# Robustly Recovering a Signal, Under a Group Action

Ankur Moitra (MIT)

Based on joint work with Alex Wein (NYU)

# APPLICATIONS IN ENGINEERING

Many inverse problems where groups play a key role

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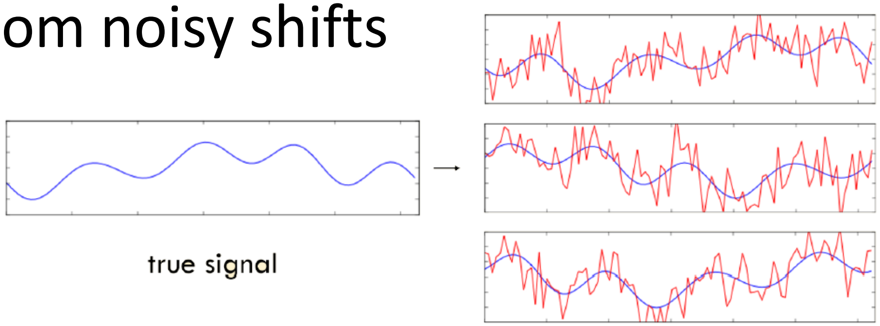
**Multireference Alignment**

Recover a signal from random noisy shifts



true signal

noisy data

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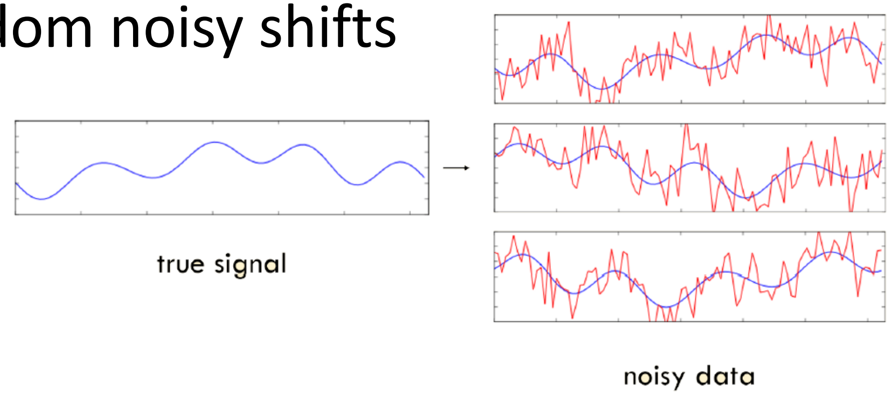
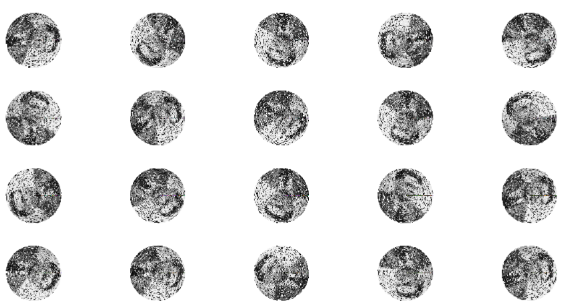



## Global Registration

Estimate positions from rigid motions



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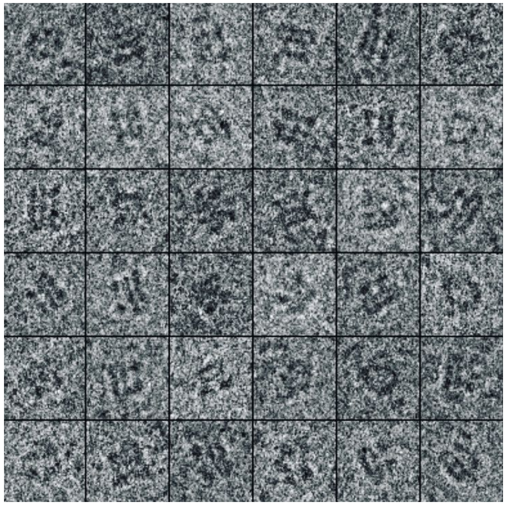
**Cryo-electron microscopy**

Determine 3D structure from random noisy 2D projections



Revolutionary technique that was awarded the **2017 Nobel Prize in Chemisitry**

# General framework for capturing inverse problems under a group action

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**Goal:** Recover some $\widehat{x}$ that is close to the orbit

$$\{\rho(g) \cdot x | g \in G\}$$

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How do popular inverse problems fit into this framework?

| Problem | Group | Notes |
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| **Cryo-EM** | $SO(3)$ | More general, apply a projection after rotation |

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How do we know we recovered the actual 3D structure, or just got stuck in some **local minimum**?

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When the noise is large, it is often **impossible** to find the relative group action

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For example, can we allow **heterogeneity**? This is natural when a molecule has multiple isomers

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**But, as we will see they only work for finite groups and ignore the group structure**

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As usual, we want algorithms with **provable guarantees**!

We will need to solve some challenging problems at the interface between combinatorics and representation theory along the way

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# WARM-UP: DISCRETE MRA

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**random cyclic shift**

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**Lemma [Perry, Weed, Bandeira, Rigollet, Singer]:** T can be estimated from samples through the relation

$$T = \mathbb{E}[(\rho(g) \cdot y)^{\otimes 3} - 3\sigma^2 \, \mathrm{sym}((\rho(g) \cdot y) \otimes I)]$$

where the expectation is over y and a random g

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# JENNRICH'S ALGORITHM

**Theorem [Jennrich]:** There is a polynomial time algorithm to decompose a tensor of the form

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We will be able to use this algorithm to solve discrete MRA

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The running time and sample complexity depend polynomially on inverse accuracy, and on the condition number of the factors

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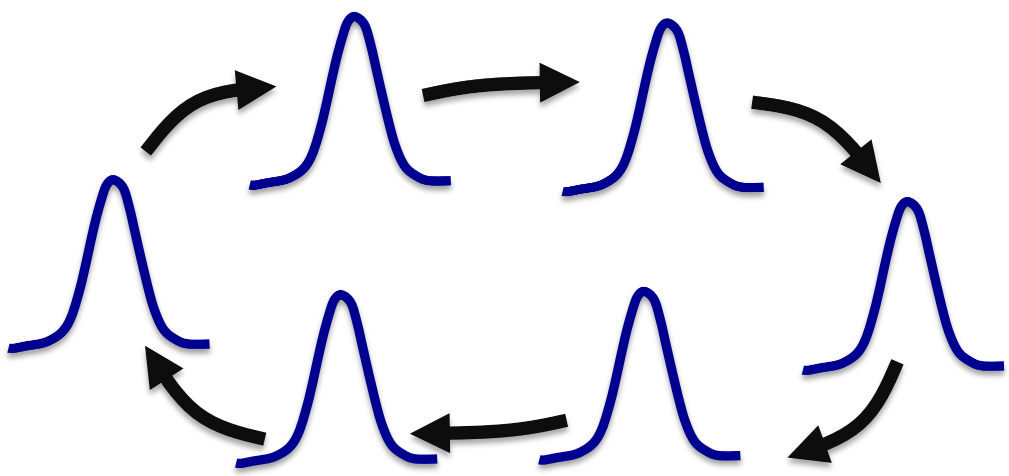

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$$\sum_{g \in G} \frac{1}{d} \mathcal{N}(\rho(g) \cdot x, \sigma^2 I)$$

and we're using tensor decomposition to learn the model, following **[Hsu, Kakade]**, but ignoring the relationship between the centers

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**We call this problem orbit tensor decomposition, and it seems to be the key to solving more general orbit recovery problems**

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**Theorem [Moitra, Wein]:** There is a polynomial time algorithm for list recovery for continuous MRA that works with additive noise and in a heterogenous setting where there are a polynomial number of random components

The algorithm is inspired by recent algorithms for rounding the **sum-of-squares hierarchy**, and an abstraction called **tensor networks**

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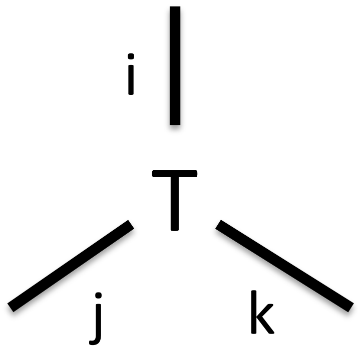


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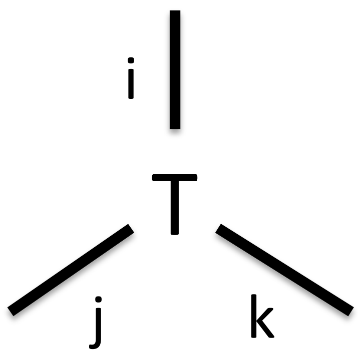


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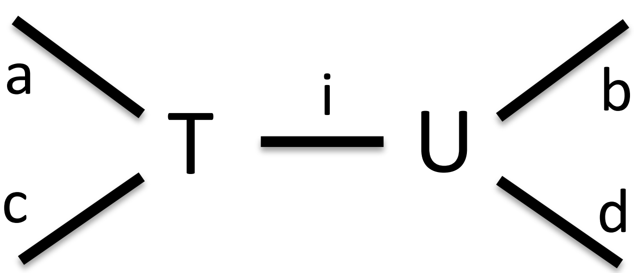
**tensors can be attached by summing over connected indices**




$$B_{a,b,c,d} = \sum_i T_{a,c,i} U_{b,d,i}$$

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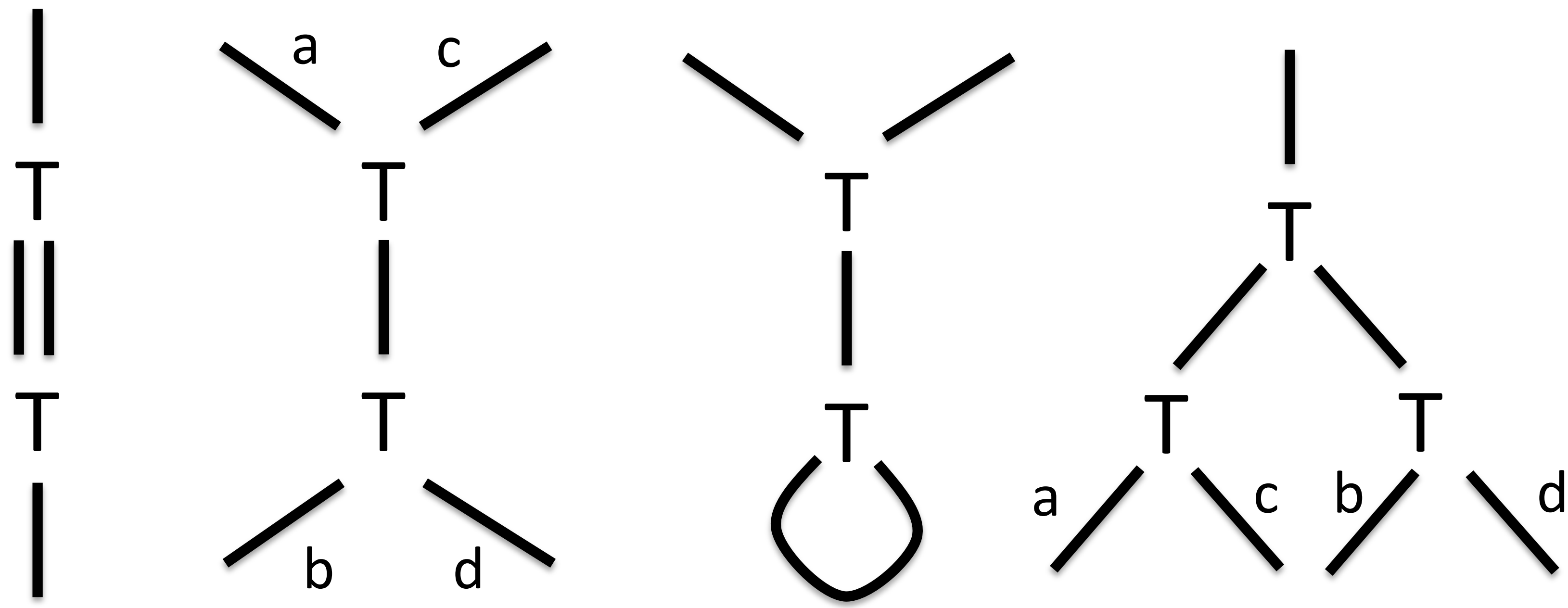
# REVISITING PRIOR WORK

Prior work implicitly uses this framework




See **[Richard, Montanari]**, **[Barak, Moitra]**, **[Hopkins, Shi, Steurer]**, **[Hopkins et al.]**, **[Hopkins, Shi, Steurer]** for applications to tensor principal component analysis, tensor completion, decomposing random overcomplete third order tensors, etc

# SPECTRAL METHODS FROM TENSOR NETS

Given input tensor T

- **Step #1:** Build a new tensor B by connecting copies of T according to the tensor network
- **Step #2:** Flatten B to form a symmetric matrix M
- **Step #3:** Compute the leading eigenvector of M

**We use the trace method to show the top eigenvector is close to the orbit of x**

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**For tensor networks, this turns into a labelled counting problem**

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- A More General Tensor Spectral Toolkit?

## Part II: Discrete vs. Continuous MRA

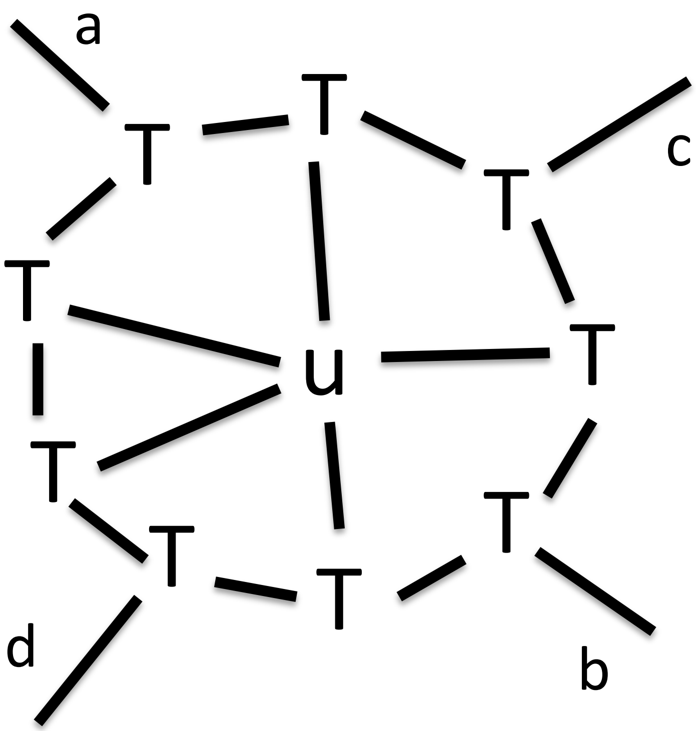
- Jennrich's Algorithm
- The Challenge of Infinite Groups
- Our Results

## Part III: Tensor Networks

- Spectral Methods from Tensor Networks
- **Our Approach, and Discussion**

# THE BLUEPRINT

We give a spectral method based on the following tensor network



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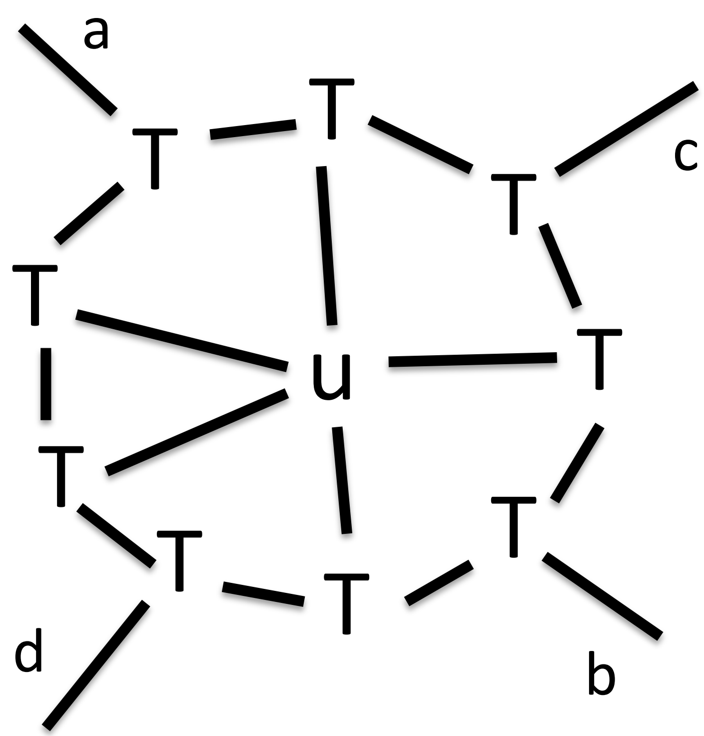


**Smaller tensor networks fail for this problem**

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With tensor networks, the trace method turns into a counting problem, **let's see some examples…**

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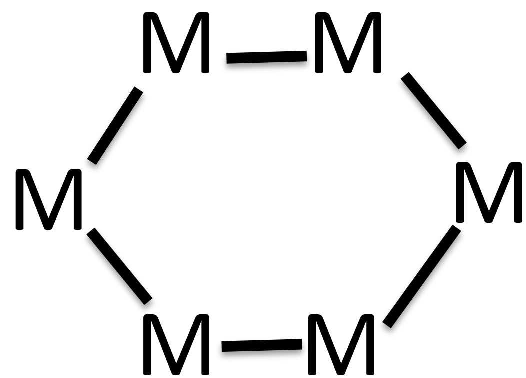
**Lemma:** $\mathbb{E}[\mathrm{Tr}(M^6)]$ is the number of ways of labeling the edges of

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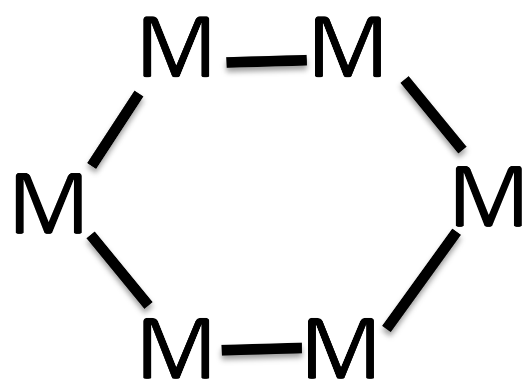


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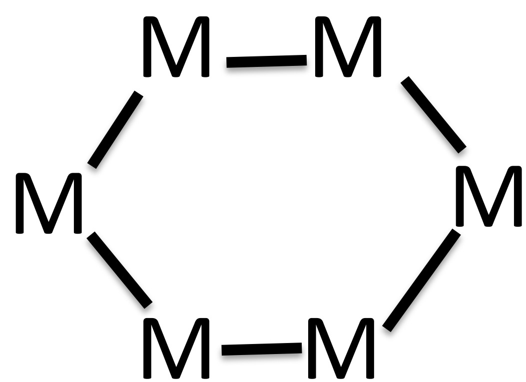


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**Proof:** First, $\mathrm{Tr}(M^6)$ is a sum over length six walks. Then observe that a term has expectation zero **unless each edge is traversed an even number of times**. $\blacksquare$

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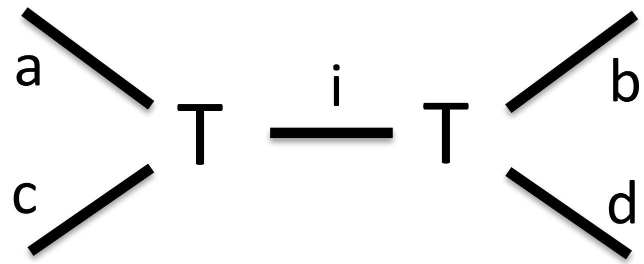

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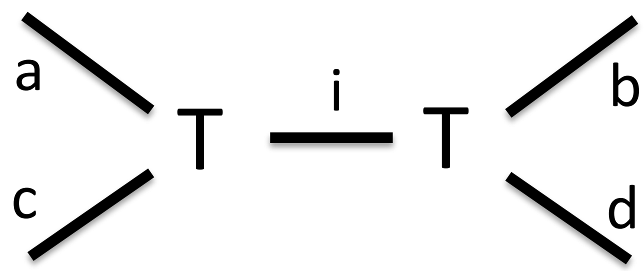


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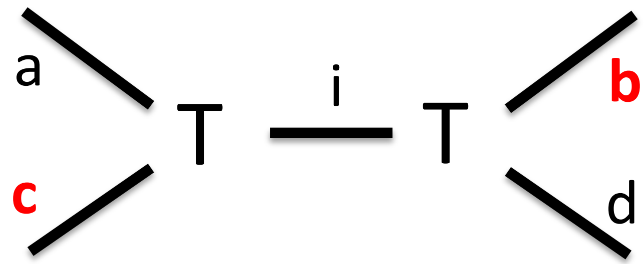


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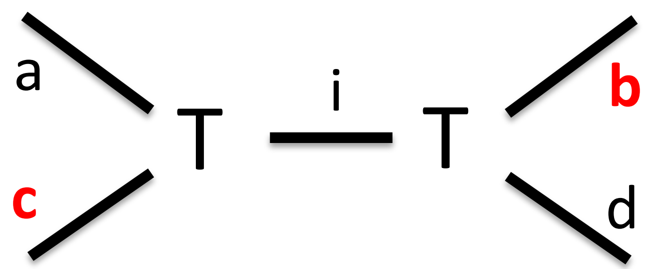


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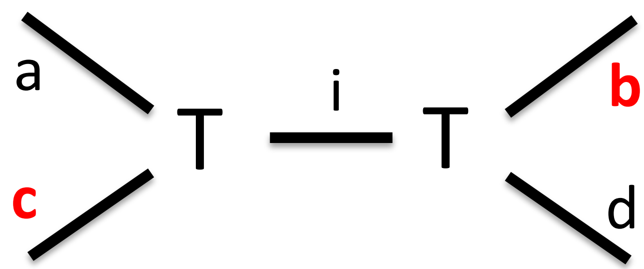


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For example, if we want to compute $\mathbb{E}[\mathrm{Tr}(M^6)]$ we can plug the tensor network into the six cycle, and we get…

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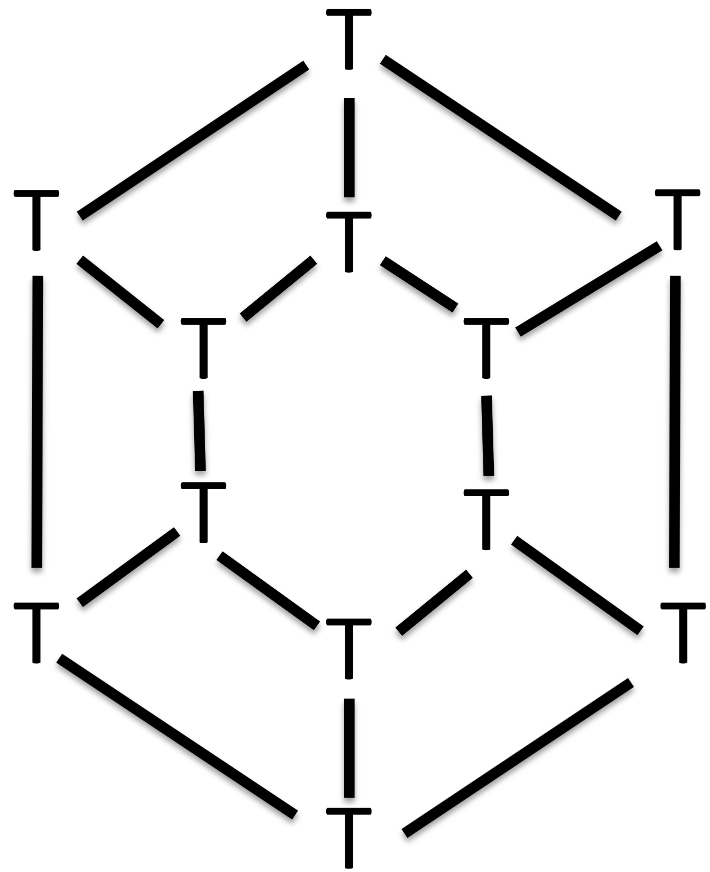

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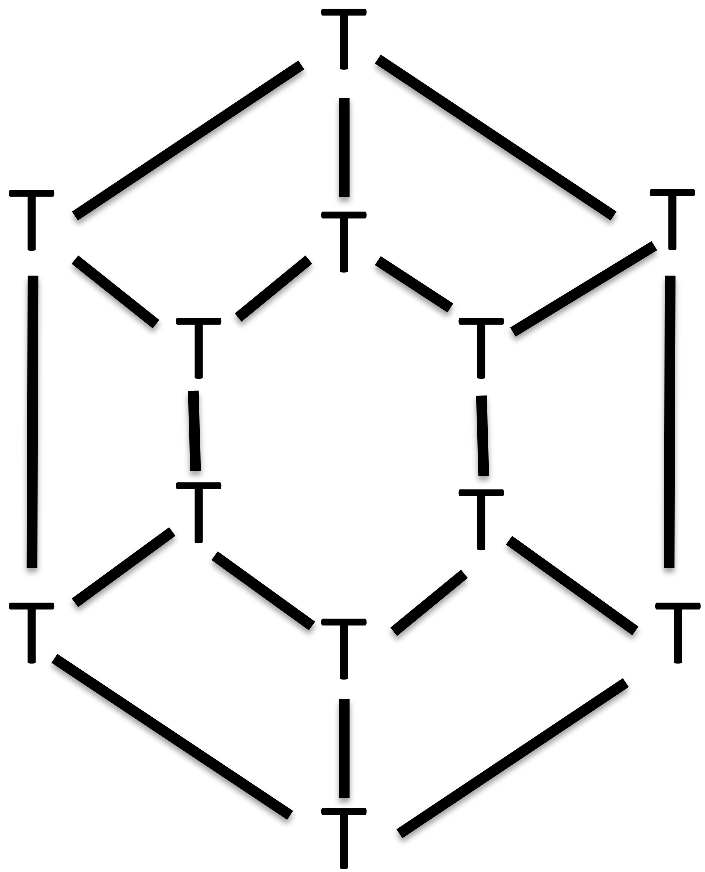


And $\mathbb{E}[\mathrm{Tr}(M^6)]$ is the number of ways to label the edges of the diagram so that **each triple {i, j, k} appears incident to an even number of T's**.

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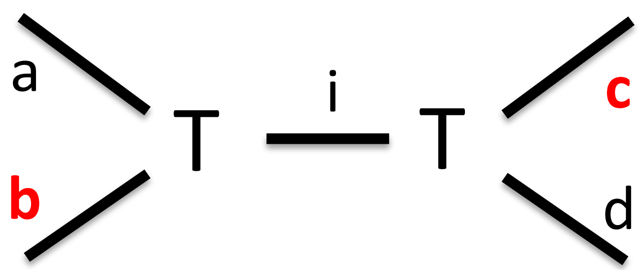

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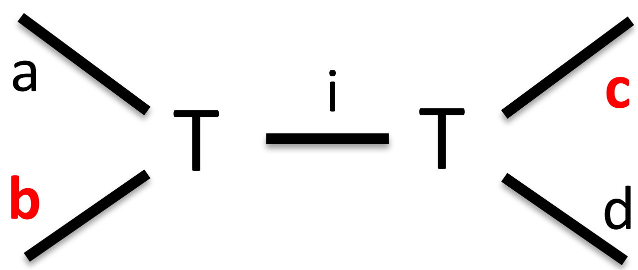


then plugging it into the six cycle we would get **something different**

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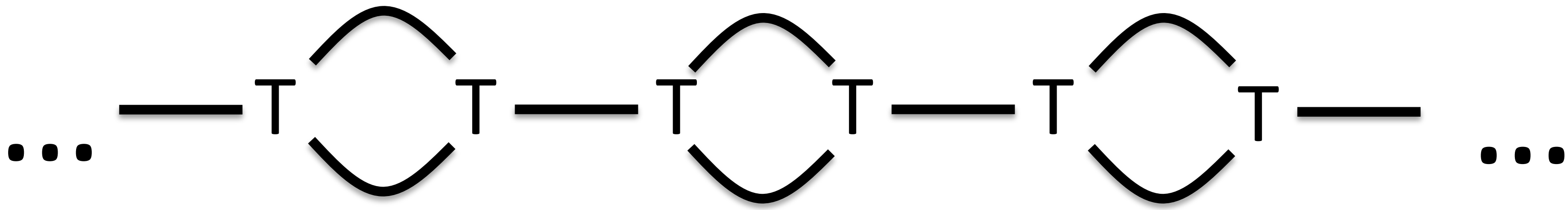

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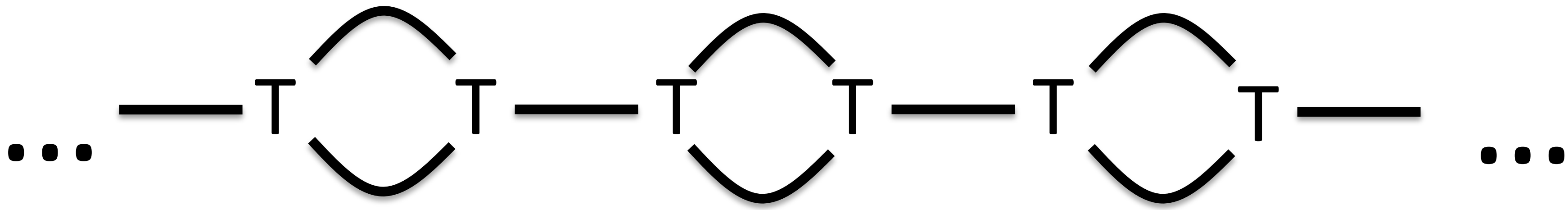


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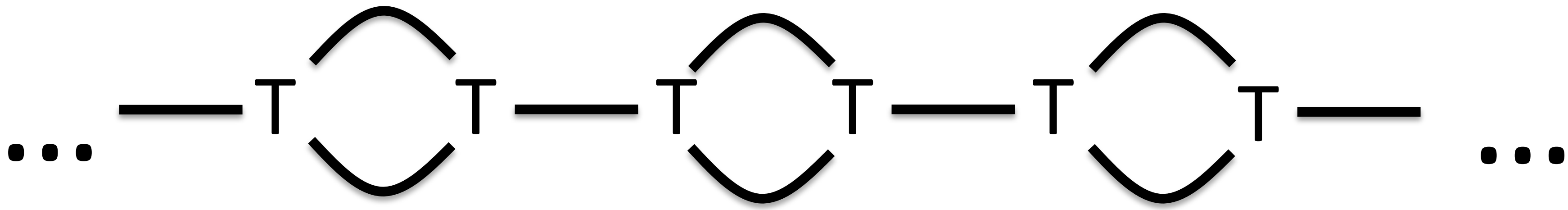


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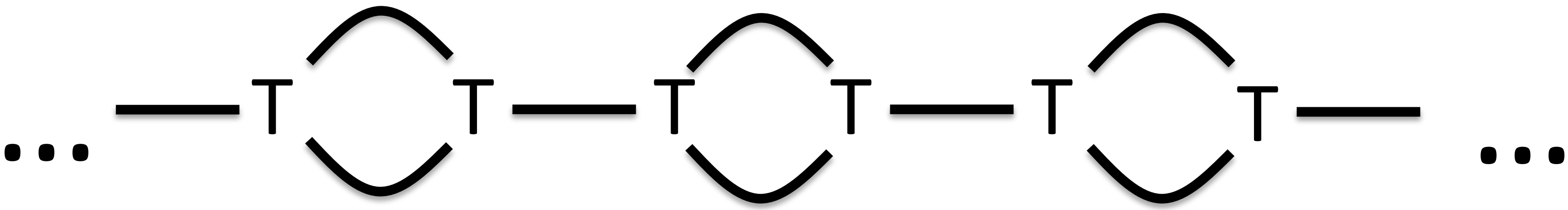


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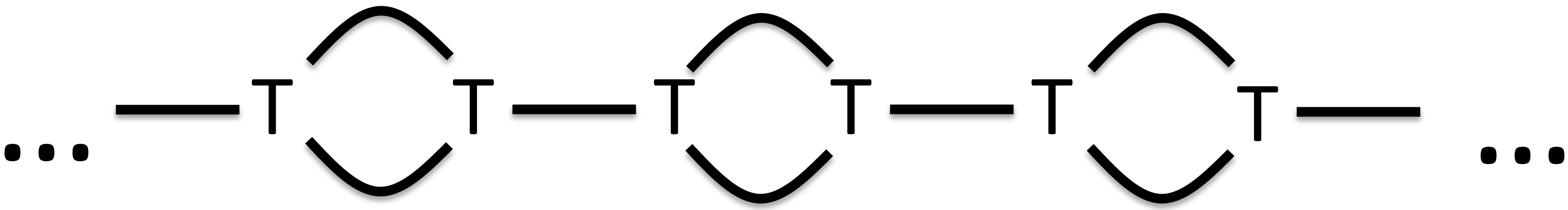


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**Tensor networks are a convenient way to think about this trick, and others that appear in the sum-of-squares literature**

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In the fourier domain, T is supported on indices (i, j, k) where

$$i + j + k = 0$$

and this becomes a constraint in our labeling problem

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For more complex groups, can we find good tensor networks?

**We understand how invariant theory governs the statistical complexity of orbit recovery, but not how to get good algorithms!**

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**Part IV: Invariant Theory**

- Generating the Ring, Generically
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When s = 3, this is called the **bispectrum** and was introduced by Tukey in a statistical context

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There is a subtlety already for discrete MRA…

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In orbit recovery on a compact group G

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**But they do not get algorithms (because they would need to solve systems of polynomial equations to find a signal that is consistent with the system of invariant polynomials)**

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**Our tensor network approach can be thought of as a first step towards making this algorithmic! (We get optimal sample complexity for continuous MRA)**

## Summary:

- Orbit recovery is a challenging family of problems over a group action, with many applications
- We gave the first algorithms that work with infinite groups, along with a general framework to design tensor spectral algorithms from tensor networks
- **Is invariant theory the new method of moments?**

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# Thanks! Any Questions?