Simple, Efficient and Neural Algorithms for Sparse Coding

Ankur Moitra (MIT)

joint work with Sanjeev Arora, Rong Ge and Tengyu Ma

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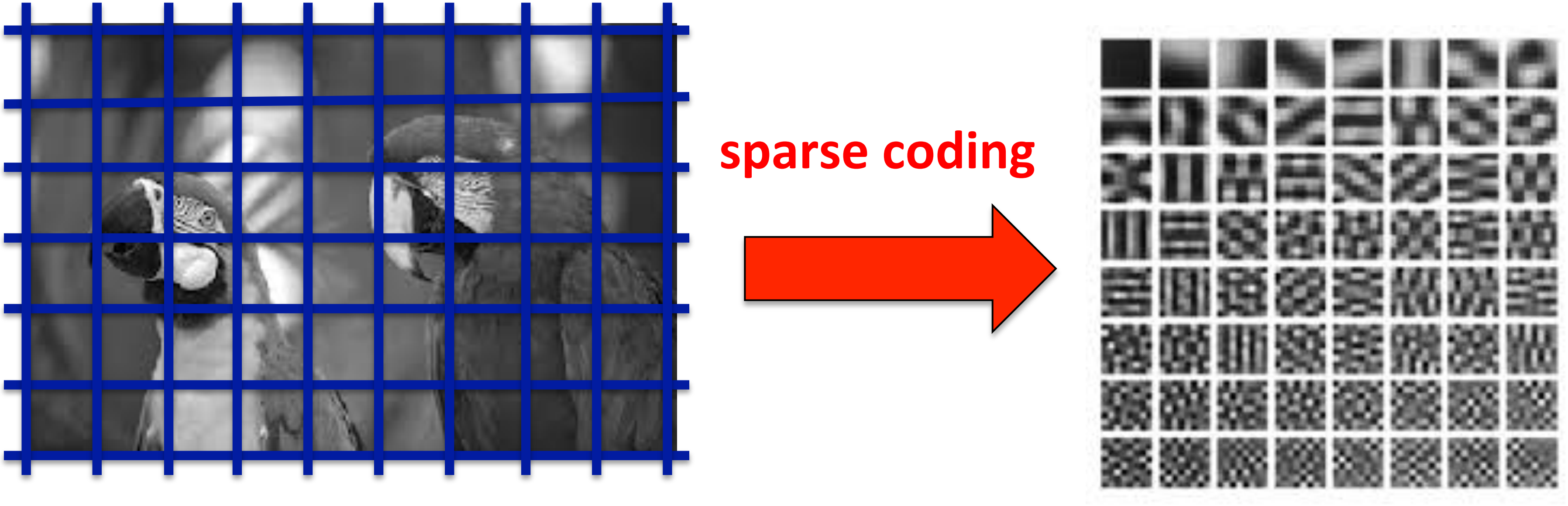

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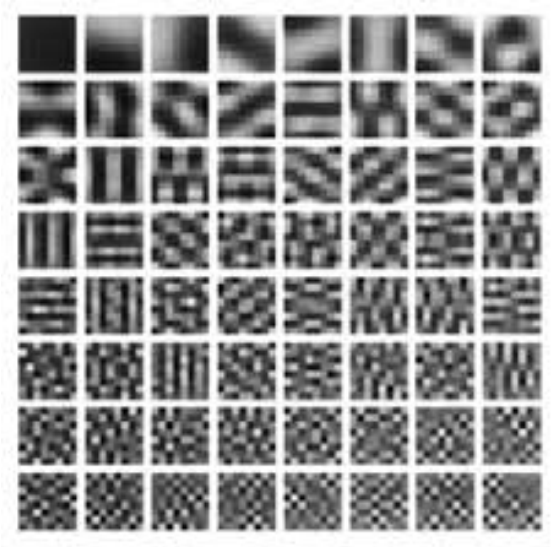

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Properties: localized, bandpass and oriented

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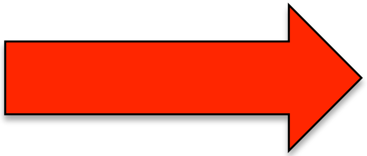

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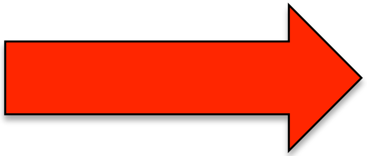

singular value decomposition

Noisy!

Difficult to interpret!

(collection of vectors)

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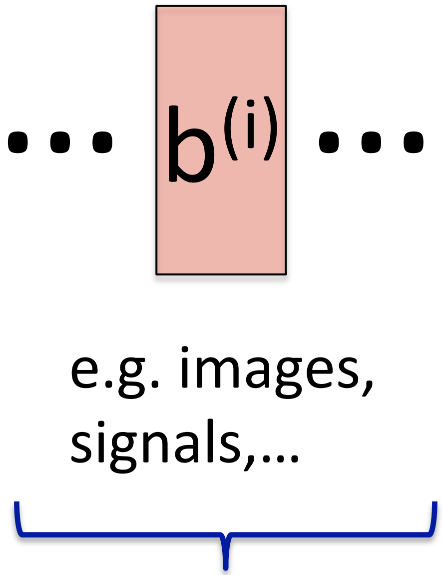
More generally, many types of data are sparse in an appropriately chosen basis:



$\cdots \; b^{(i)} \; \cdots$

e.g. images, signals,…

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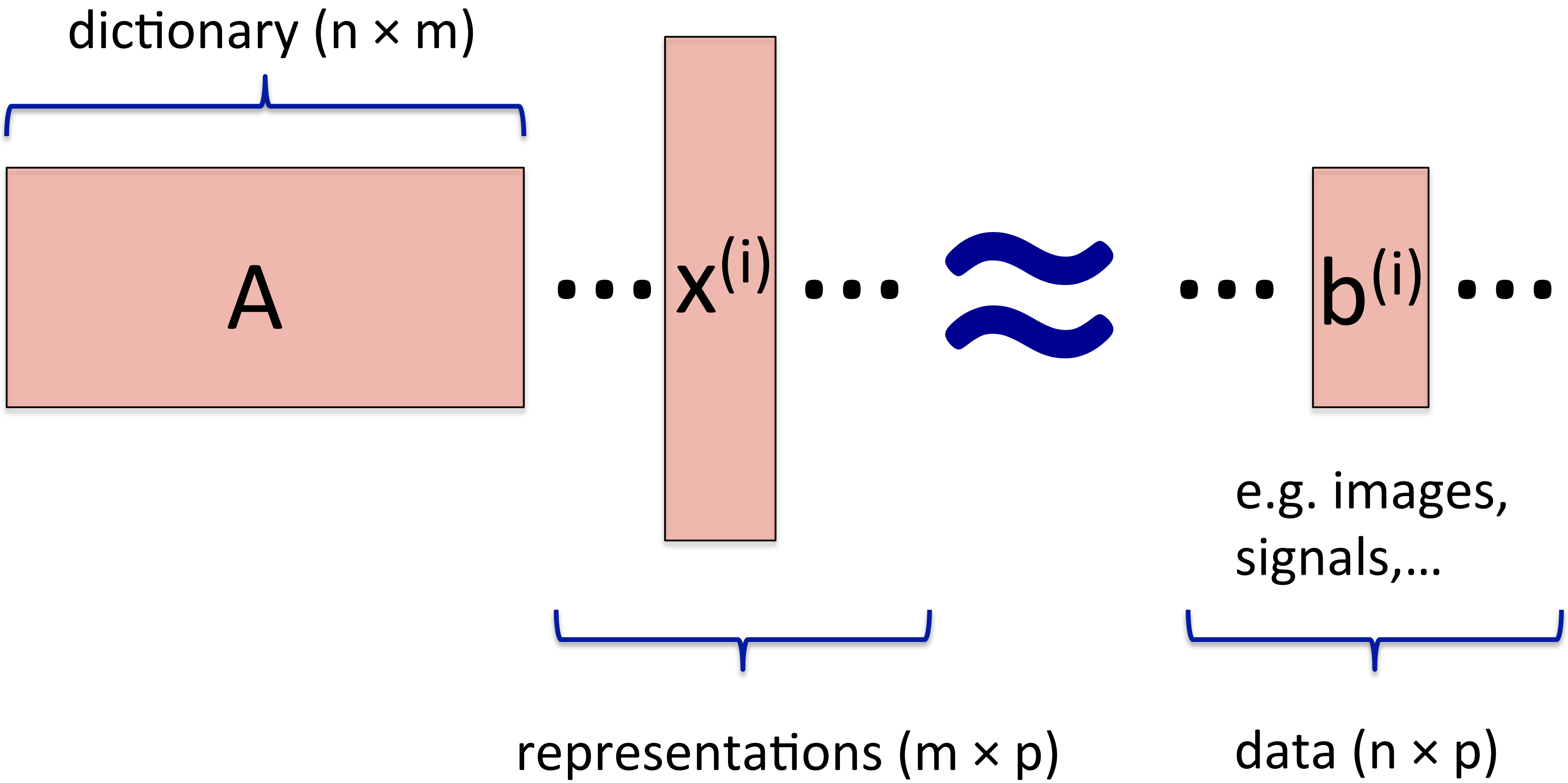

dictionary (n × m)

$A \cdots x^{(i)} \cdots \approx \cdots b^{(i)} \cdots$

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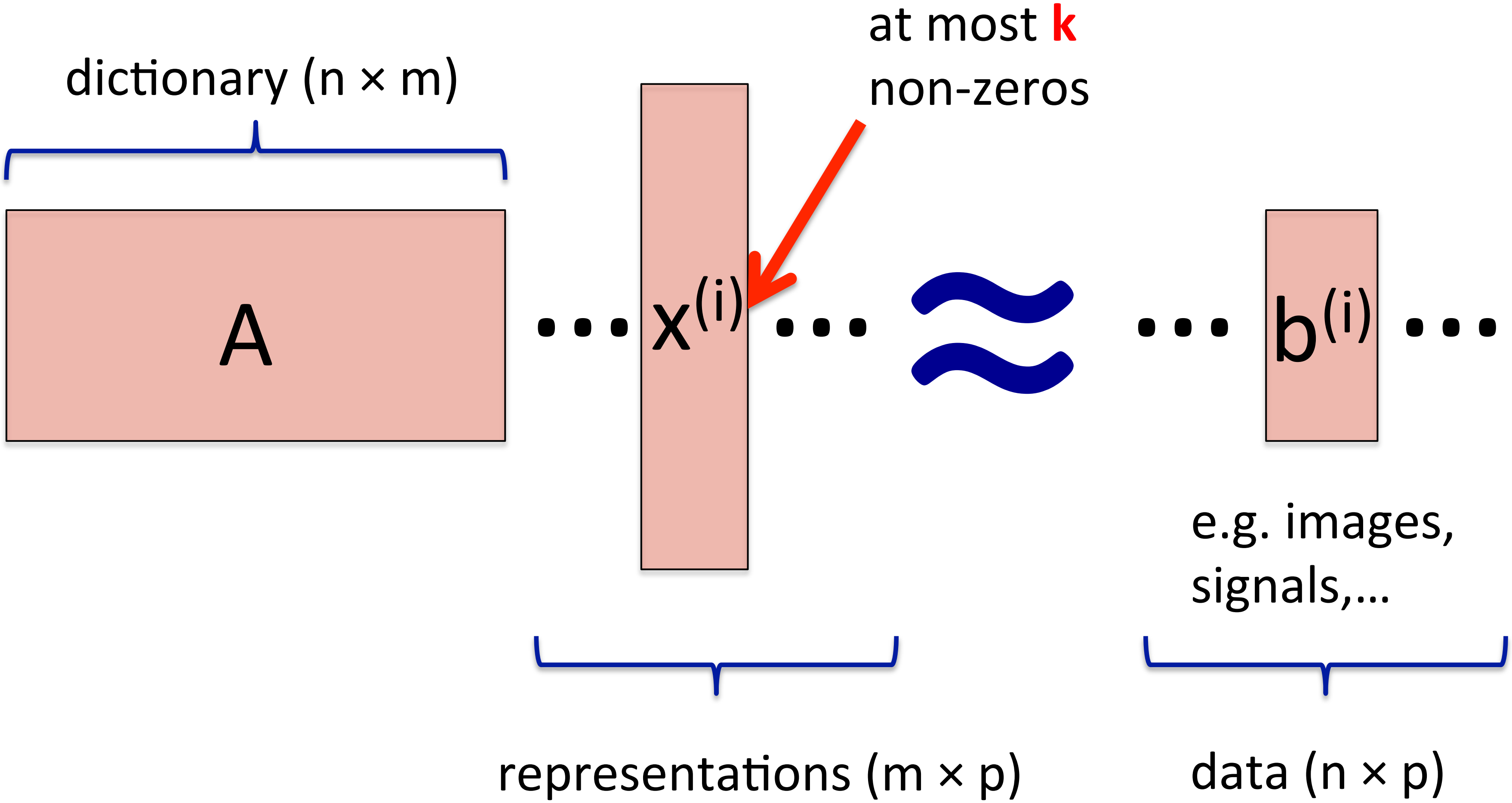

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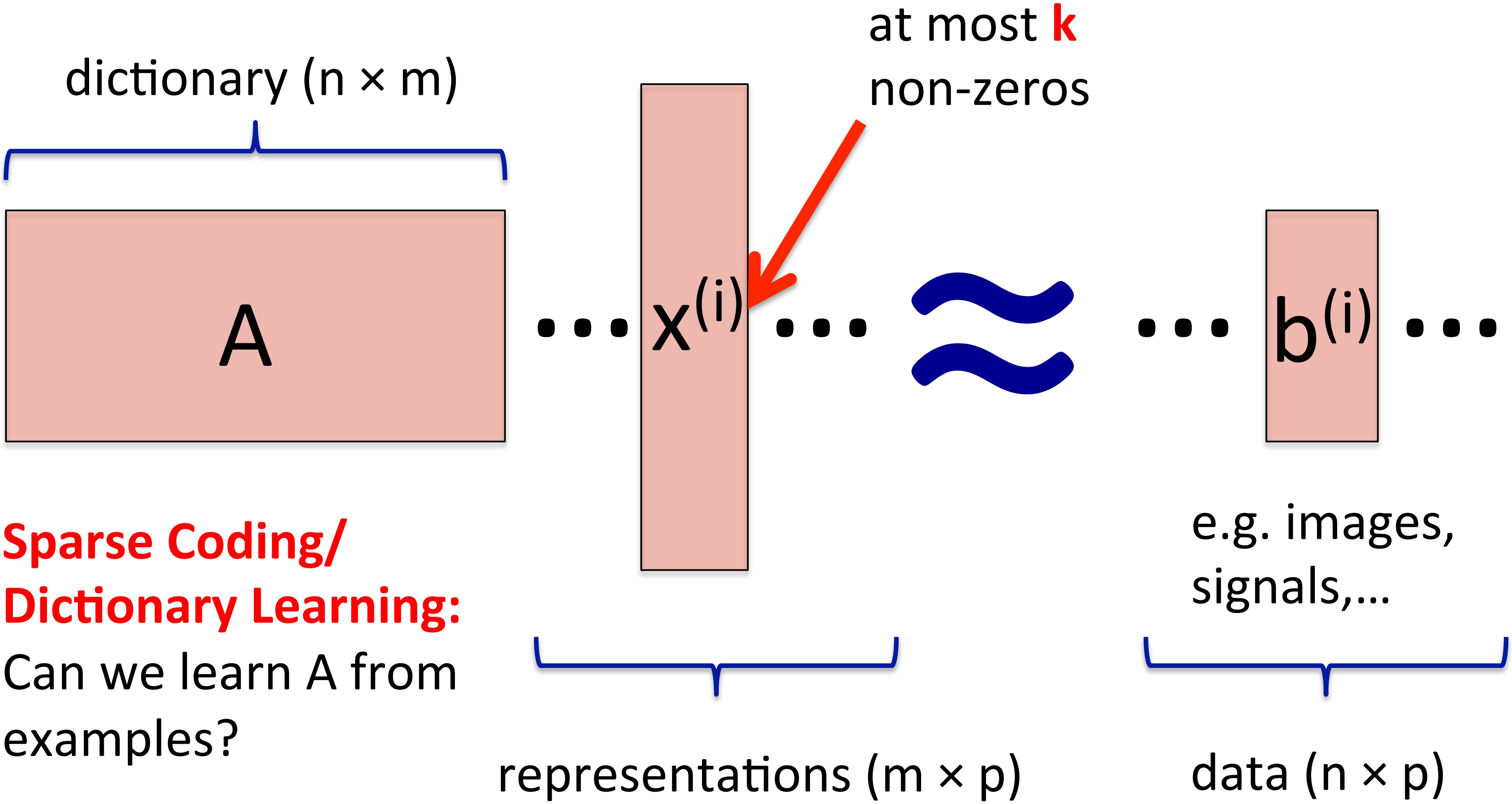

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Sparse Coding/ Dictionary Learning: Can we learn A from examples?

NONCONVEX FORMULATIONS

Usual approach, minimize **reconstruction error**:

$$\min_{A,\, x^{(i)}\text{'s}} \sum_{i=1}^{p} \| b^{(i)} - A\, x^{(i)} \| + \sum_{i=1}^{p} L(x^{(i)})$$

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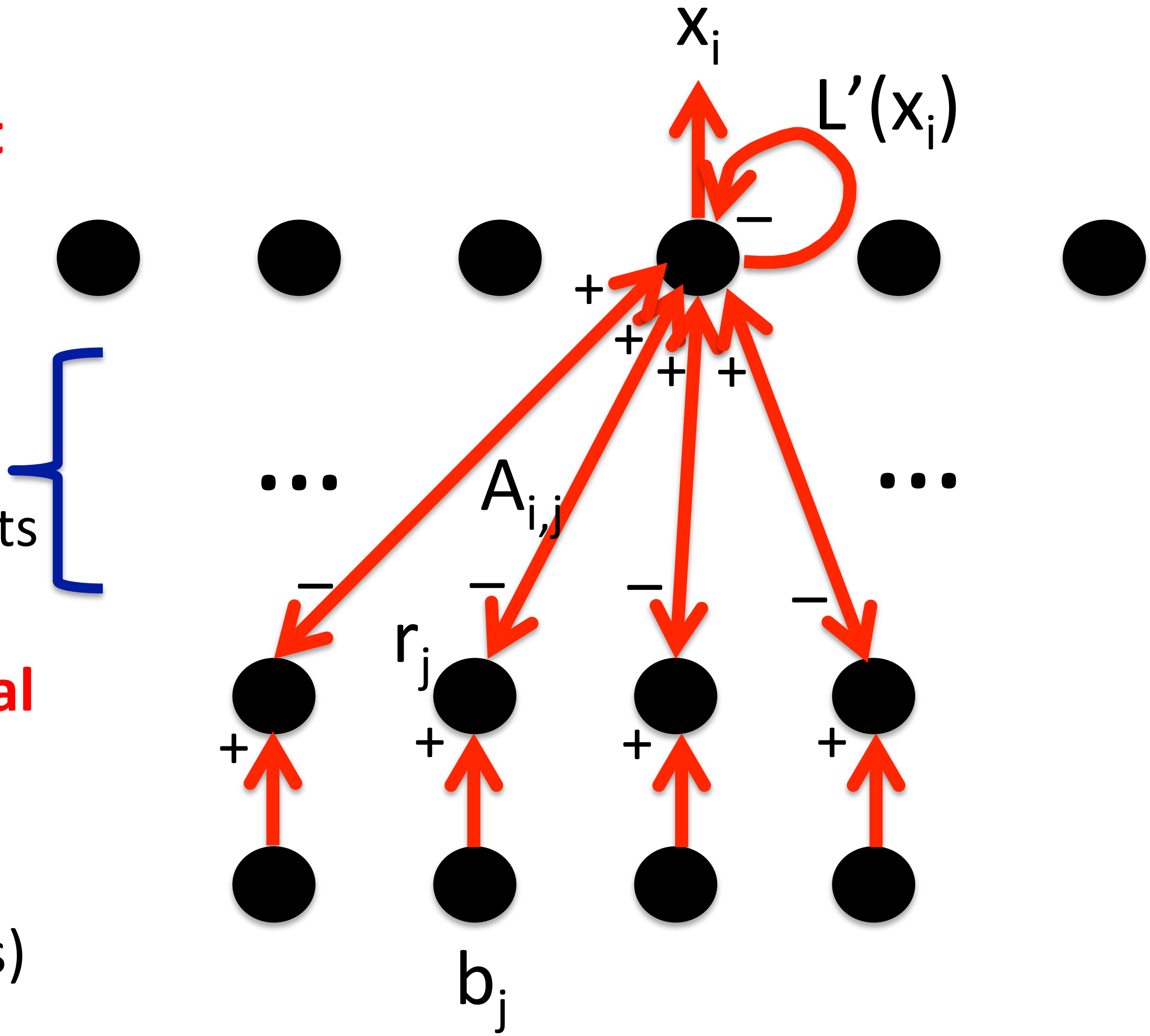
This optimization problem is **NP-hard**, can have many local optima; but **heuristics** work well nevertheless…

A NEURAL IMPLEMENTATION

[Olshausen, Field]:



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stored as
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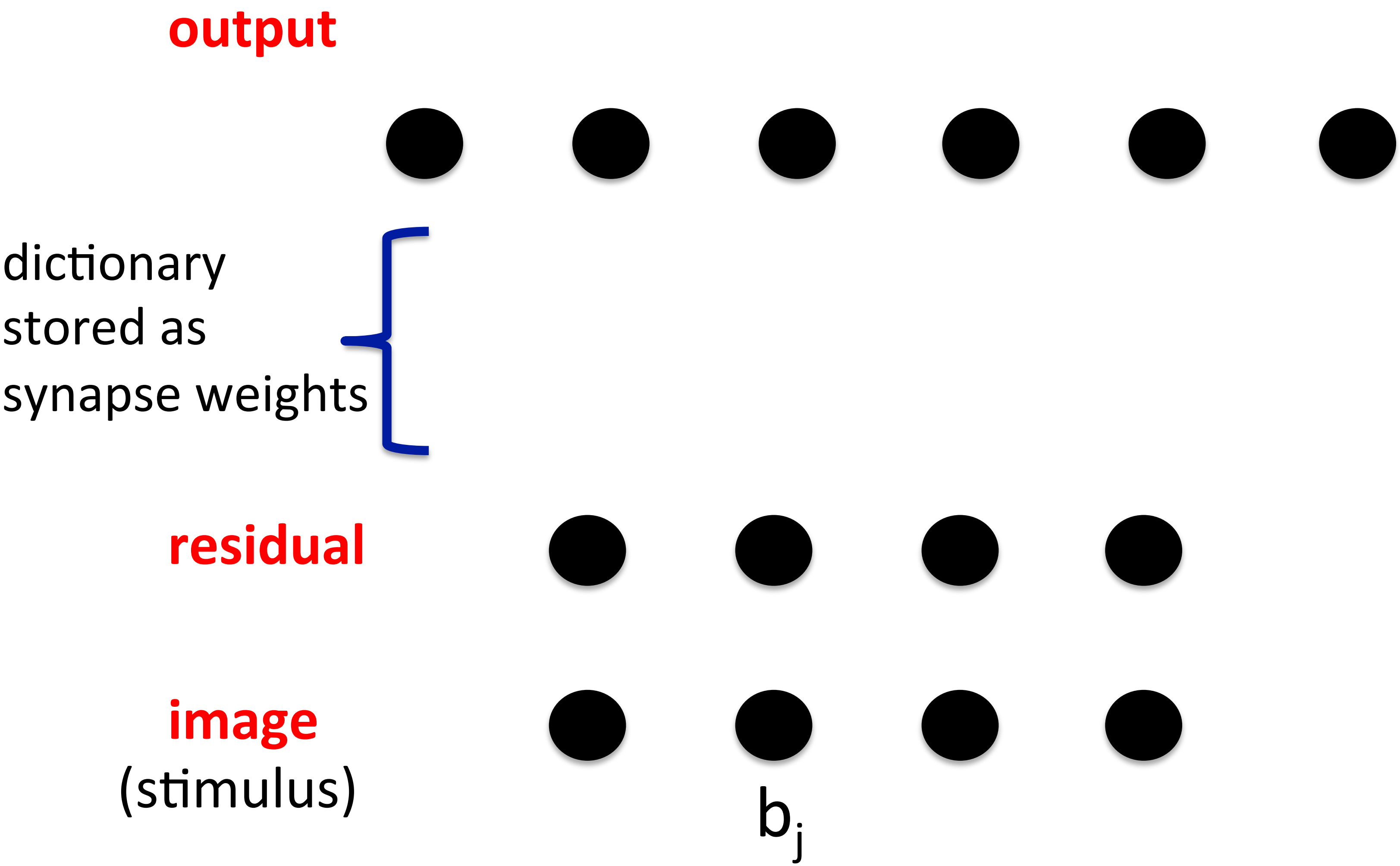

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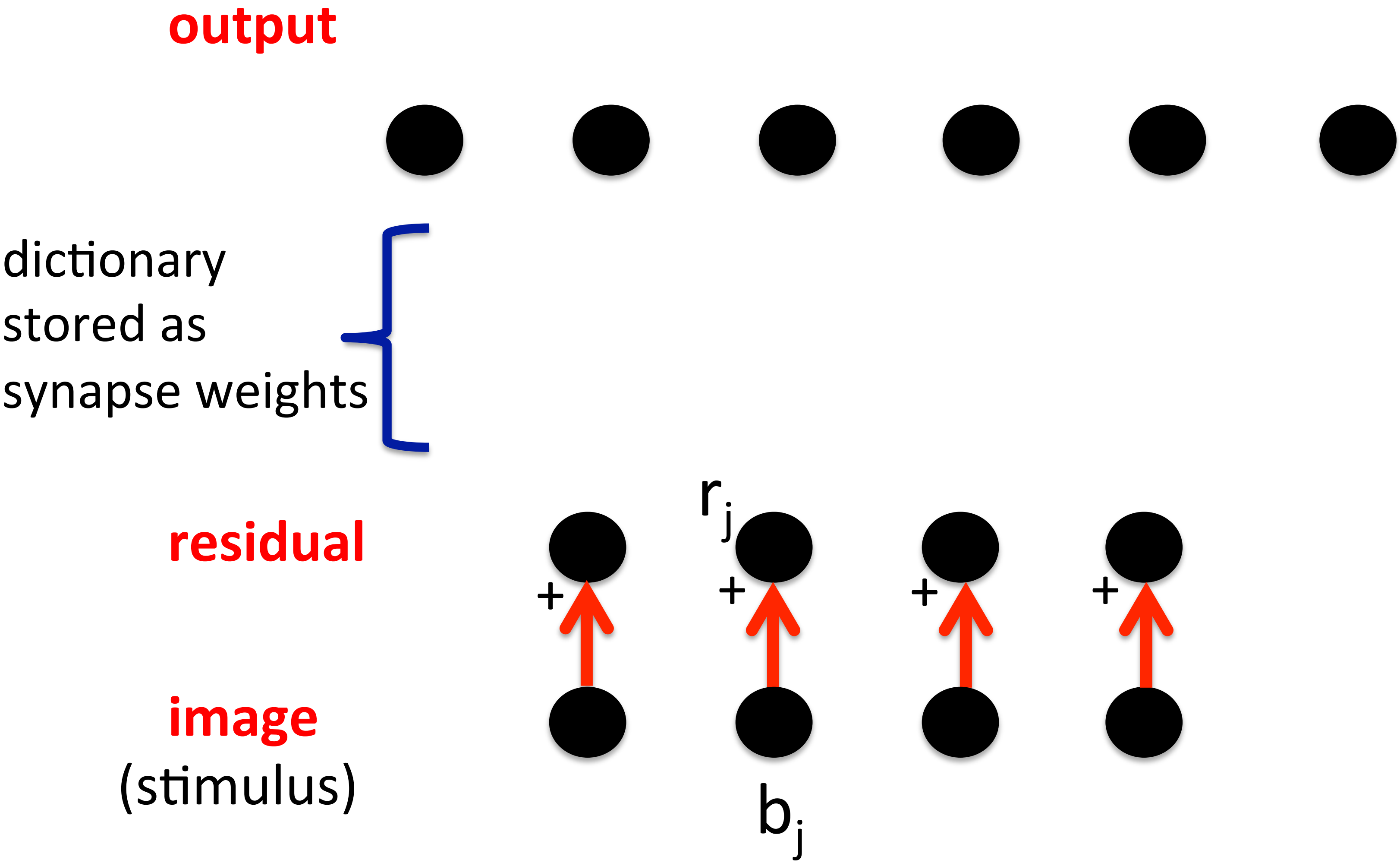

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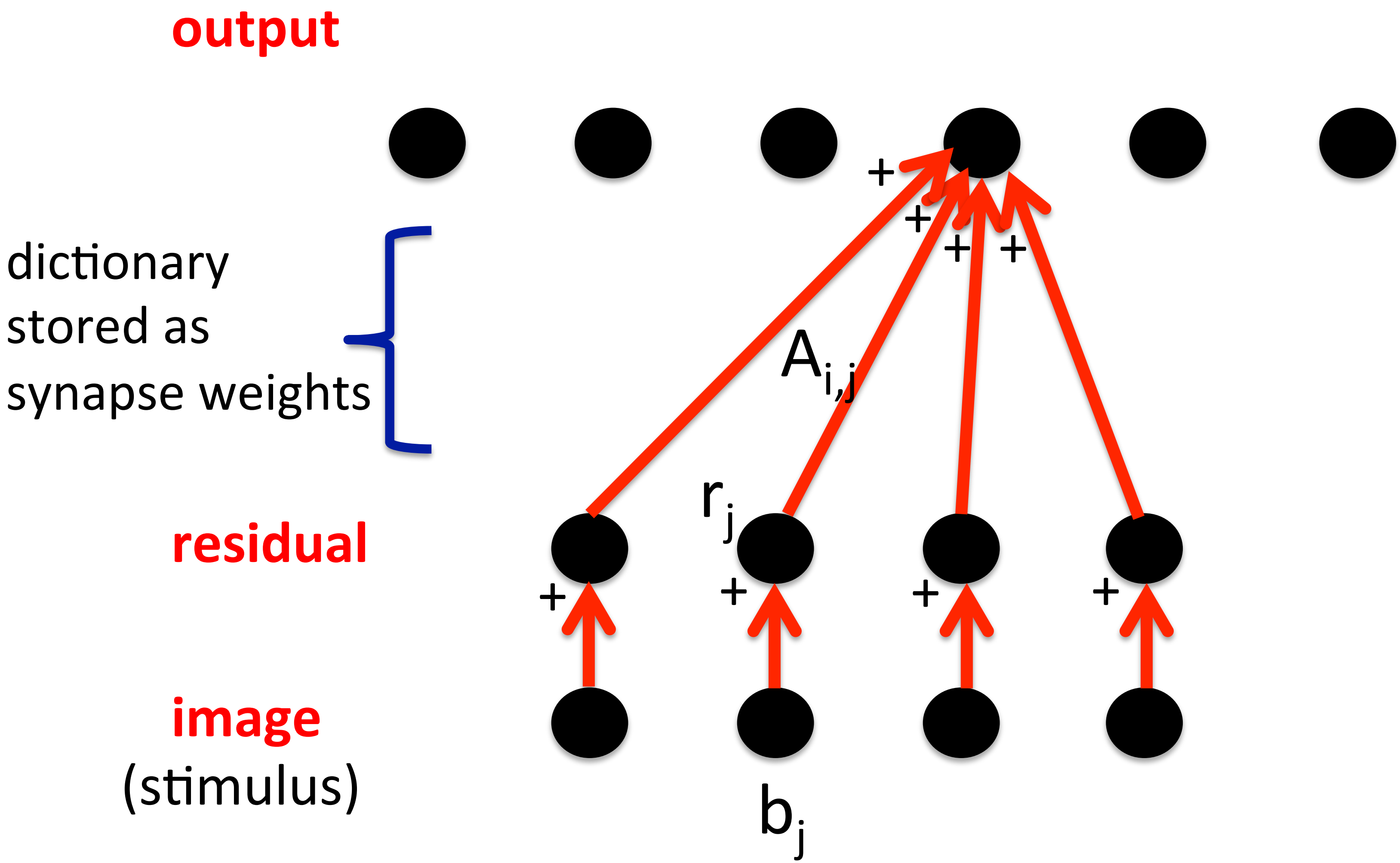

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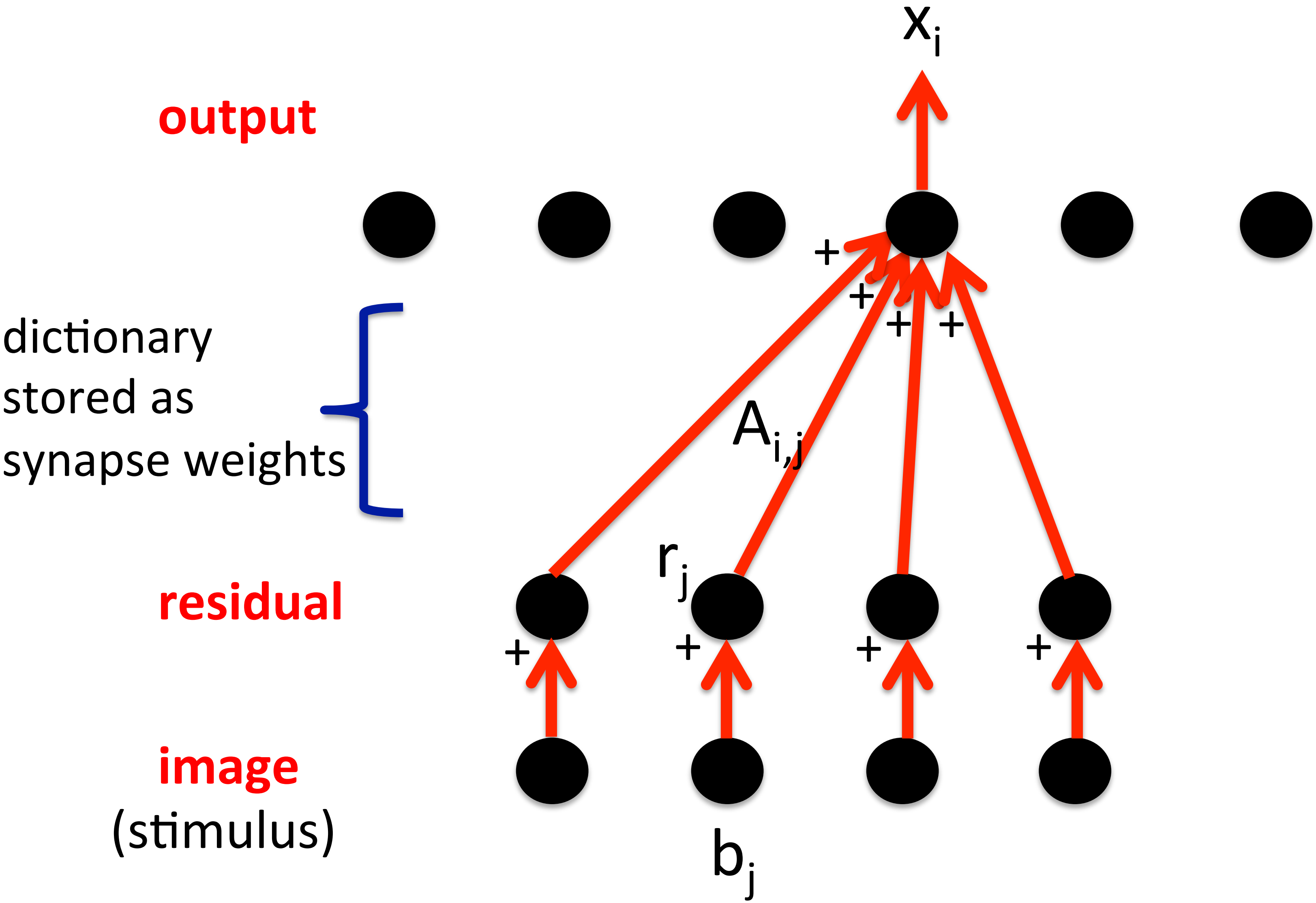

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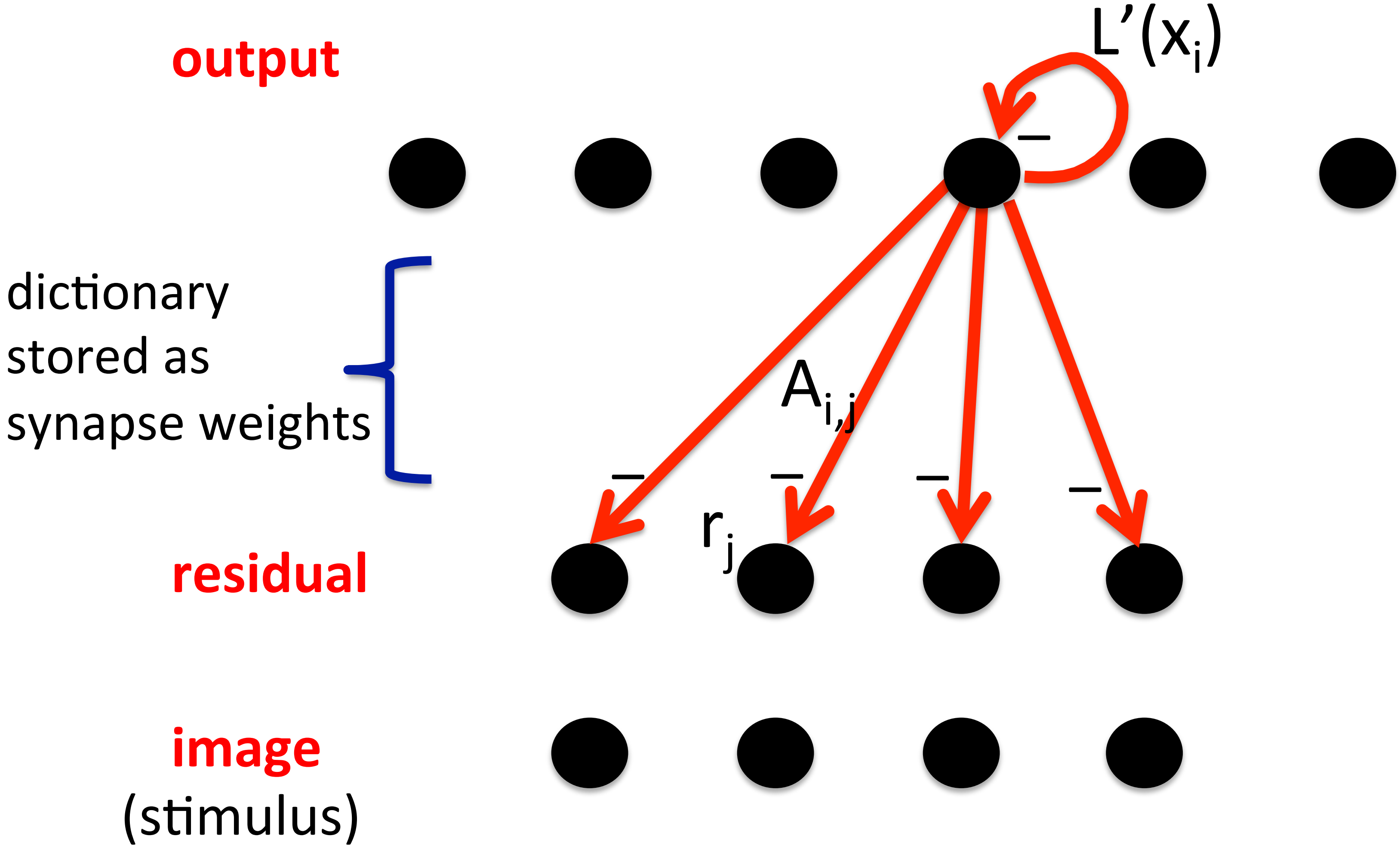

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Are simple, local and Hebbian rules sufficient to find **globally** optimal solutions?

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Theoretical Computer Science (SWW13, AGM14, AAJNT14):

- New algorithms with **provable** guarantees, in a natural **generative model**

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- unknown dictionary A
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[Barak, Kelner, Steurer ‘14]: works for overcomplete A up to sparsity roughly $n^{1-\varepsilon}$, but running time is **exponential** in accuracy

OUR RESULTS

Suppose $k \leq \sqrt{n}/\mu$ polylog(n) and $||A|| \leq \sqrt{n}$ polylog(n)

Suppose $\hat{A}$ that is column-wise δ-close to A for $\delta \leq 1/$polylog(n)

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i.e. if $k > \sqrt{n}/2\mu$, then x is not necessarily the sparsest soln to $Ax = b$

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Alternate between the following steps (size q batches):

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In contrast, previous (provable) algorithms might need to compute a new estimate **from scratch**, when new samples arrive

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In particular, the output is a thresholded, weighted sum of activations

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"neurons that fire together, wire together"

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The update to a weight $\hat{A}_{i,j}$ is the product of the activations at the residual layer and the decoding layer

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But ours is **online**, **local** and **Hebbian**, all of which are basic properties to require

The surprise is that such simple building blocks can find **globally optimal** solutions to **highly non-trivial** algorithmic problems!

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New Goal: Prove that (with high probability) the step **(2)** approximates the gradient of this function

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Theorem: If g^s is $(\alpha, \beta, \varepsilon_s)$-correlated with z^*, then

$$\| z^s - z^* \|^2 \leq (1-2\alpha\eta)^s \| z^0 - z^* \|^2 + \frac{\max_s \varepsilon_s}{\alpha}$$

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$$\langle g^s, z^s - z^* \rangle \geq \alpha \| z^s - z^* \|^2 + \beta \| g^s \|^2 - \varepsilon_s$$

Theorem: If g^s is $(\alpha, \beta, \varepsilon_s)$-correlated with z^*, then

$$\| z^s - z^* \|^2 \leq (1-2\alpha\eta)^s \| z^0 - z^* \|^2 + \frac{\max_s \varepsilon_s}{\alpha}$$

This follows immediately from the usual proof…

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$$\hat{A}_j \leftarrow \hat{A}_j + \xi (I - \hat{A}_j\hat{A}_j^T)A_j + \xi \mathbf{E}_R[\hat{A}_R\hat{A}_R^T]A_j + \text{error}$$

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Auxiliary Lemma: $\|\hat{A} - A\| \leq 2$, remains true throughout if η is small enough and q is large enough

Proof: Let ζ denote any vector whose norm is negligible (say, $n^{-\omega(1)}$).

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However, in those settings A and $\hat{A}$ are so close that the objective function is **essentially convex**

We show that it converges even from **mild** starting conditions

As a result, our bounds improve on existing algorithms in terms of **running time**, **sample complexity** and **sparsity** (all but SOS)

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Any Questions?

Summary:

- **Online**, **local** and **Hebbian** algorithms for sparse coding that find a globally optimal solution (whp)
- Introduced a framework for analyzing iterative algorithms by thinking of them as trying to minimize an **unknown**, **convex** function
- The key is working with a generative model
- Is **computational intractability** really a barrier to a rigorous theory of neural computation?