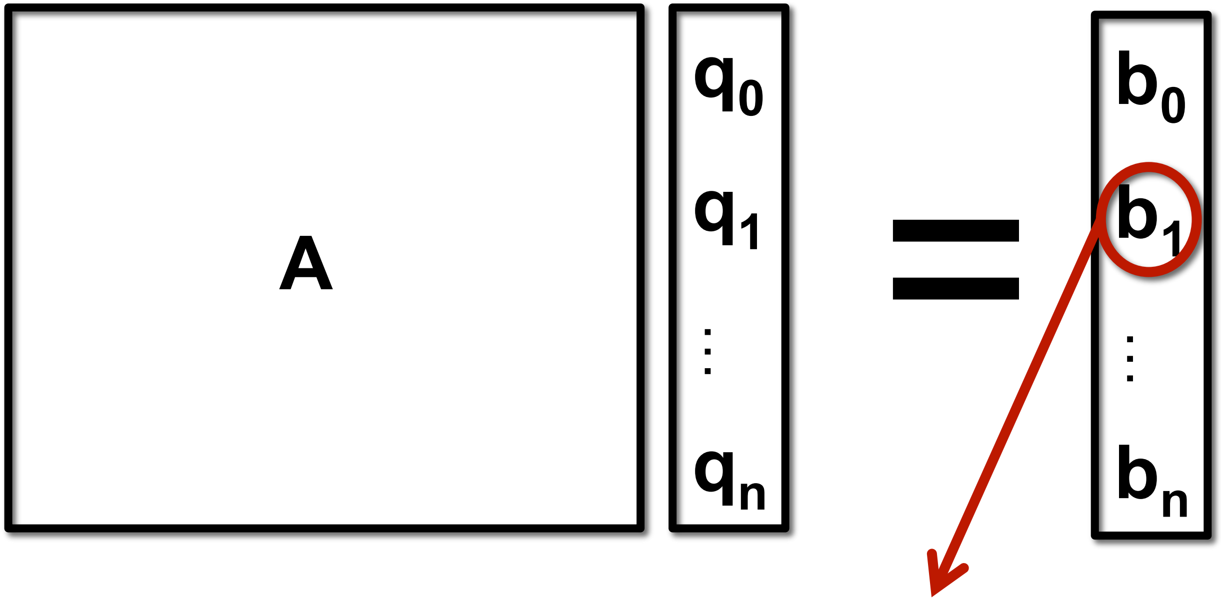

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probability of one ‘1’
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0	?	0	?	?	?	?	1	?	0

The Issue…

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Can we perturb e_0 s.t. $(e_0+\eta)A^{-1}$ has bdd norm?

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Idea: Write a linear program for computing a good RLI, and prove that the dual has no solution

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Why is this basis natural for population recovery?

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If we can prove no such polynomial exists

⬇

There is a good RLI, which we can find via an LP

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The dual program wants to construct **p(x)** s.t.

$$p(0) \geq \varepsilon \, ||p||_{coeff} + C \, ||q||_{coeff}$$

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Conversely, for a polynomial are its coefficients large in at least one of the two representations?

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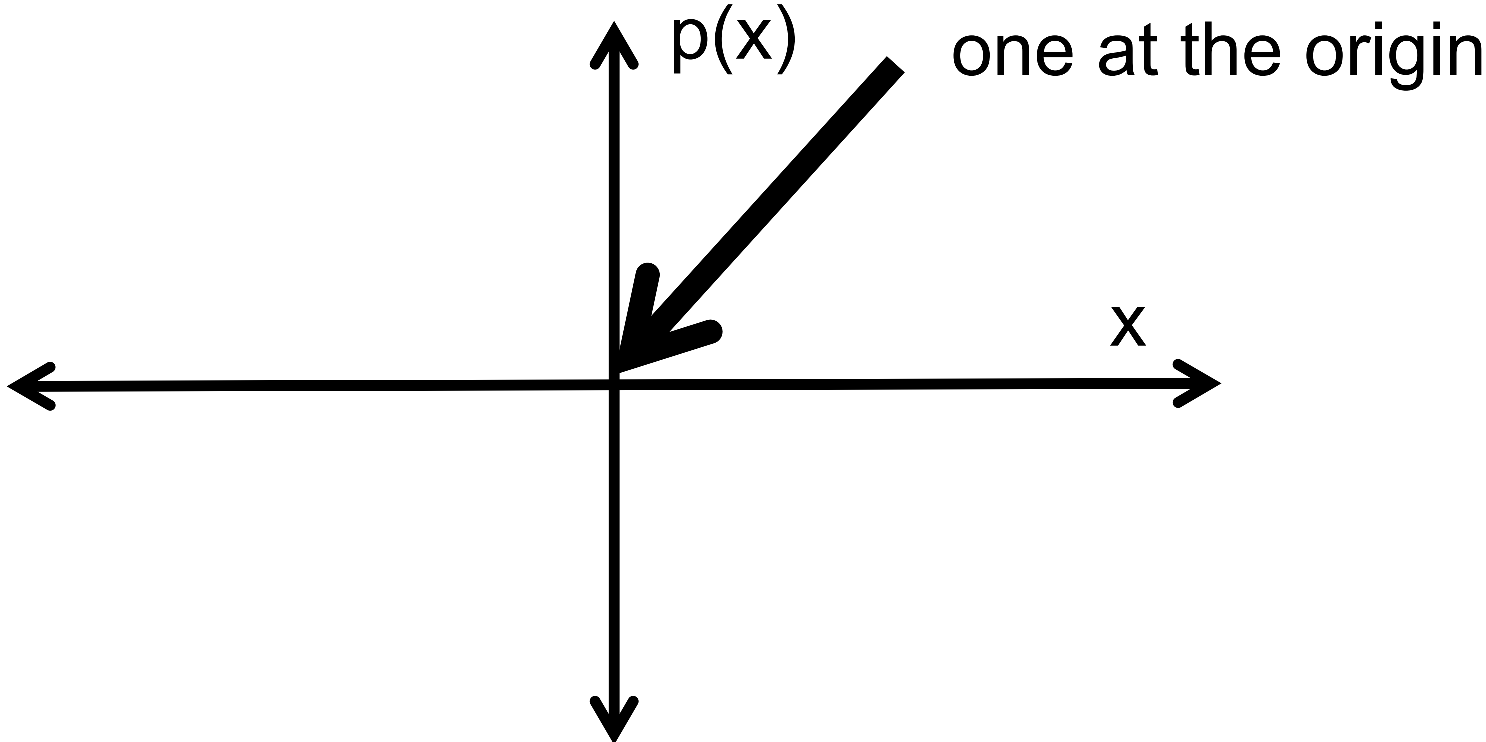

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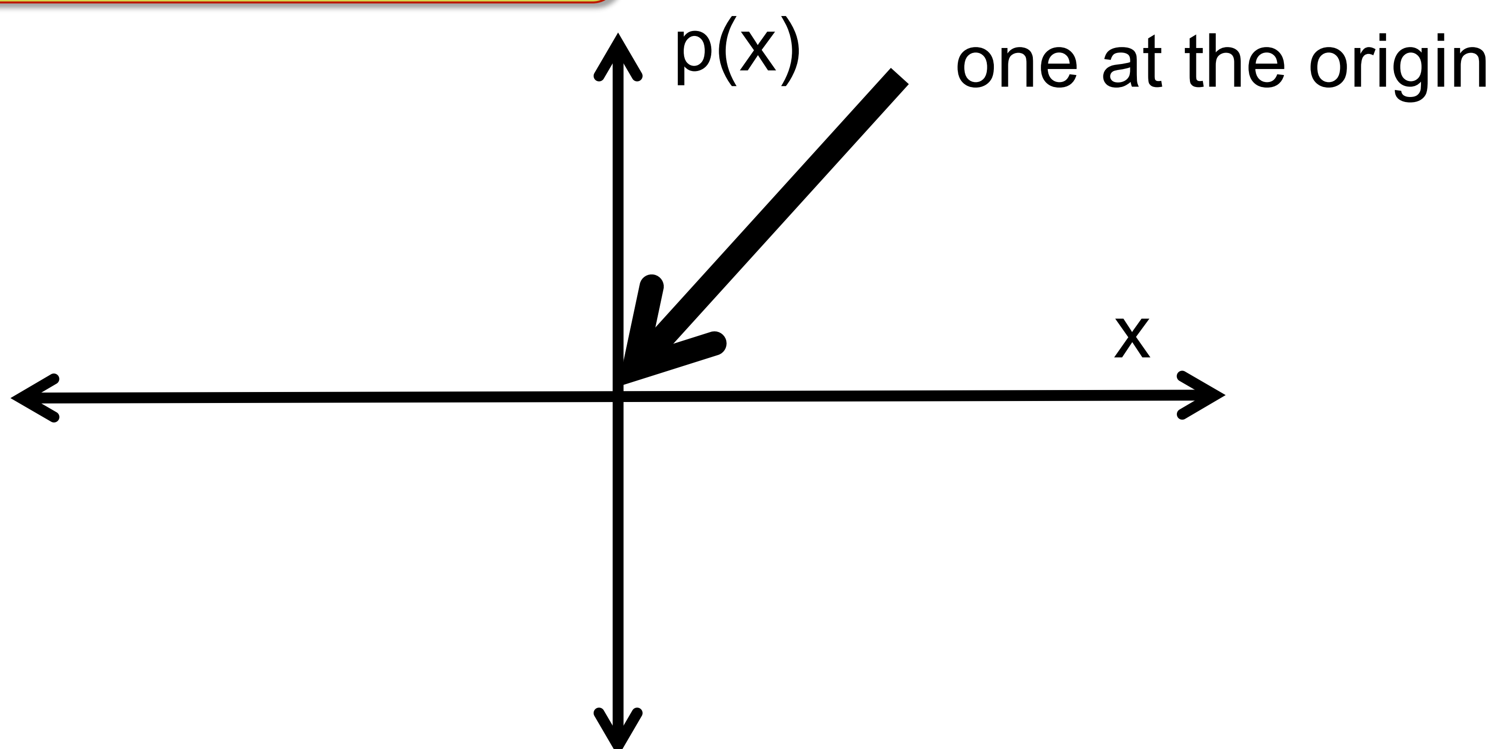

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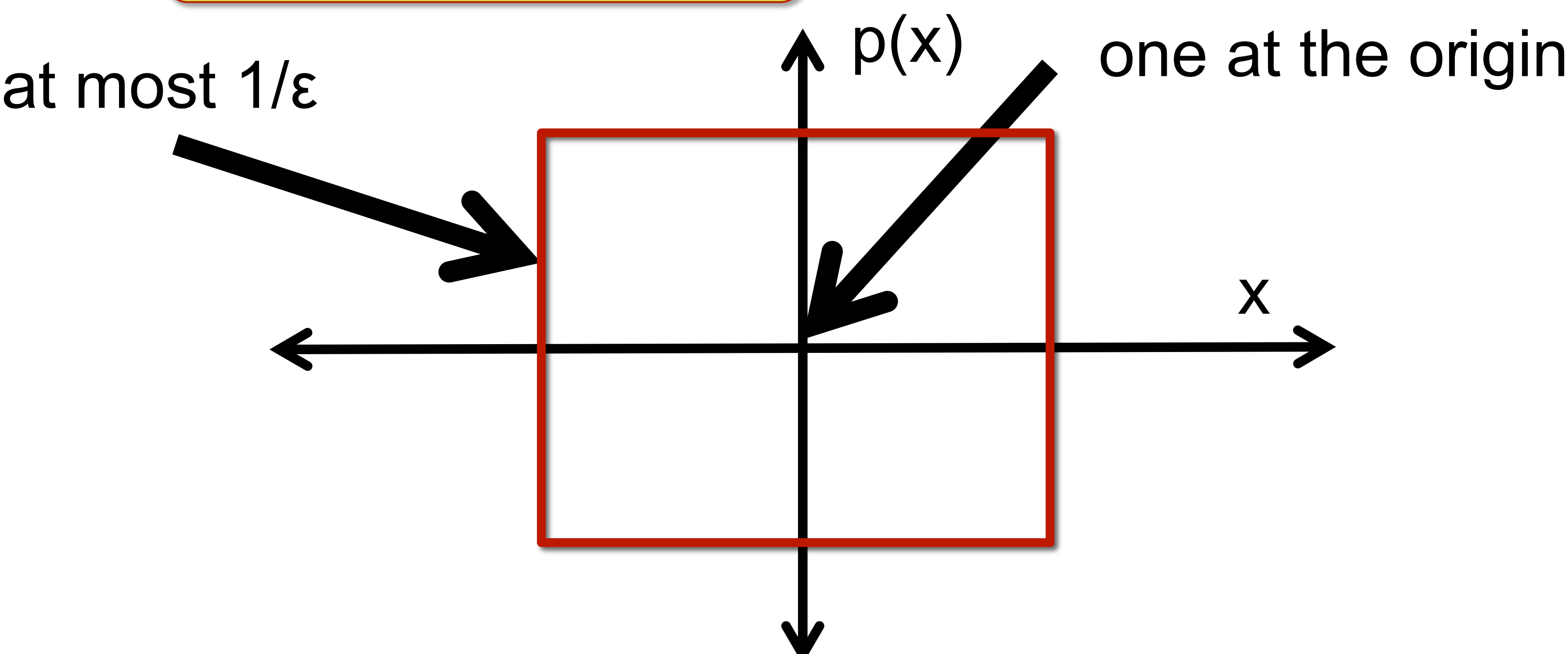

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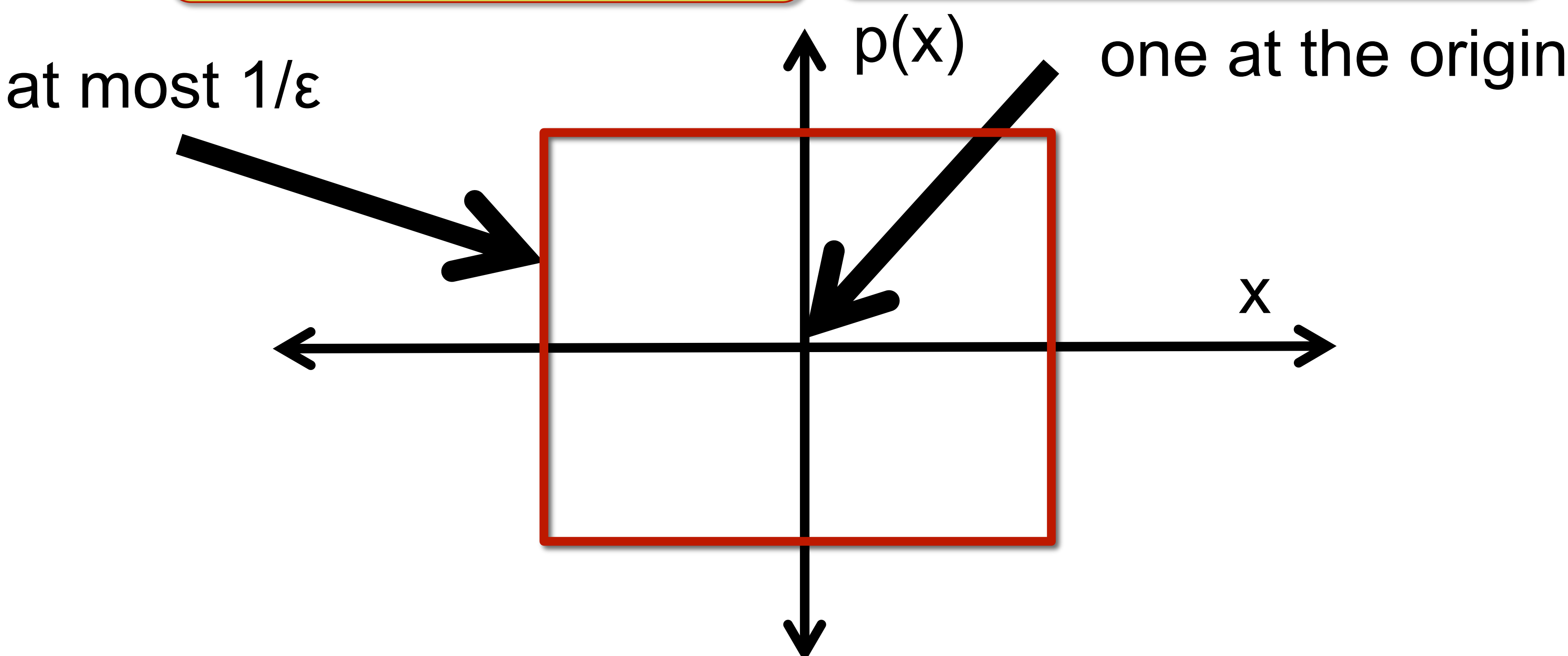

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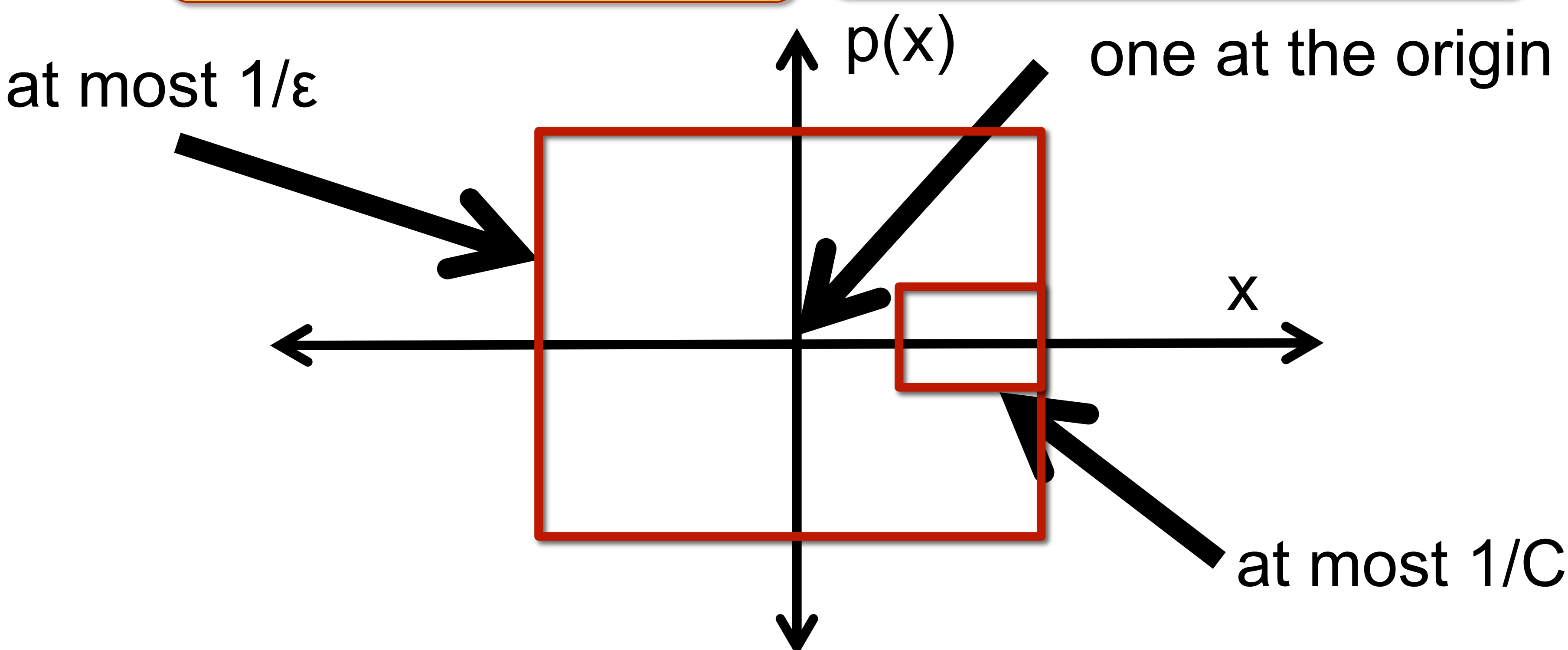

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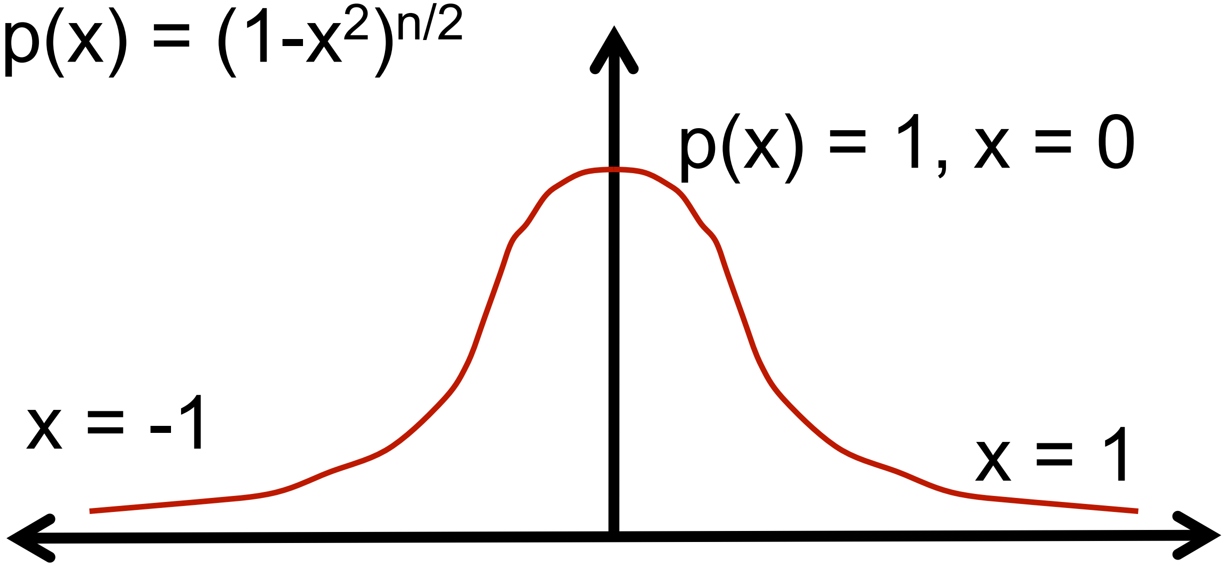

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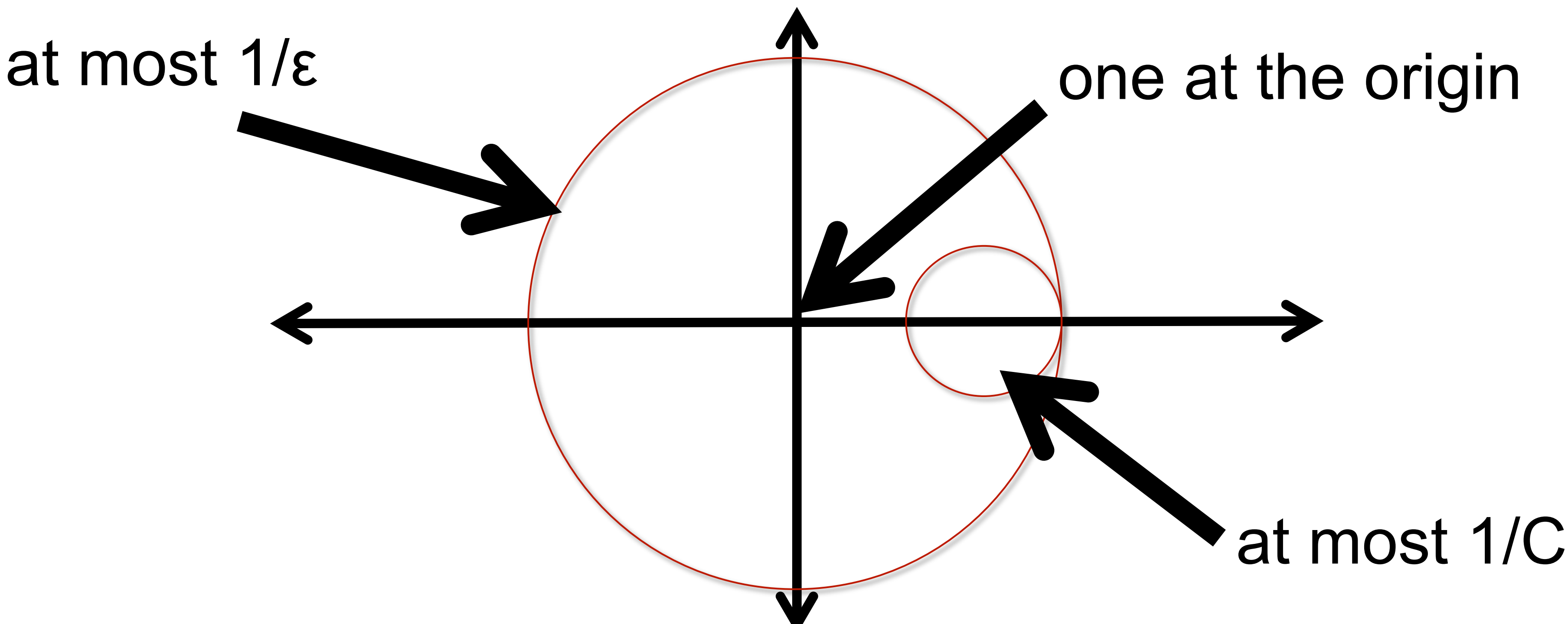

Hadamard Three Circle Theorem

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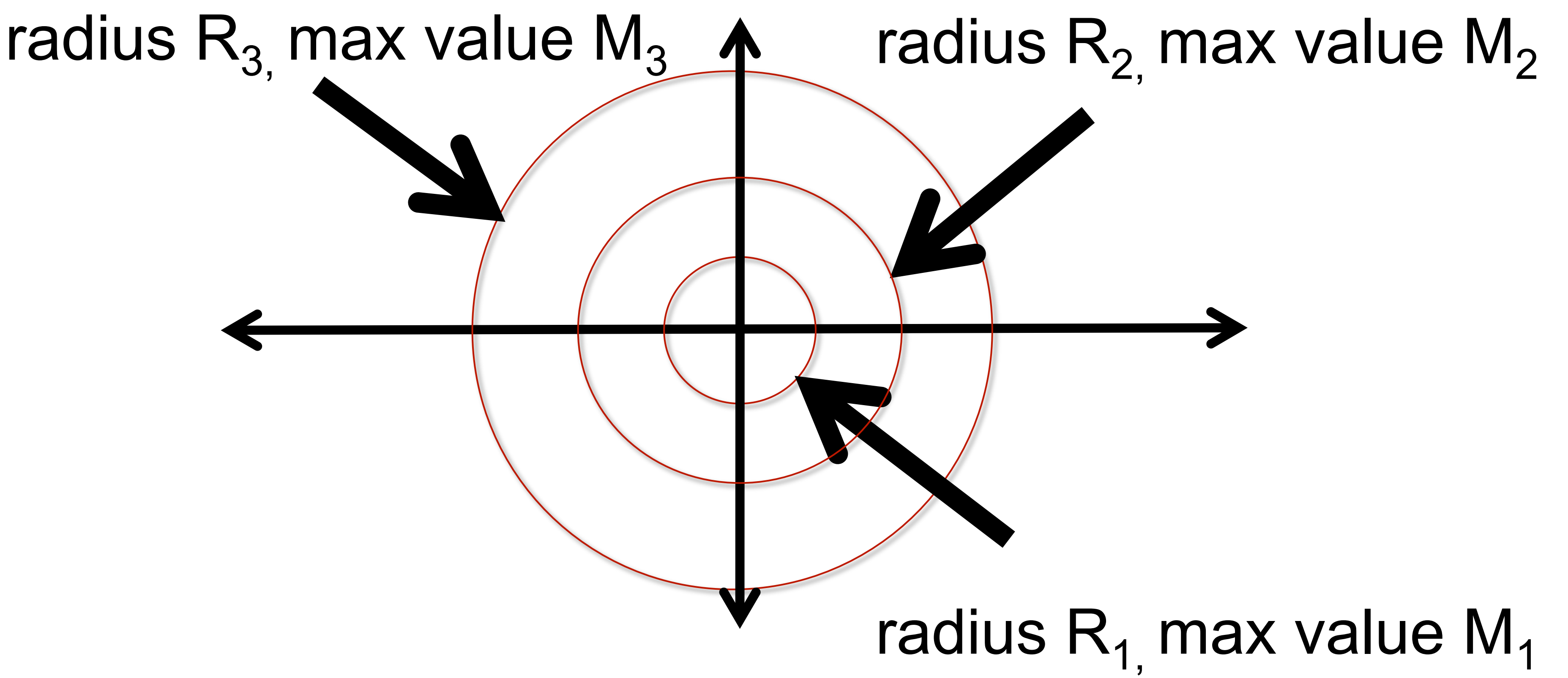
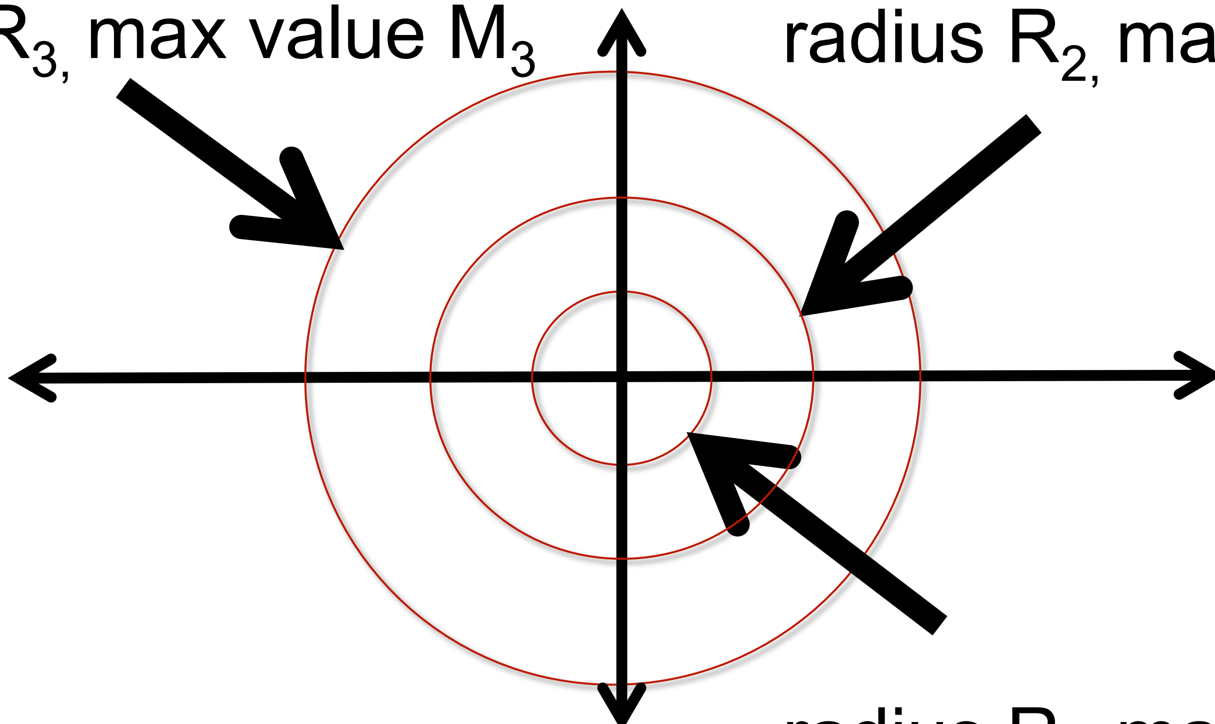


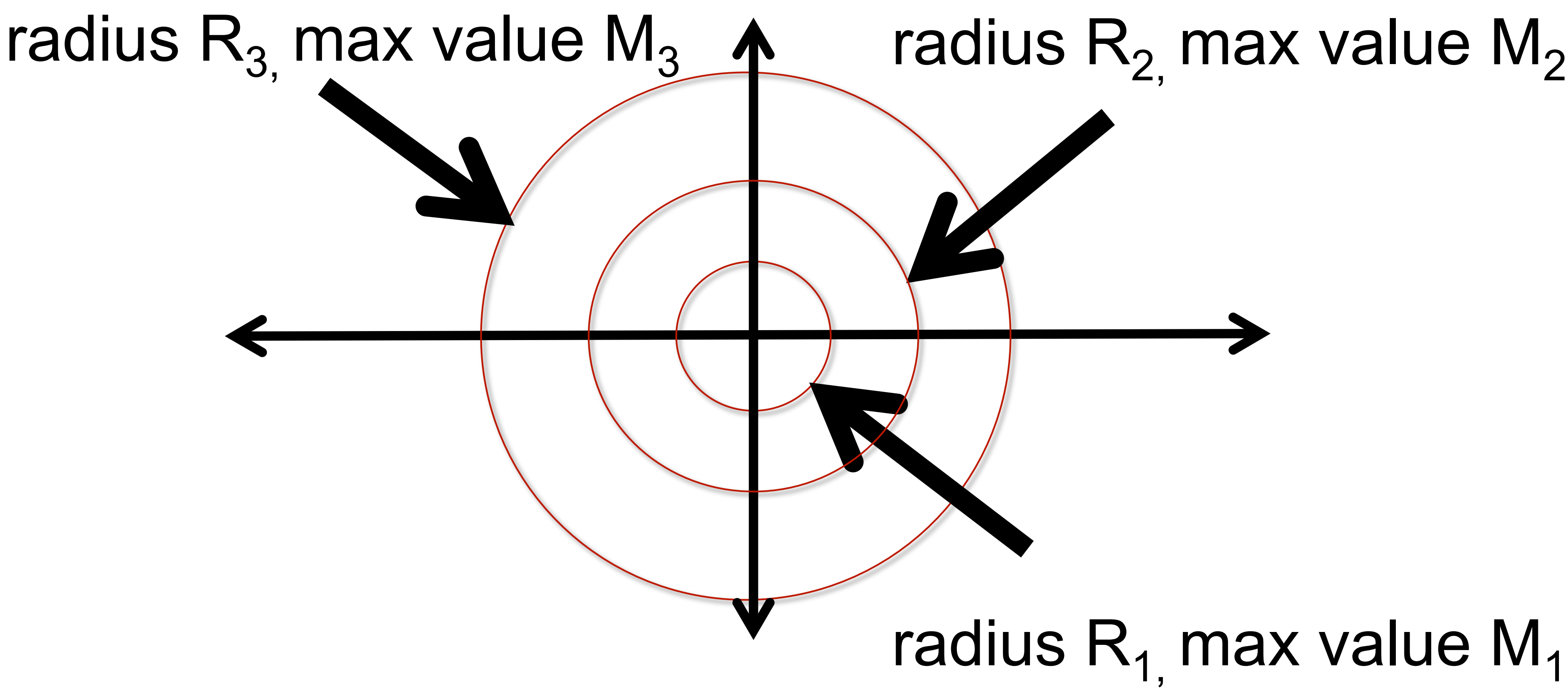
Hadamard Three Circle Theorem

$$\log\frac{R_3}{R_1}\log M_2 \le \log\frac{R_2}{R_1}\log M_3 + \log\frac{R_3}{R_2}\log M_1$$



Hadamard Three Circle Theorem

Hence M_2 is bounded by a geometric average of M_1 and M_3 (that depends on the radii)!



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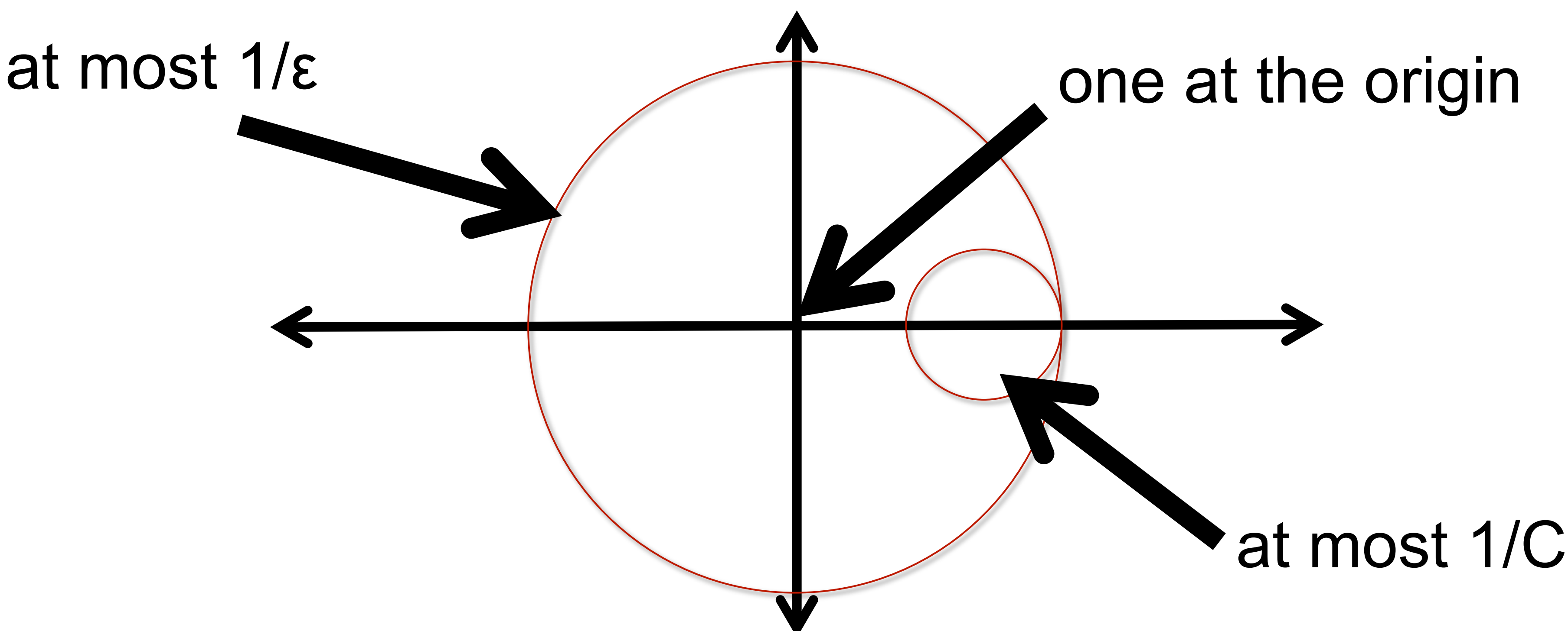

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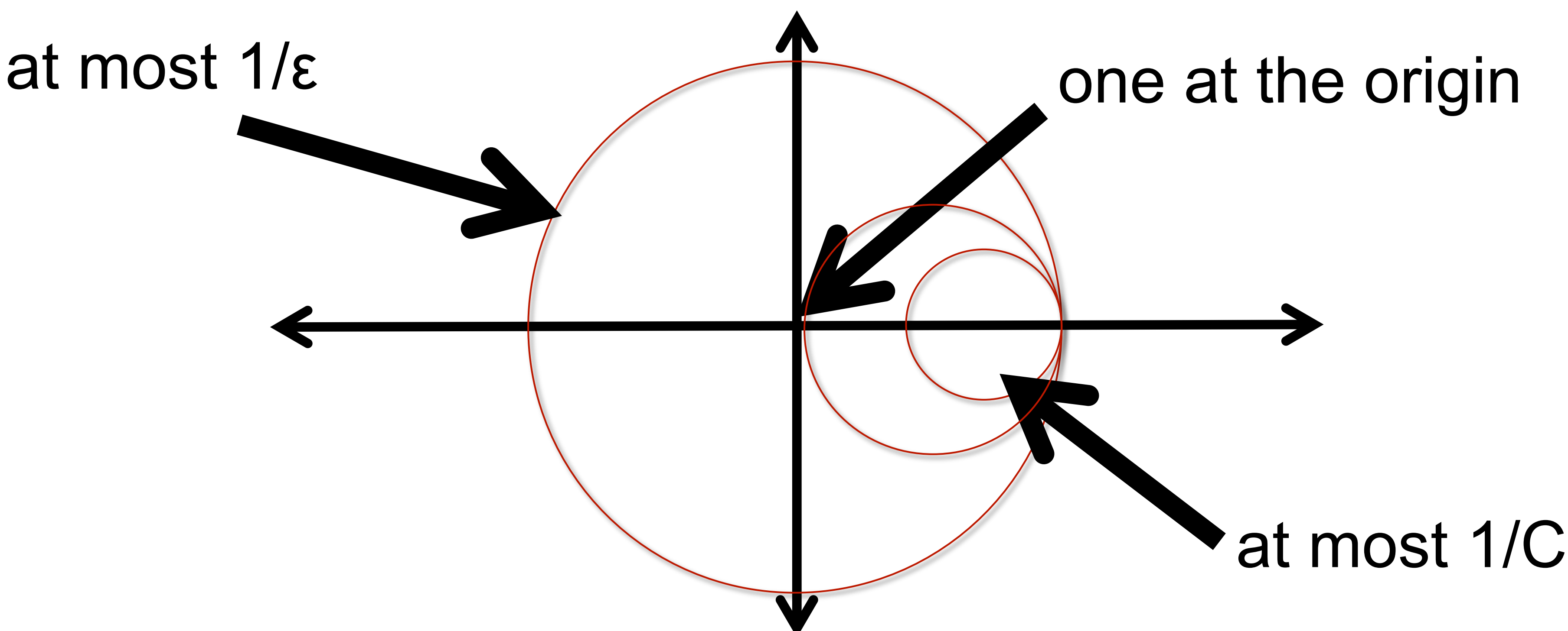
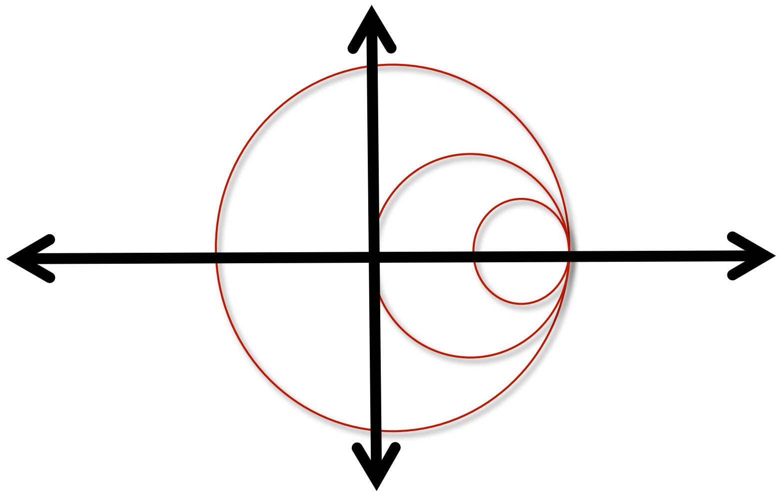
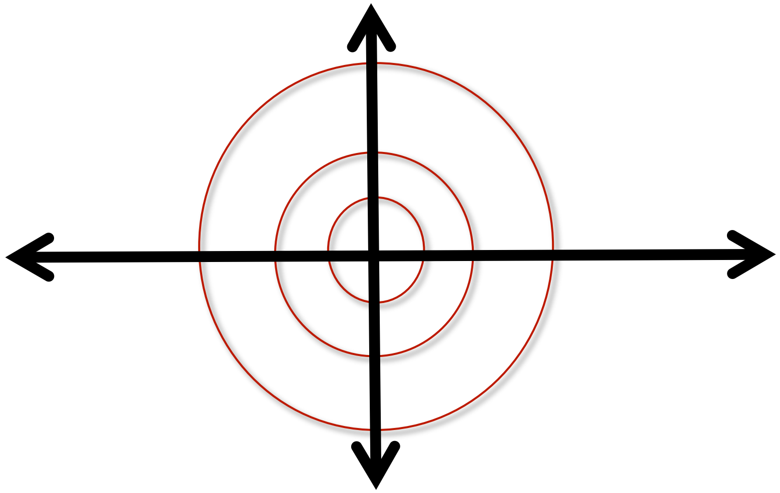


Is there a holomorphic map between these two pictures?





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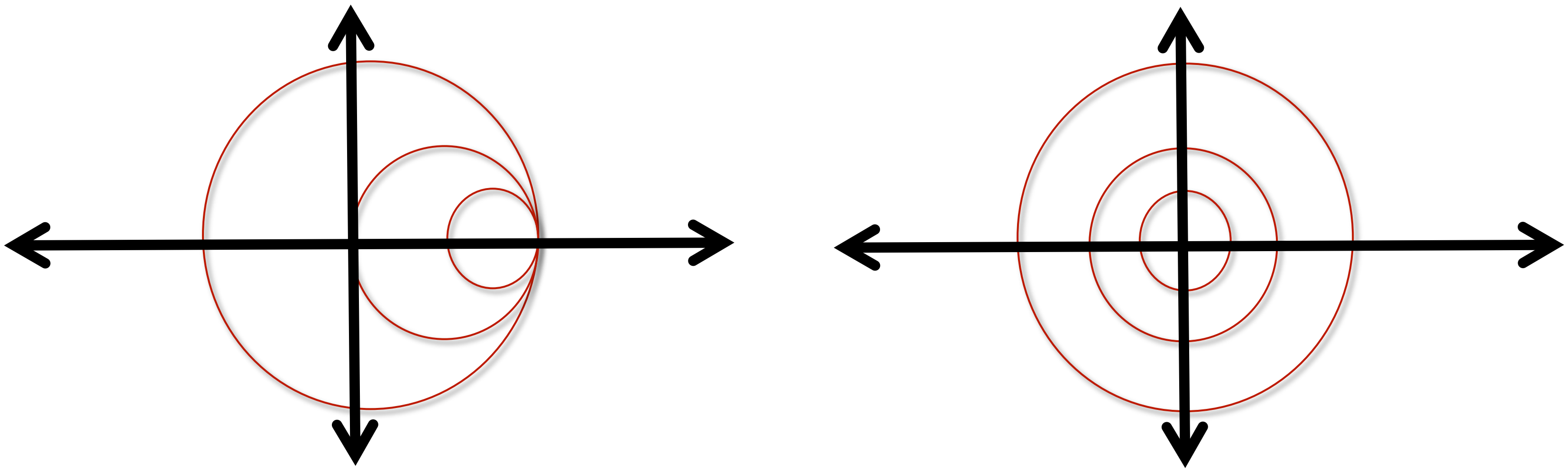

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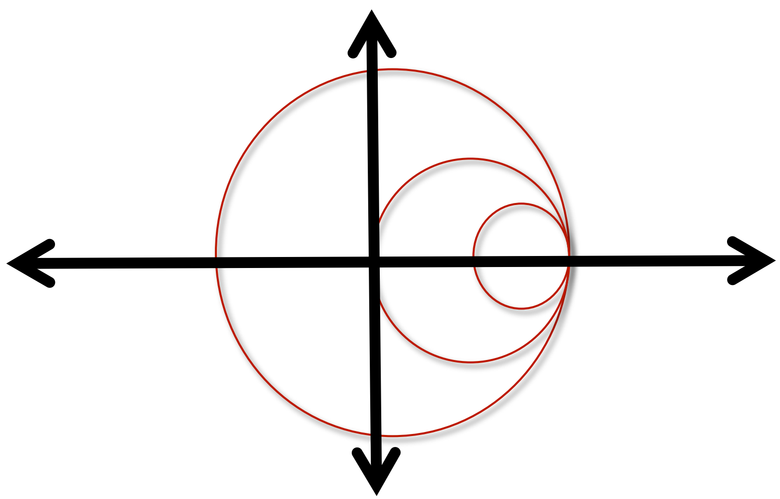

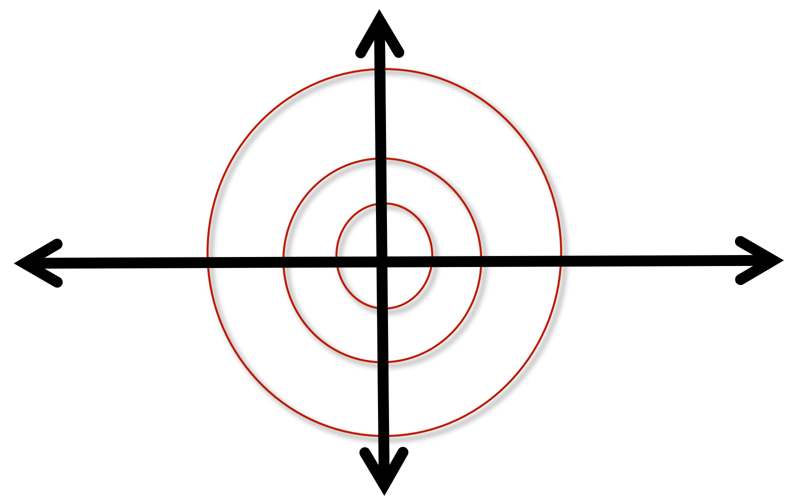

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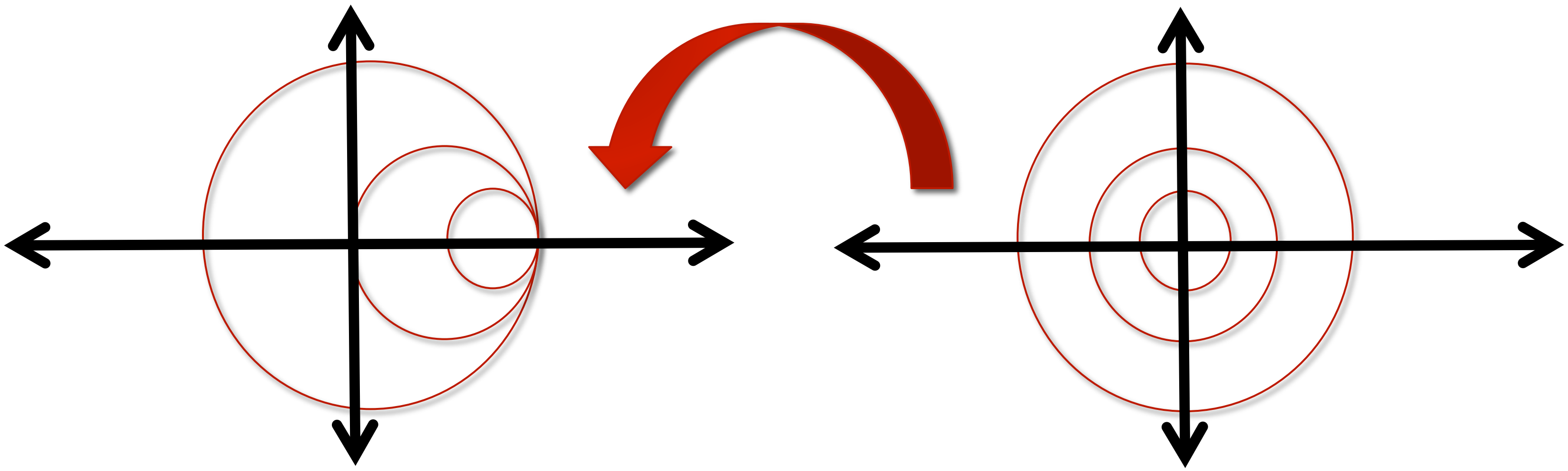

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Yes! And it is called the Mobius Transform

Outline

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Uncertainty Principle (via complex analysis)

Outline

Is the Linear Program feasible?

Uncertainty Principle (via complex analysis)

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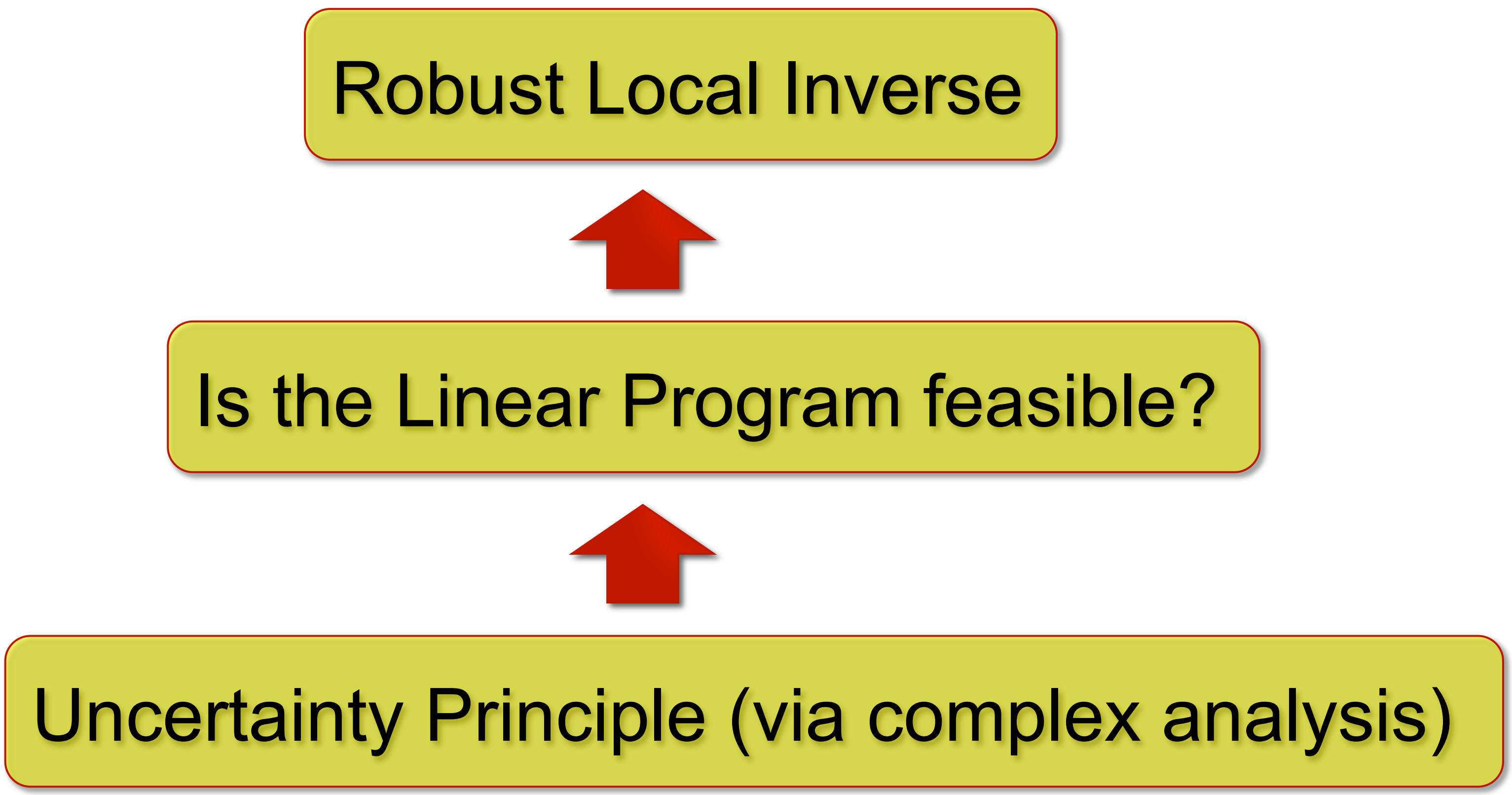
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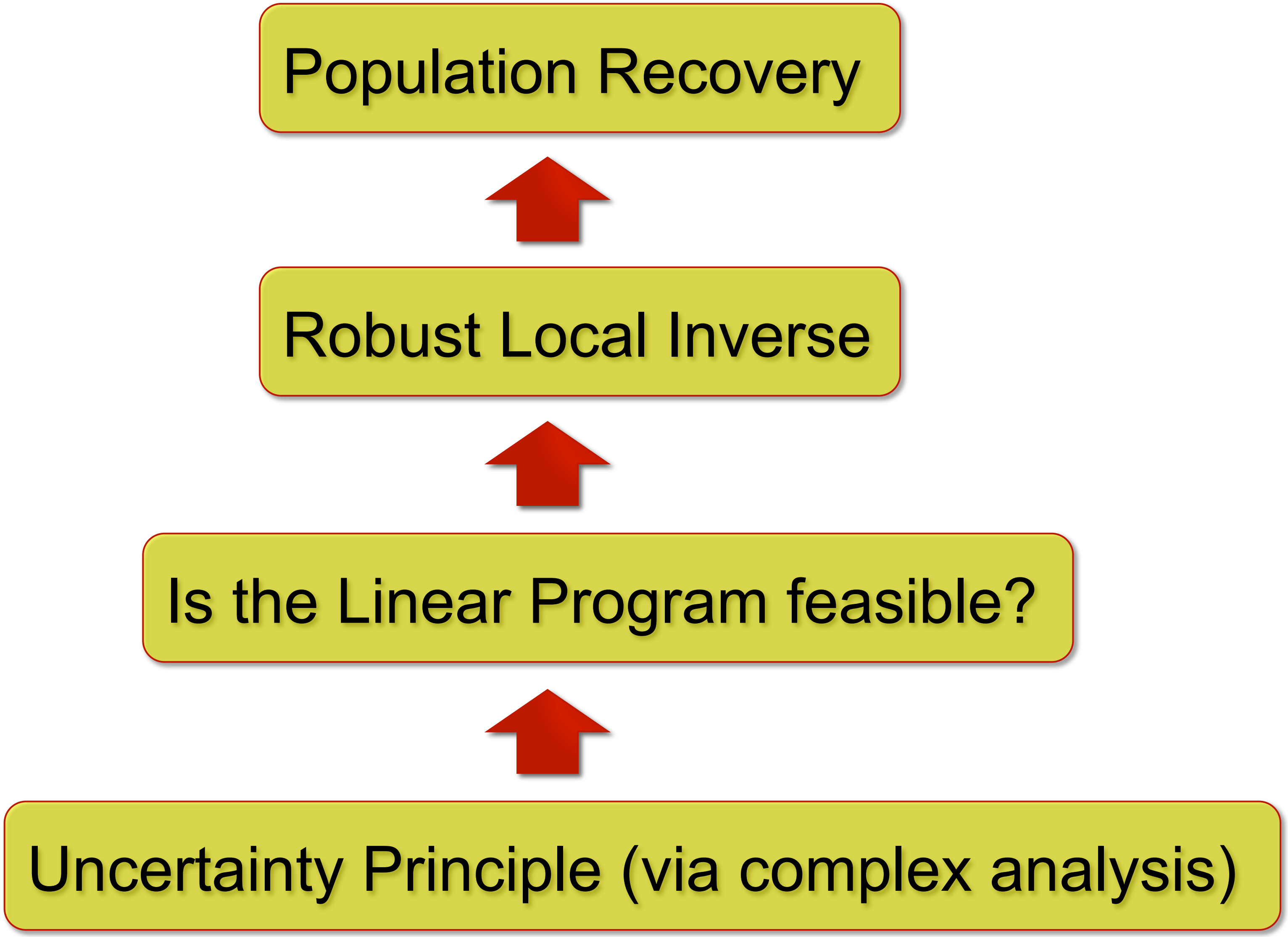
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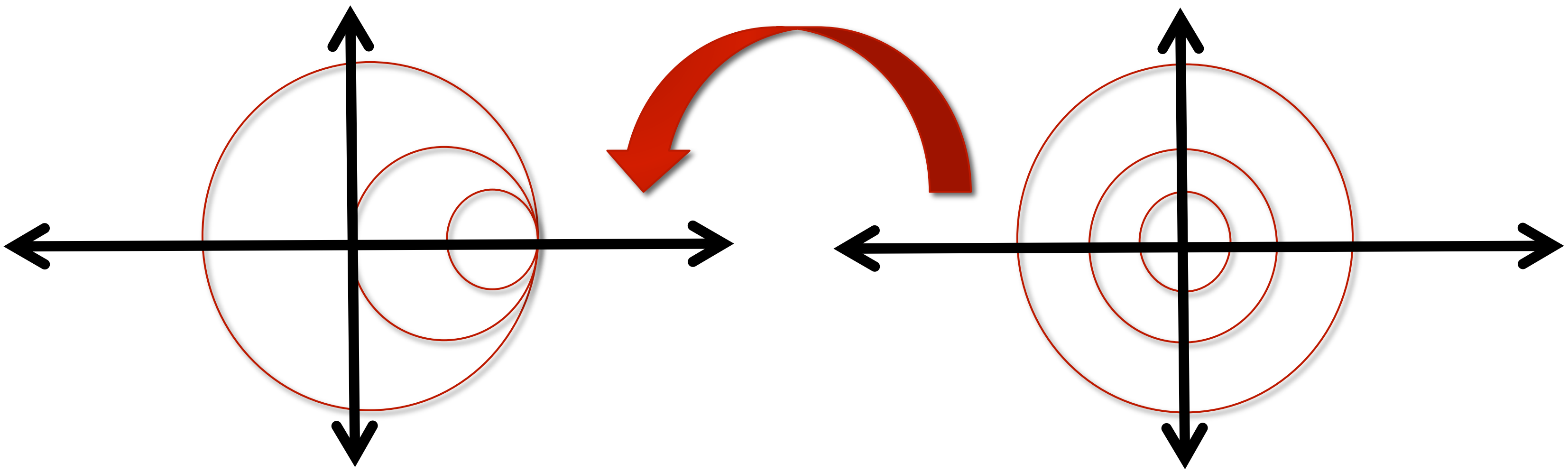
Thanks!



Any Questions?