Learning Restricted Boltzmann Machines

Ankur Moitra (MIT)

joint work with Guy Bresler and Frederic Koehler

GRAPHICAL MODELS

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Generalizations: larger alphabet (Potts model), higher-order interactions (Markov Random Field), directed (Bayesian network)

CONDITIONAL INDEPENDENCE

Often helpful to look at their graph structure:

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Can we learn graphical models from random samples?

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[Bresler et al. ‘08], **[Ravikumar et al. ‘10]**: Better algorithms when there are no long range correlations

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Part I: Introduction

- Learning Ising Models
- Latent Variables and Higher-Order Dependencies
- Our Results

Part II: Learning Ferromagnetic RBMs

- The Discrete Influence Function
- A Greedy Algorithm
- The Griffiths-Hurst-Sherman Inequality

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Scientific theories that explain data in a more parsimonious way can be learned/tested

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Are there efficient algorithms for learning RBMs?

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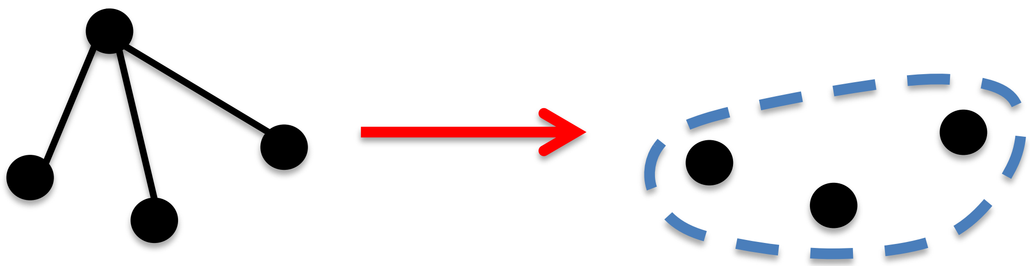

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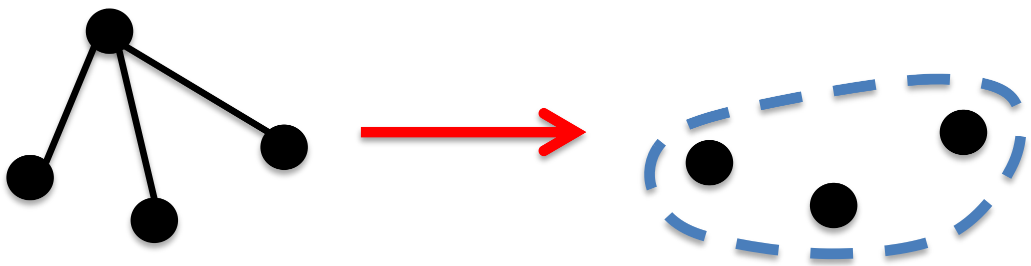

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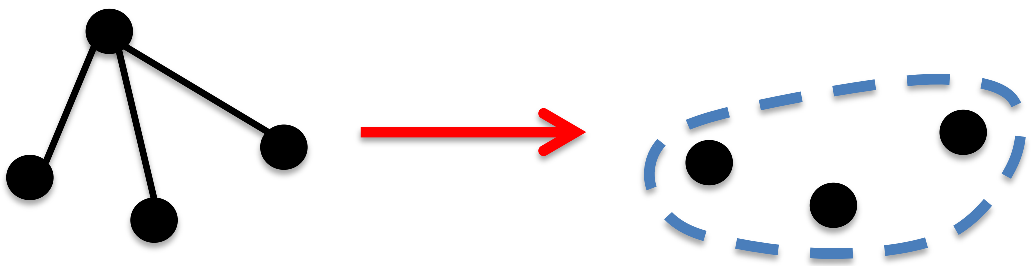

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So what type of distribution is it?

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Even worse, the reduction produces bounded degree MRFs

Main question (revised): Let d be the maximum degree

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These algorithms are close to trivial, because we can always brute-force search for the two-hop neighbors of a node in n^{d^2} time

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Earlier work of **[Martens et al. '13]** showed that dense RBMs can represent parity (more generally, any predicate depending on # 1s)

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Earlier work of **[Bogdanov et al. '08]** required a large number of latent variables, one for each gate in a given circuit

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In our context, it prevents hidden nodes from cancelling out each other's lower-order interactions

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Theorem: There is a greedy algorithm with running time f(d) n^2 and sample complexity f(d) log n for learning ferromagnetic RBMs, with upper and lower bounds on the interaction strength

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Everything generalizes to ferromagnetic Ising models with latent variables, under conditions on diameter of latent nodes

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It turns out that the **concavity of magnetization** is analogous to properties of the **multilinear extension**

A HINT AT THE CONNECTION

Definition: The **average magnetization** is

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(2) is called **concavity of magnetization**, and follows from the famous **Griffiths-Hurst-Sherman inequality** and captures diminishing returns

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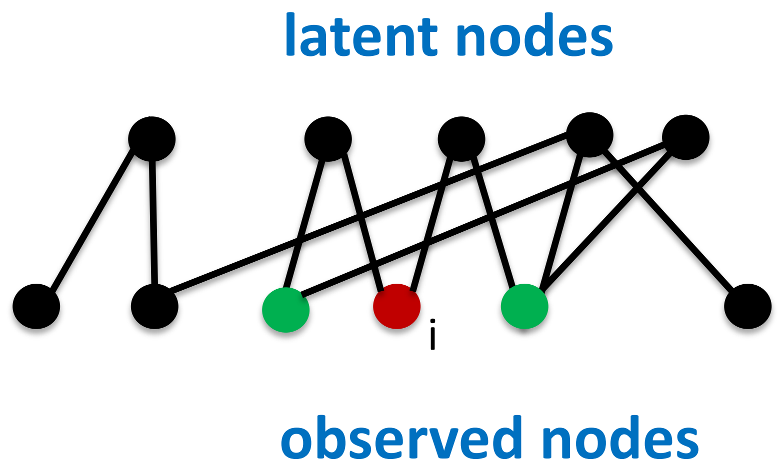

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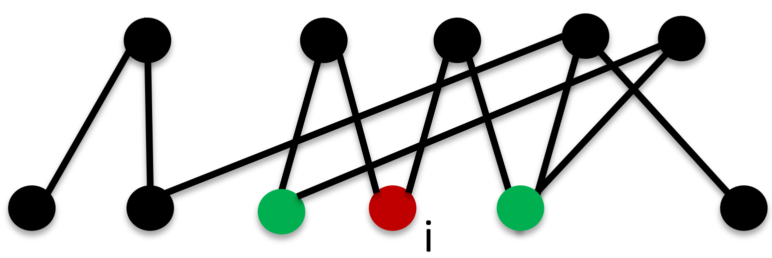

observed nodes

Because the two-hop neighbors separate i from all the other observed nodes

QUANTITATIVE BOUNDS

We say that an Ising model is (α, β)-nondegenerate if

(1) $J_{i,j} \neq 0 \Rightarrow |J_{i,j}| \geq \alpha$

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Key Lemma: If S does not contain the two-hop neighbors of i, then there is a node j such that

$$I_i(S \cup \{j\}) - I_i(S) \geq \left(\frac{2\alpha^2}{1 + e^{2\beta}}\right)(1 - \tanh(\beta))^2$$

KEY IDEAS, CONTINUED

Classic result in approximation algorithms:

Theorem [Nemhauser et al. '78]: The greedy algorithm achieves a $1 - 1/e$ factor approximation for maximizing a monotone submodular function subject to a cardinality constraint

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Idea #3: Run the greedy algorithm to learn a small superset of the two-hop neighbors

KEY IDEAS, CONTINUED

Finally when we have a small superset of the two-hop neighbors, we can learn the induced MRF

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The key is, each node no longer participates in n^d possible order d interactions, but rather at most f(d)

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In retrospect, it is the **multilinear extension** of I_i

THE GHS INEQUALITY

The Griffith-Hurst-Sherman inequality states

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Each of these terms arises as a partial derivative of the log partition function, and so does the smooth influence function

SOME IMPLICATIONS

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submodularity

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Alternatively, are there other natural classes of RBMs that admit efficient learning algorithms?

Summary:

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- Greedy algorithm for learning ferromagnetic RBMs based on submodularity

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Thanks! Any Questions?