Extensions and Limits to Vertex Sparsification

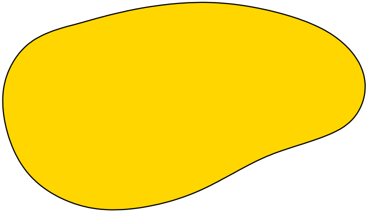
Ankur Moitra, MIT

joint work with Tom Leighton

June 5, 2010

Flow Sparsification

$G = (V, E)$



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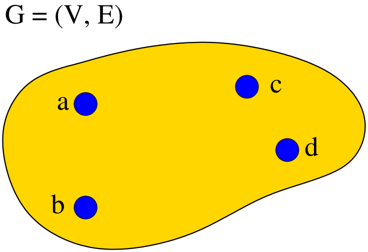

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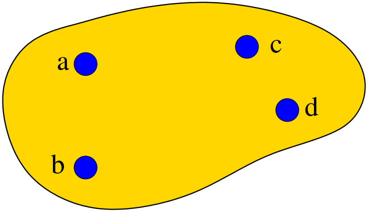

$K = \{a, b, c, d\}$

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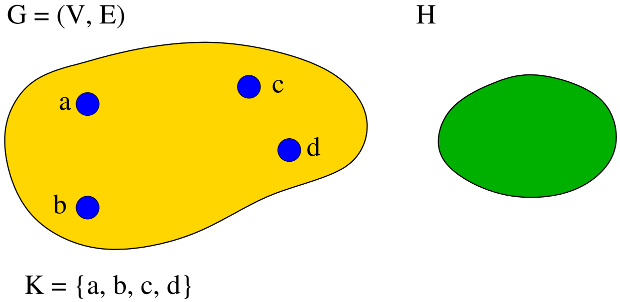
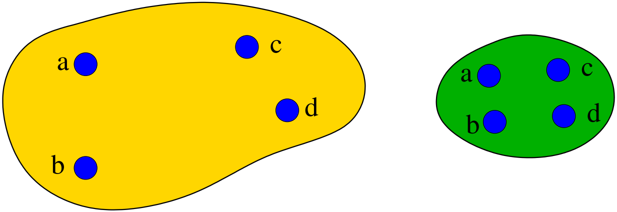


Flow Sparsification

$G = (V, E)$ $\quad$ $H = (K, E_H)$



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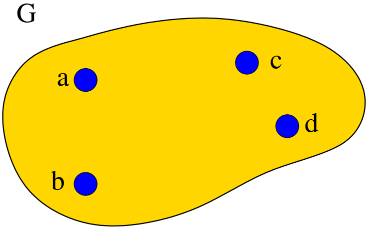
Multicommodity Flow and Congestion



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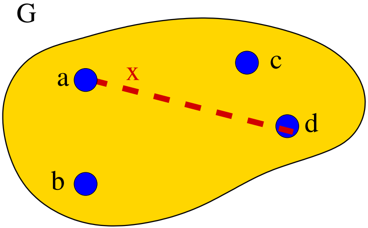

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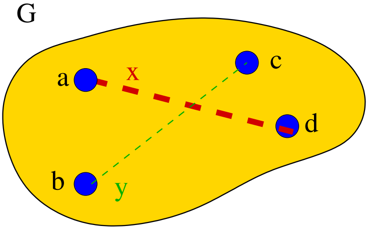

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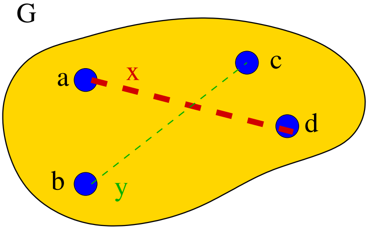

$$\vec{f}=\begin{bmatrix}0\\0\\{\color{red}x}\\{\color{green}y}\\0\\0\end{bmatrix}\begin{matrix}a\longleftrightarrow b\\a\longleftrightarrow c\\a\longleftrightarrow d\\b\longleftrightarrow c\\b\longleftrightarrow d\\c\longleftrightarrow d\end{matrix}$$

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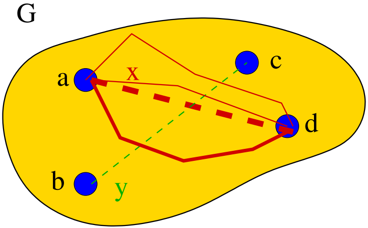

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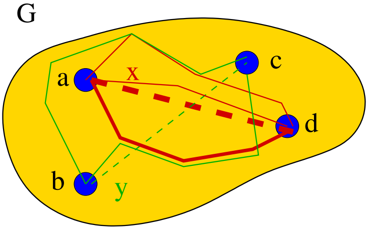

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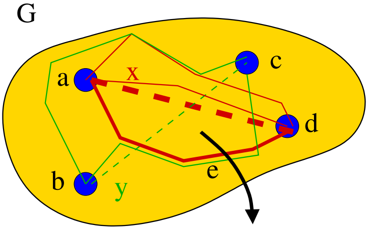

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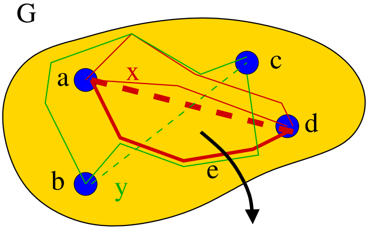

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$$\text{cong(routing)} = \max_{e} \frac{\text{flow}(e)}{\text{capacity}(e)}$$

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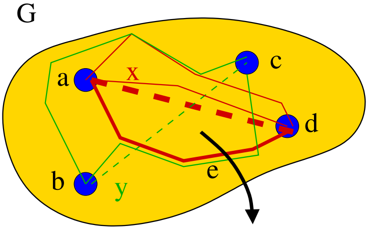

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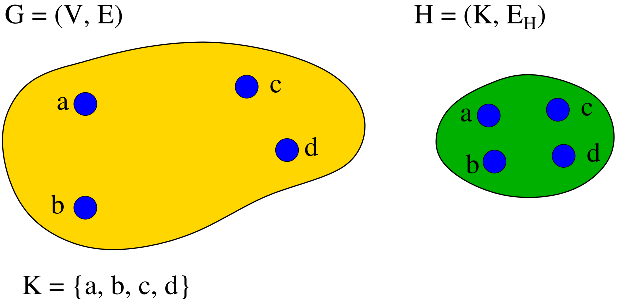
$$\text{cong}_G(\vec{f}) = \min_{\text{routings}} \max_{e} \frac{\text{flow}(e)}{\text{capacity}(e)}$$

Flow Sparsification



Definition

A graph $H = (K, E_H)$ <u>on just the terminal set</u> is a **Flow-Sparsifier** if for all demands $\vec{f} \in \Re^{\binom{K}{2}}$

$$cong_H(\vec{f}) \leq cong_G(\vec{f})$$

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Quality measures how faithfully H approximates G as a communication network

Our Results I

Theorem

For any undirected, capacitated graph $G = (V, E)$ and any set $K \subset V$ of k terminals, there is an $O(\log k/\log\log k)$-quality flow-sparsifier $H = (K, E_H)$.

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There is a polynomial (in n and k) time algorithm to compute a $O(\log^2 k / \log\log k)$-quality flow-sparsifier

Previous Results

[M, 09]: There is a graph $H = (K, E_H)$ so that the cut-function of H approximates minimum cuts in G (separating subsets of terminals)

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Flow Sparsification is <u>harder</u> than Cut Sparsification

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Motivation for vertex sparsification: obtain approximation algorithms with guarantees independent of n

Approximation algorithms that reduce to a k-terminal graph must lose a **super-constant** factor in the approximation guarantee

Outline

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Computing Quality

Suppose we are given a flow-sparsifier H

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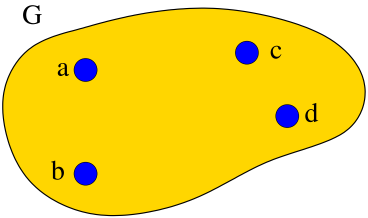
Question

Can we compute the quality of H?

Computing Quality



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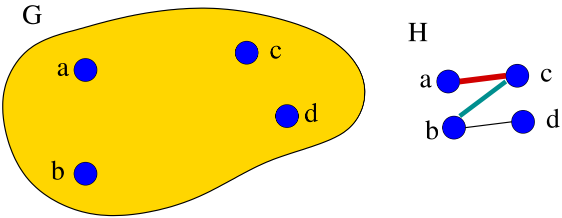

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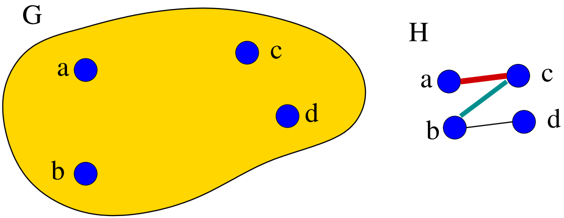

$$\vec{f}_H = \begin{bmatrix} c_H(a,b) \\ c_H(a,c) \\ c_H(a,d) \\ c_H(b,c) \\ c_H(b,d) \\ c_H(c,d) \end{bmatrix}$$

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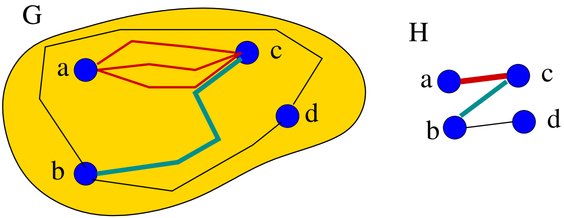

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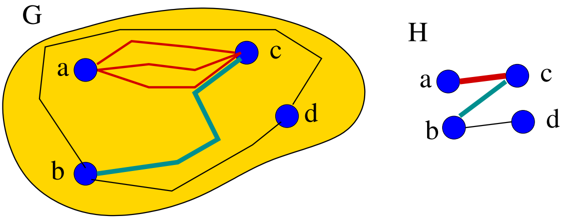

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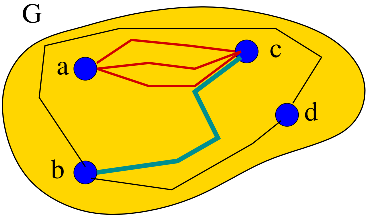

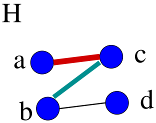

$$\vec{f} = \begin{bmatrix} x \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

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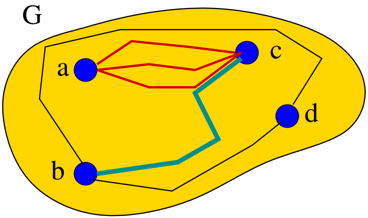

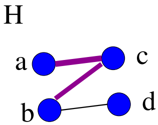

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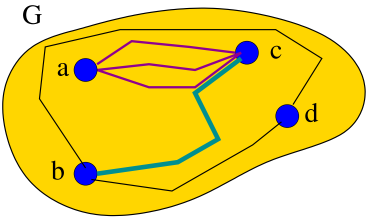

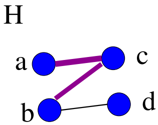

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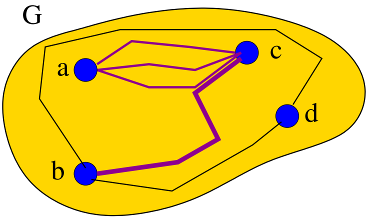

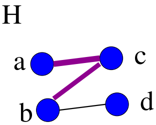

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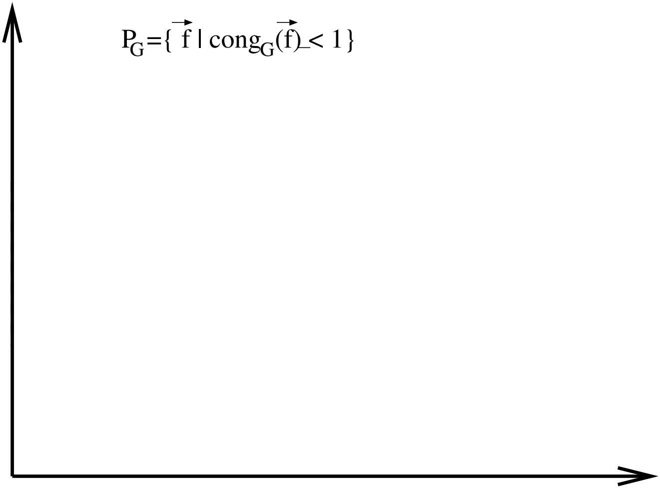

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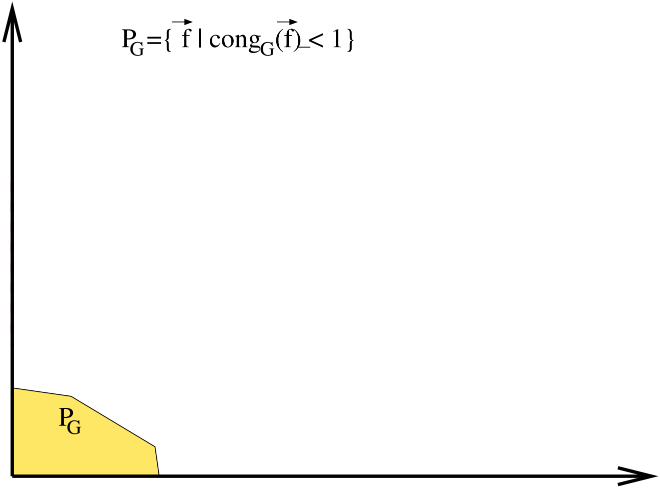

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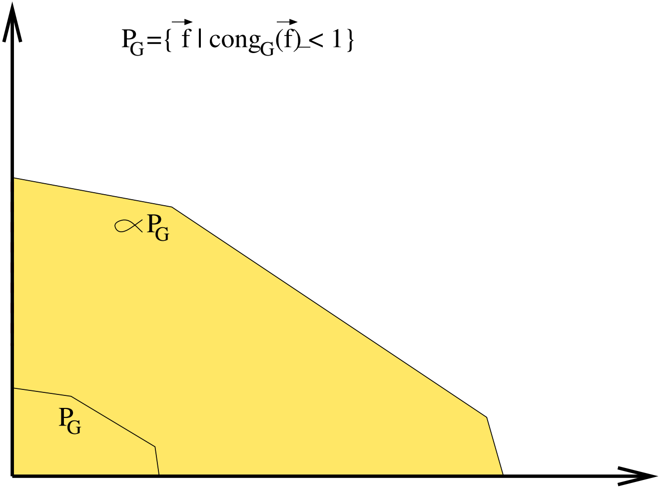

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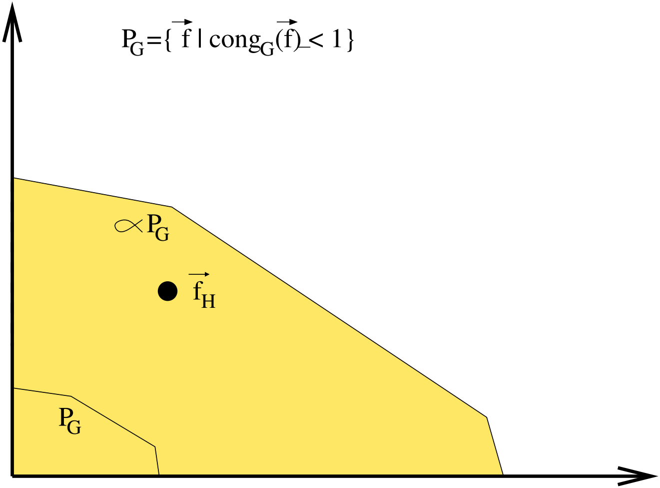

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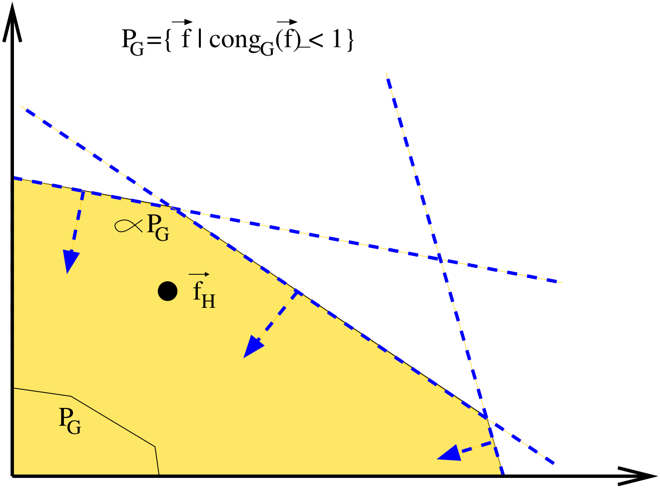

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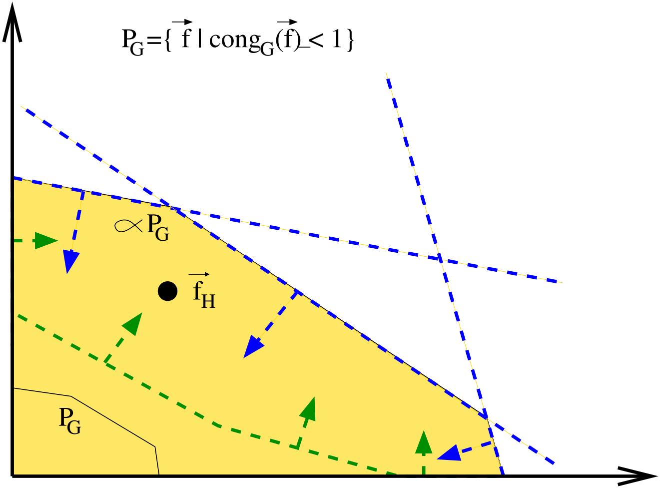

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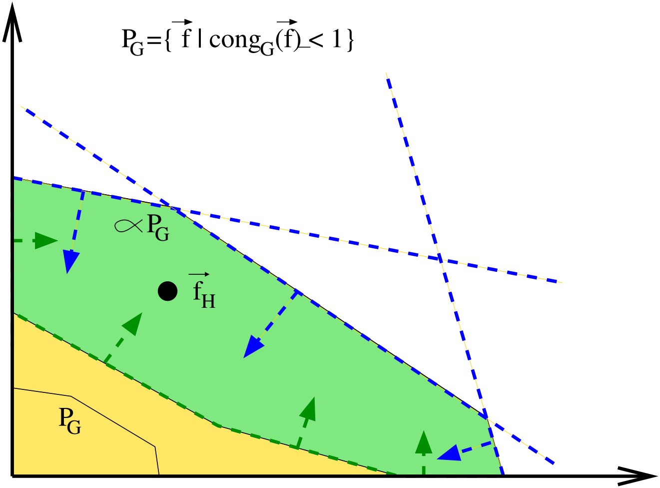

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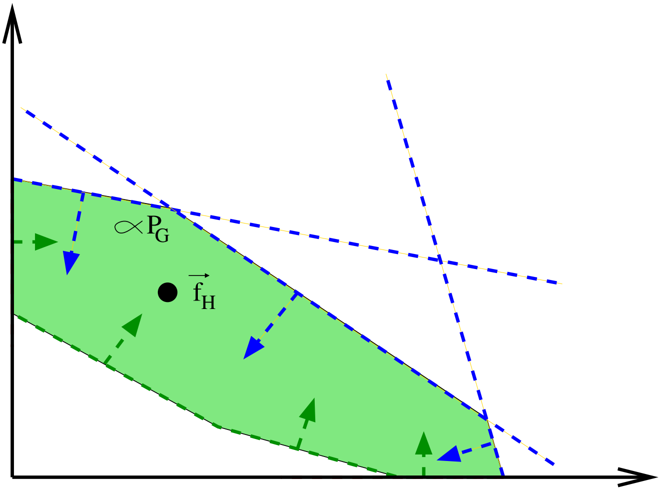

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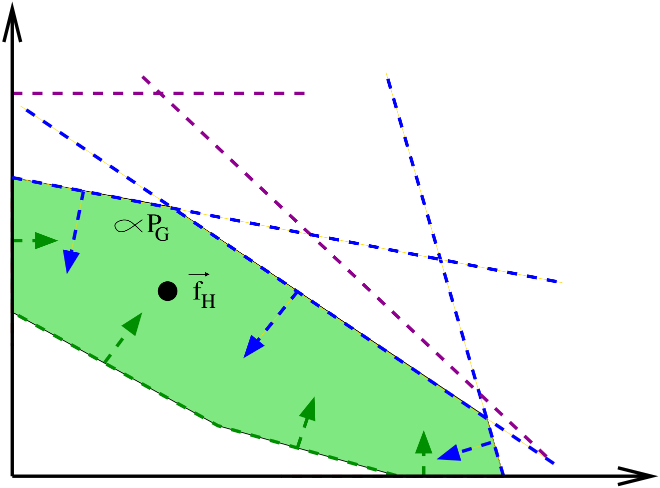

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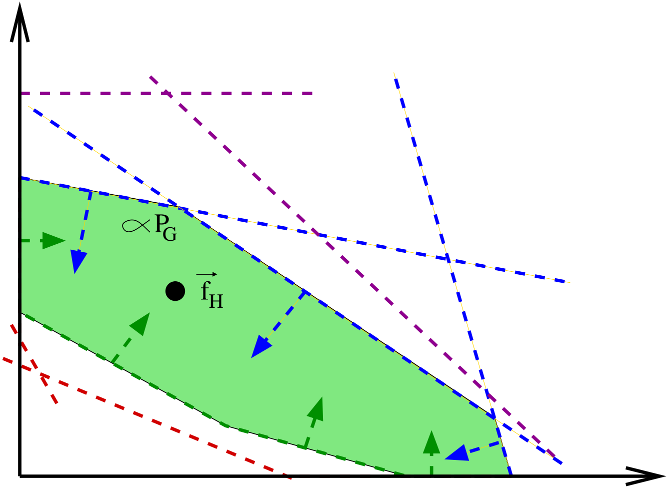

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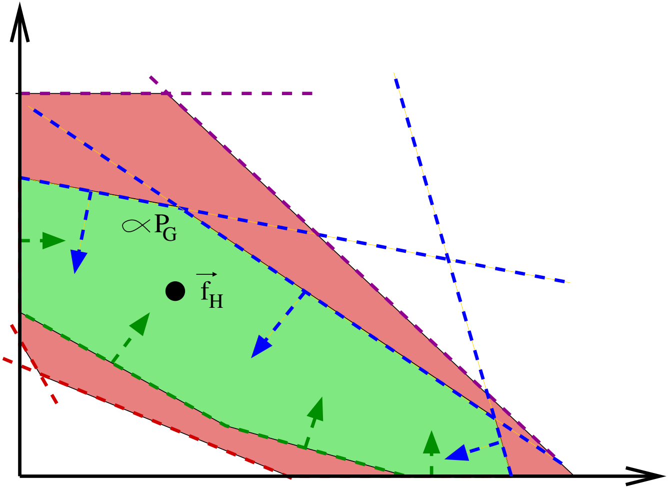

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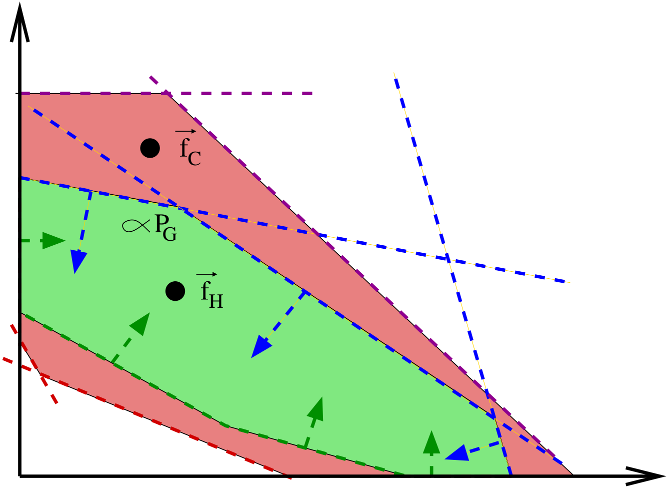

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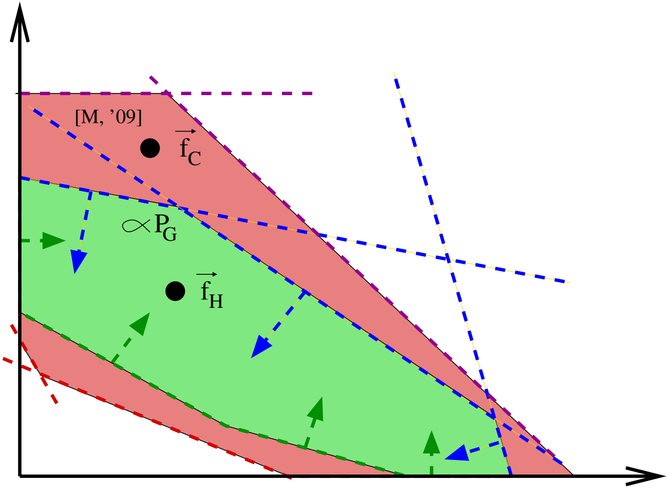

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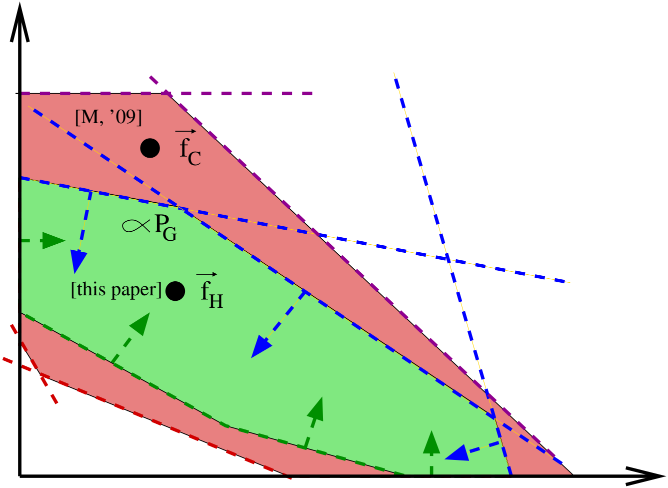

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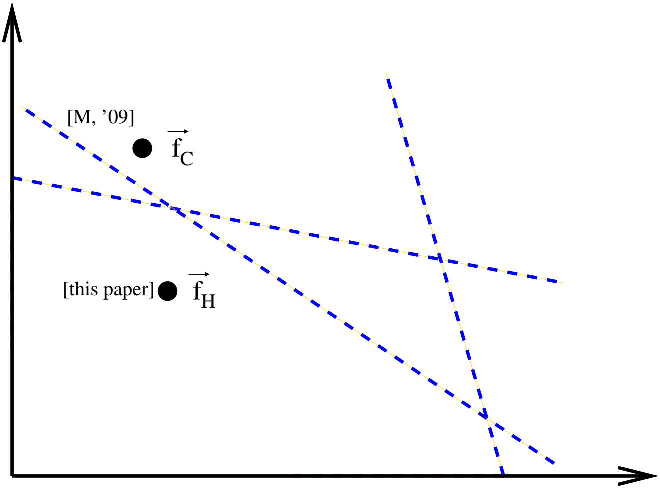

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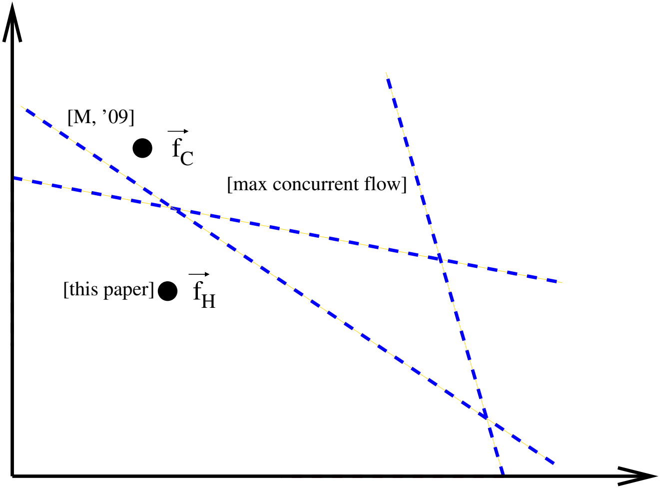

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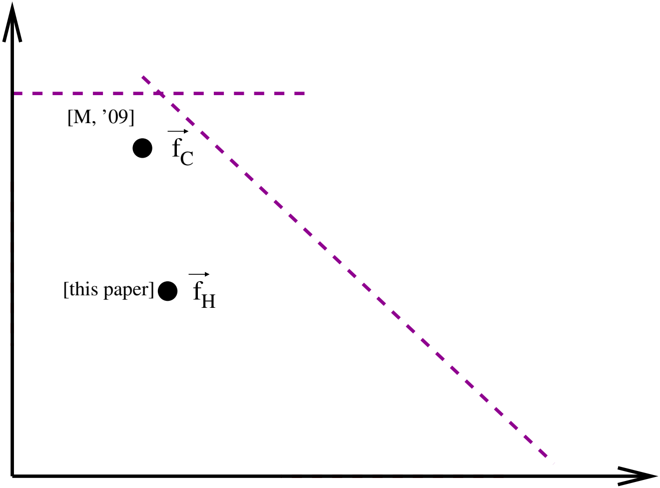

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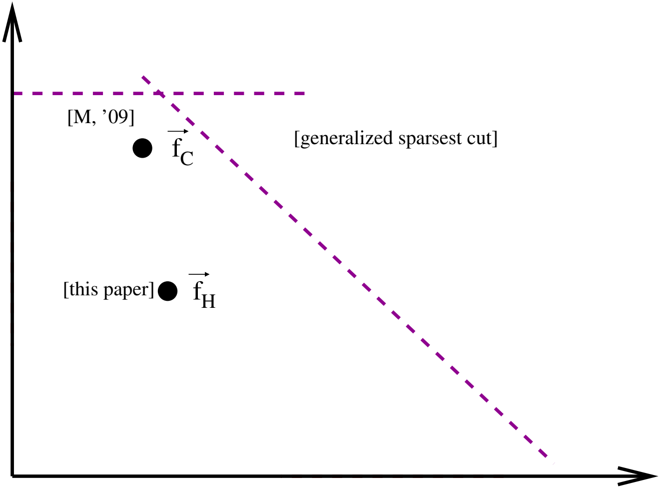

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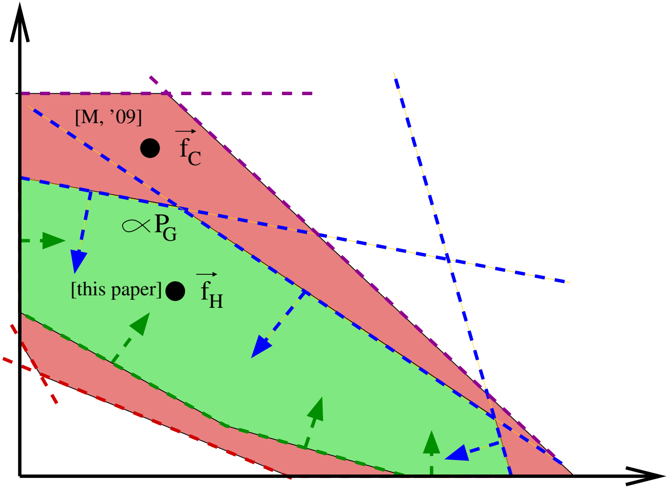

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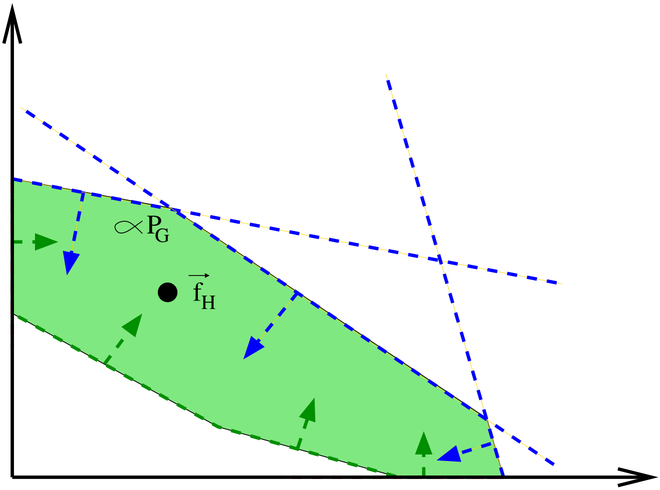

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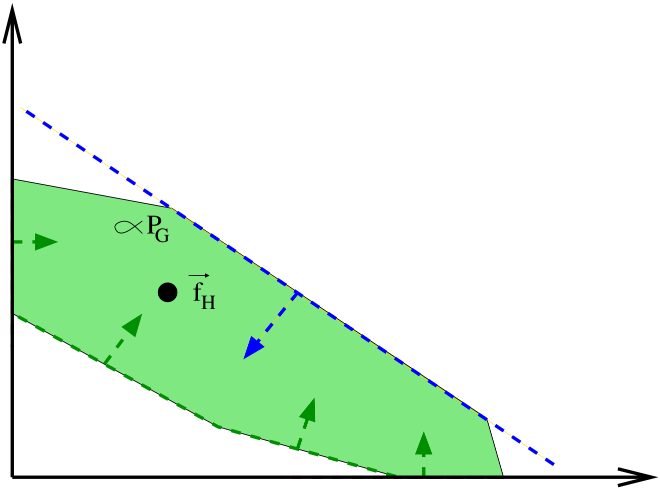
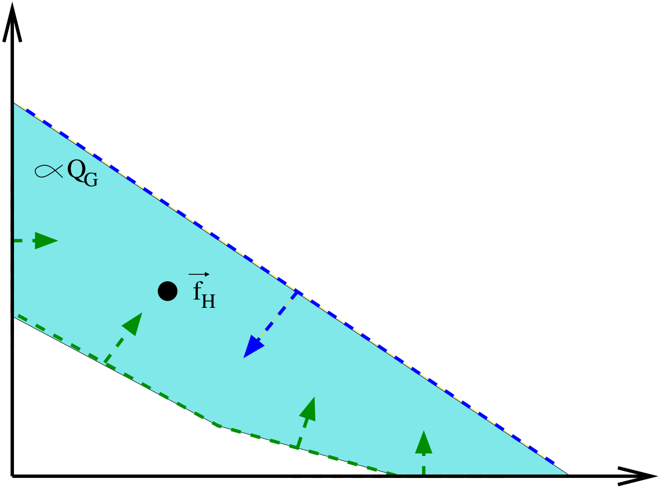
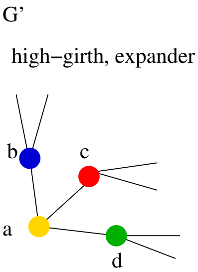


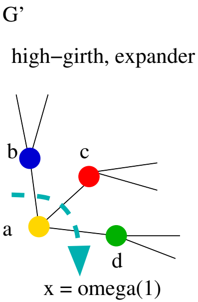
Proving Lower Bounds on Expanders



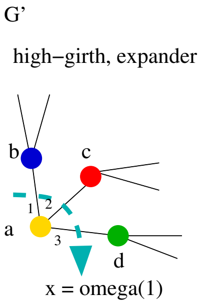
The Graph



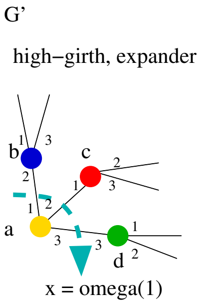
The Graph



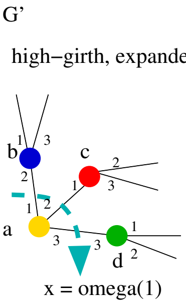
The Graph

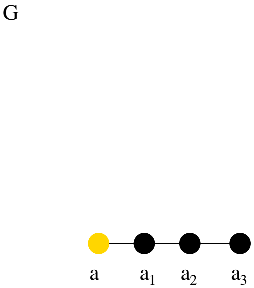


The Graph

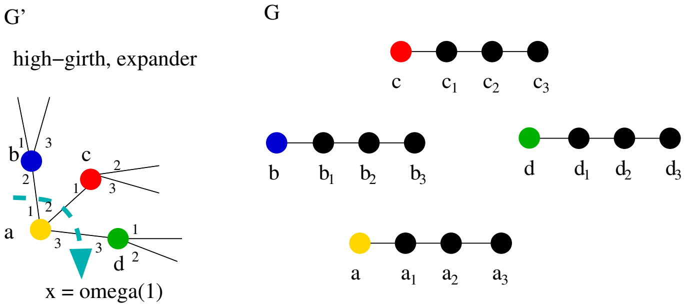


The Graph

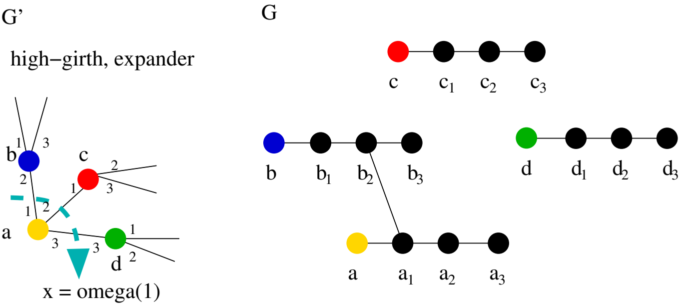




The Graph


G'
high−girth, expander
b
c
a
d
1
3
2
1
2
3
1
2
3
1
2
3
x = omega(1)
G
c
c_1
c_2
c_3
b
b_1
b_2
b_3
d
d_1
d_2
d_3
a
a_1
a_2
a_3

The Graph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1
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c
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c2
c3
b
b1
b2
b3
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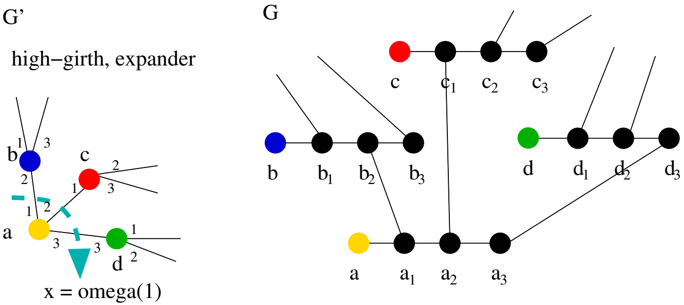
d1
d2
d3
a
a1
a2
a3

The Graph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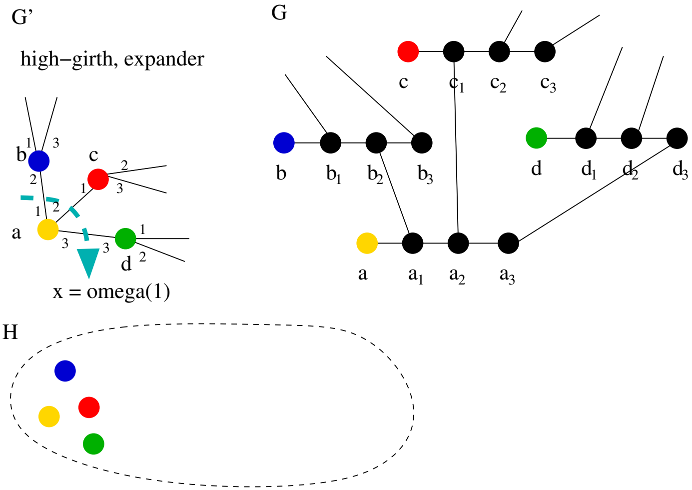
c2
c3
b
b1
b2
b3
d
d1
d2
d3
a
a1
a2
a3

A Thought Experiment



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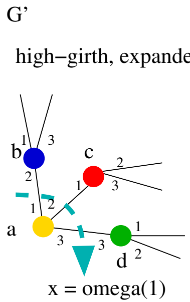

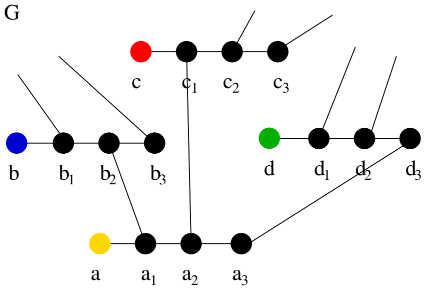

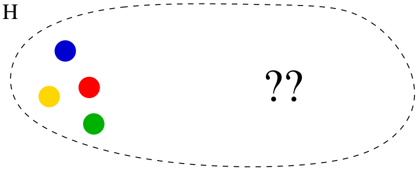

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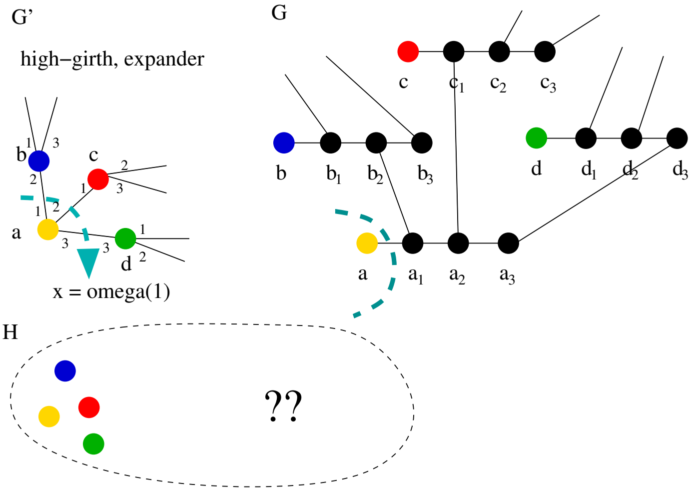

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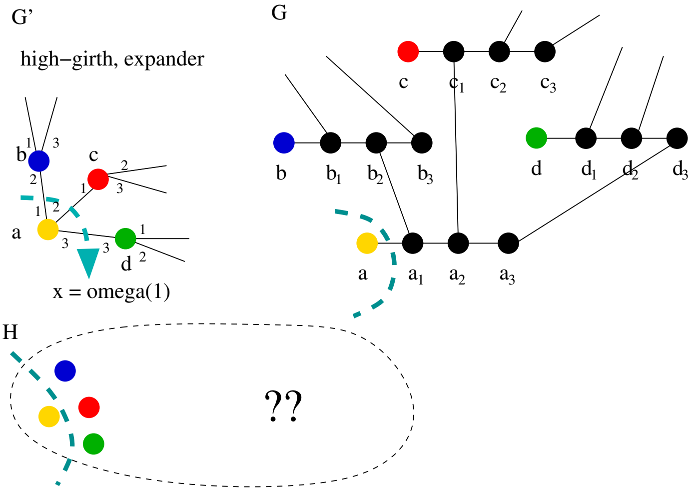

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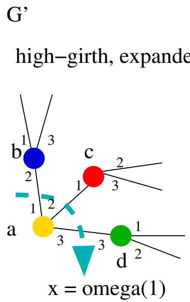

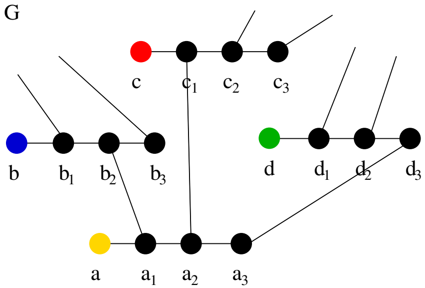

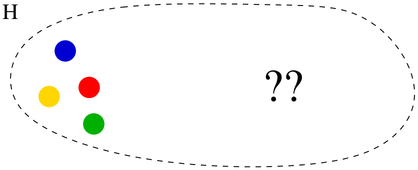

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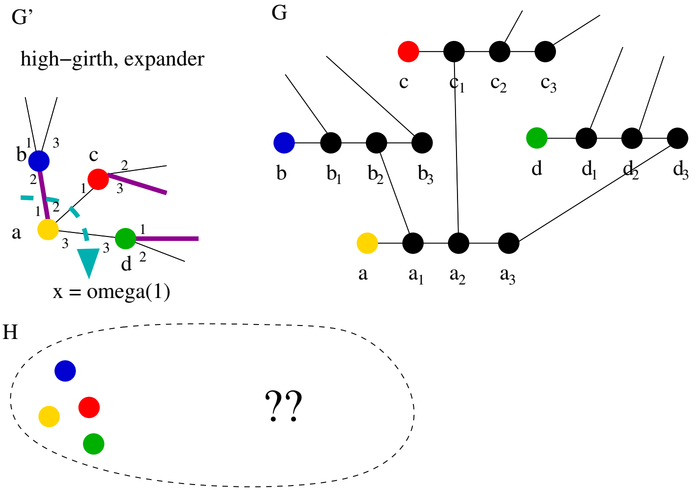
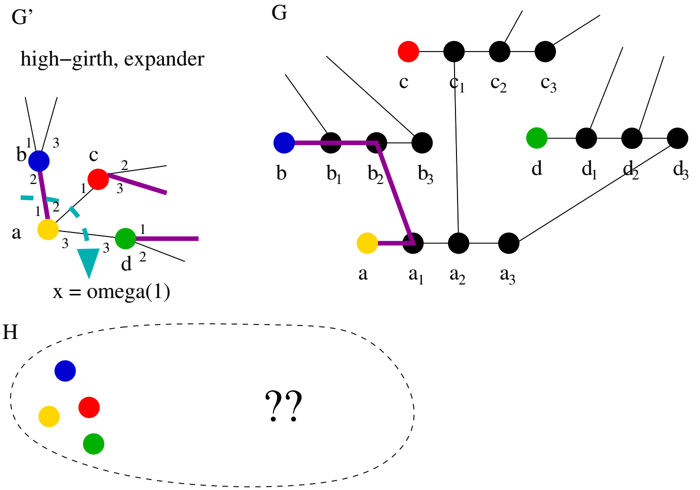
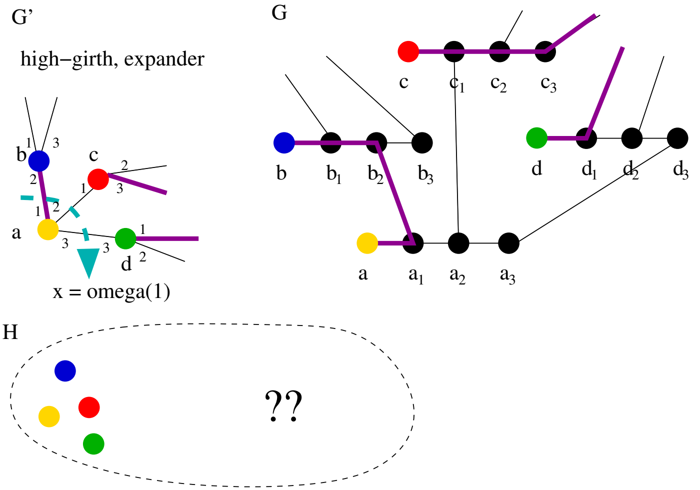


A Thought Experiment



A Thought Experiment



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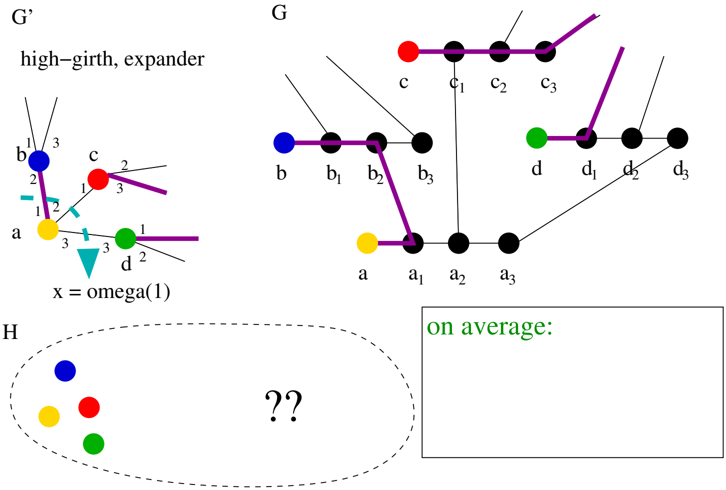

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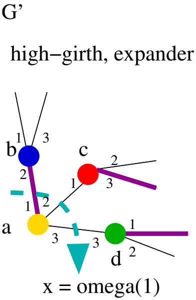
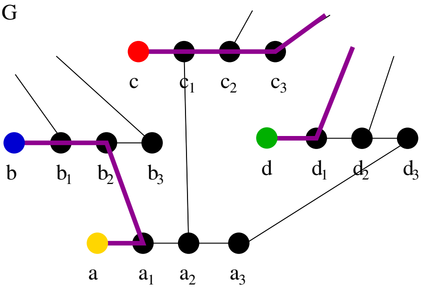
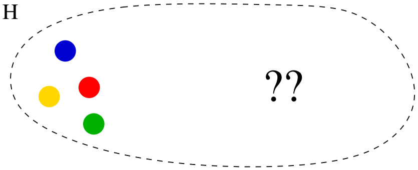


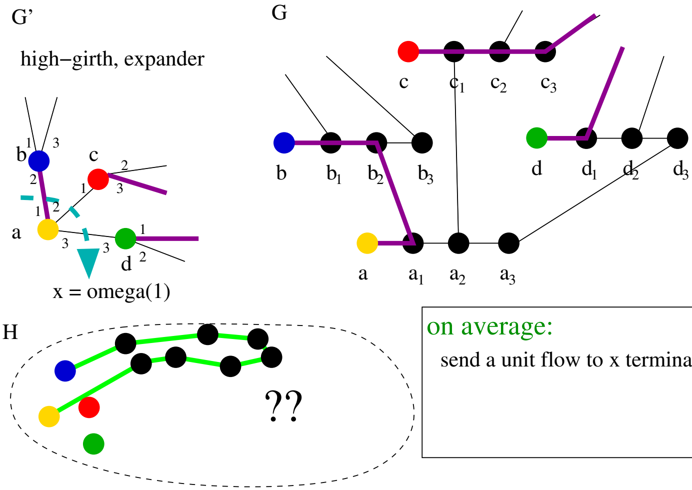




on average:

send a unit flow to x terminals

A Thought Experiment



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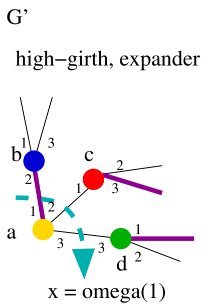
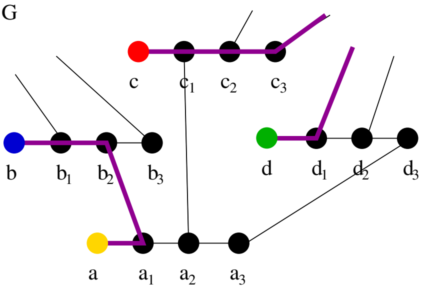
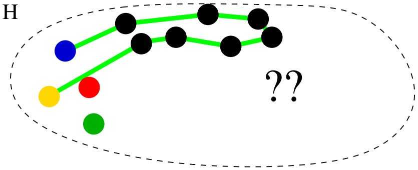




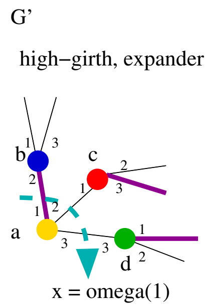


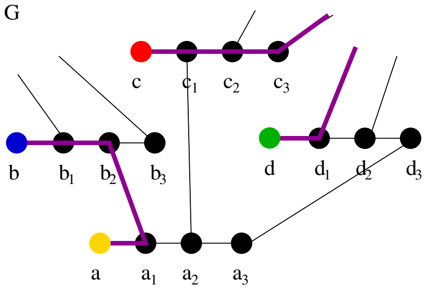
on average:

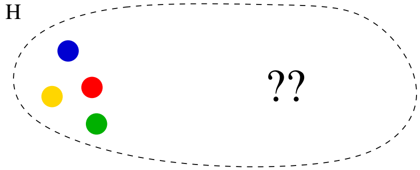
send a unit flow to x terminals

using short paths

A Thought Experiment







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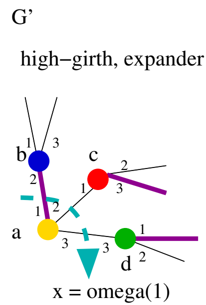

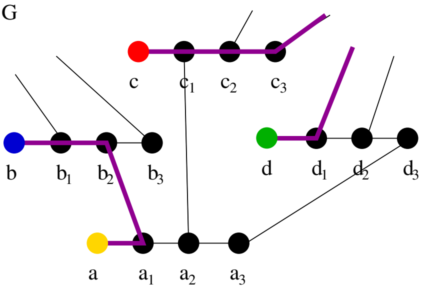

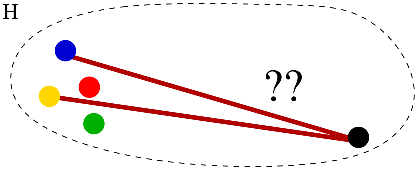

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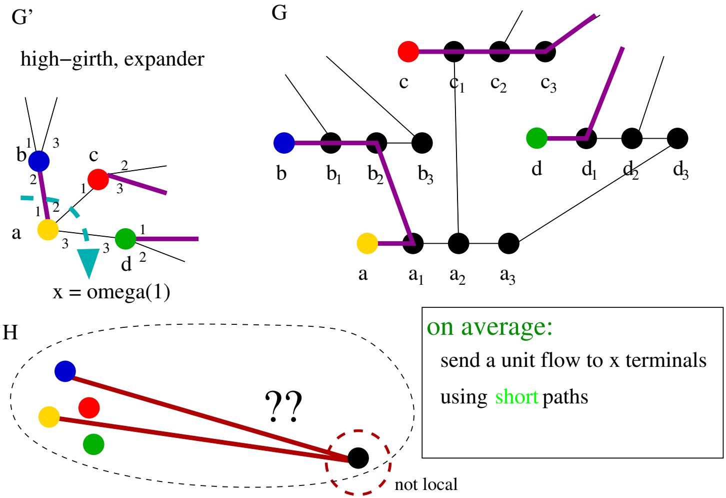

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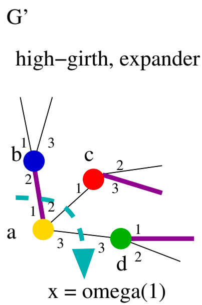
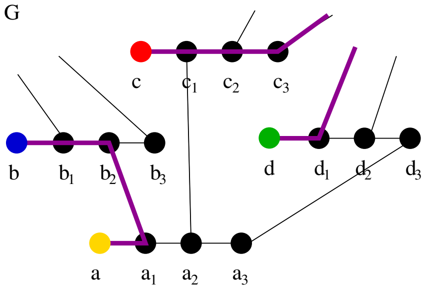
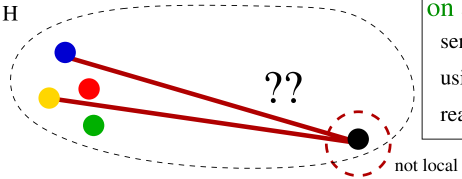






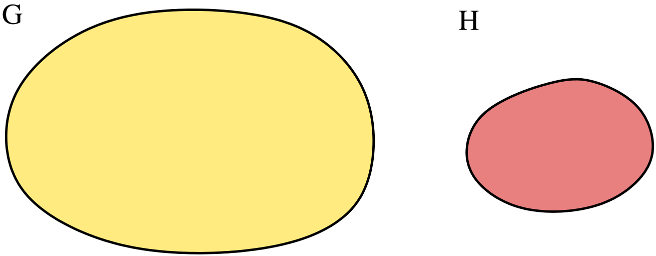
on average:

send a unit flow to x terminals

using short paths

reaching only local terminals

Girth-Routed Edges



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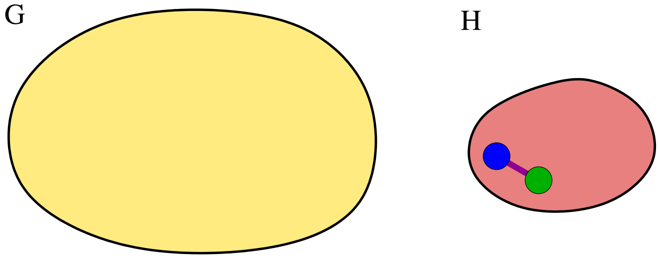
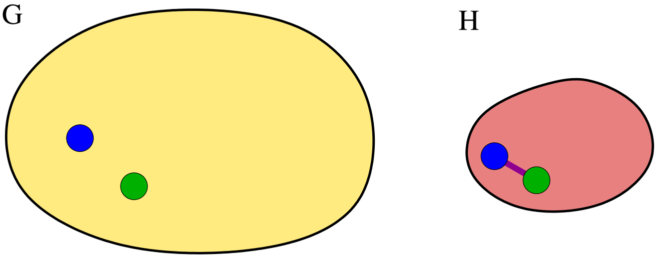
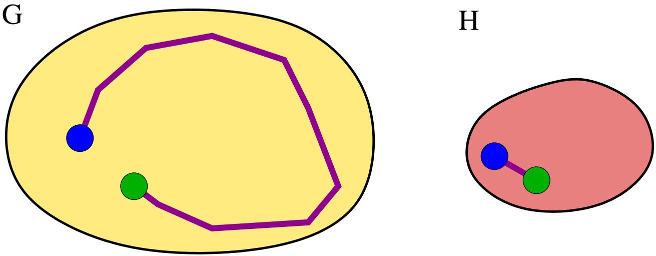


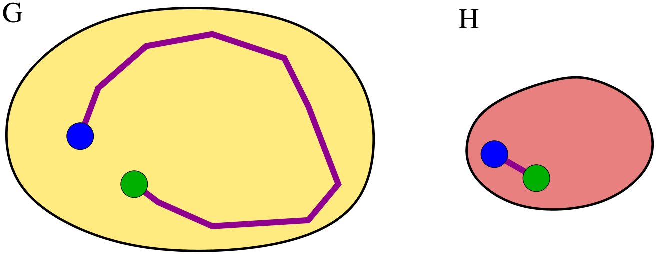
Girth-Routed Edges



Girth-Routed Edges



Girth-Routed Edges



not too many edges can be routed on girth paths

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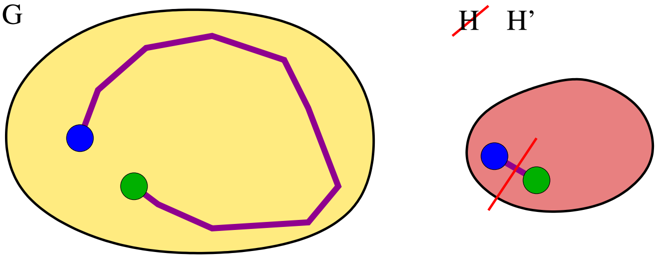

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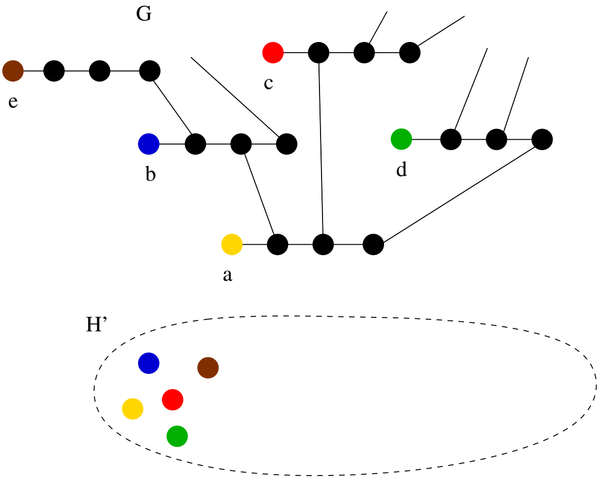
A Reduction to Cut-Width



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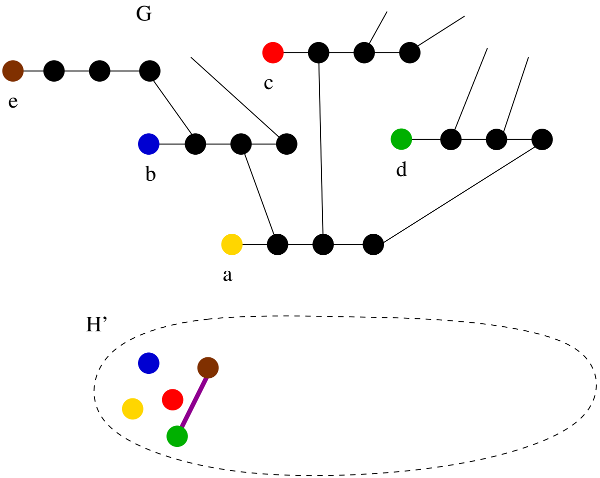

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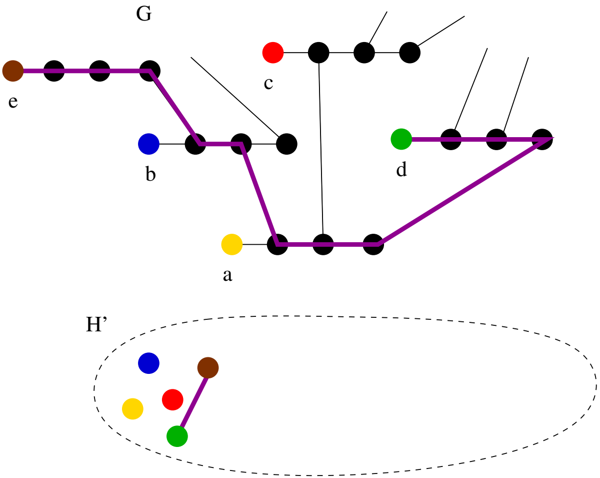

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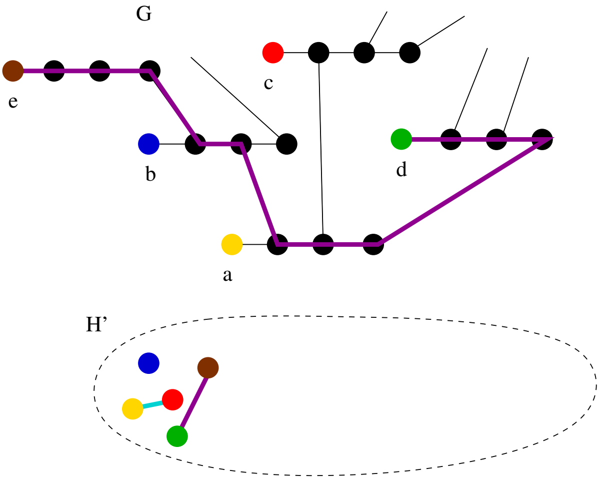

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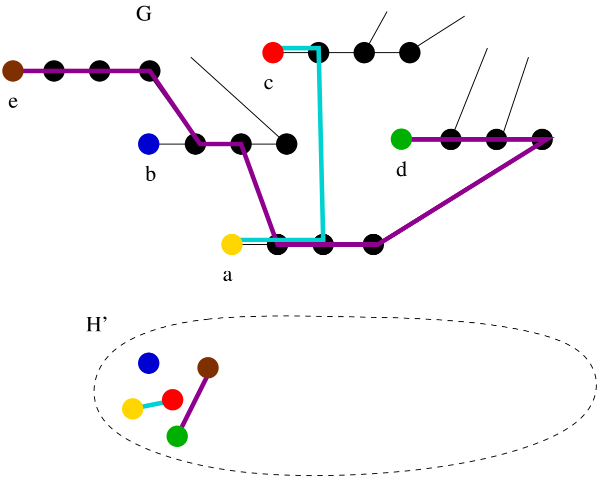

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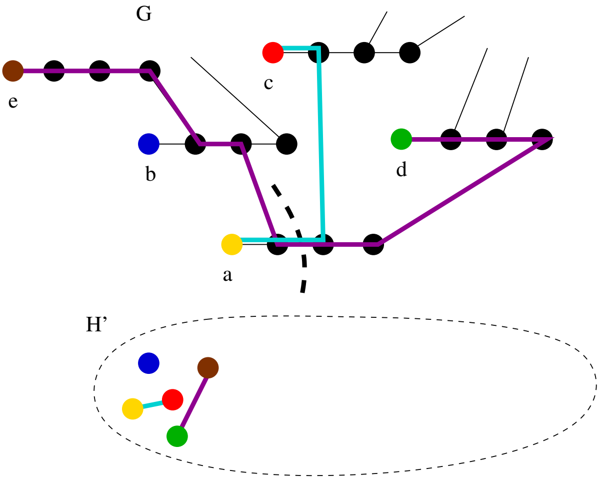

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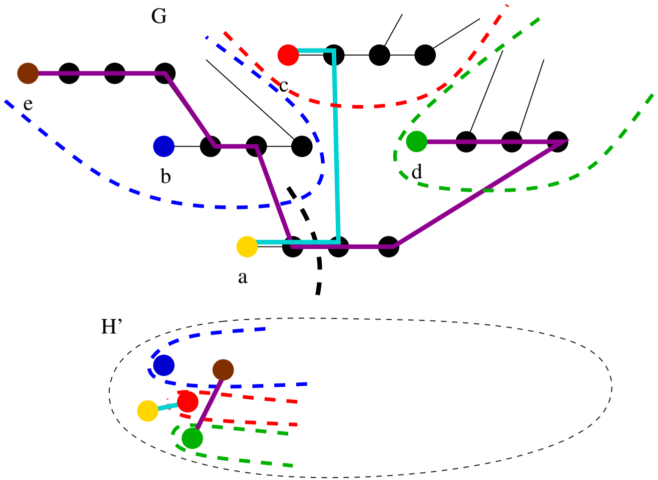

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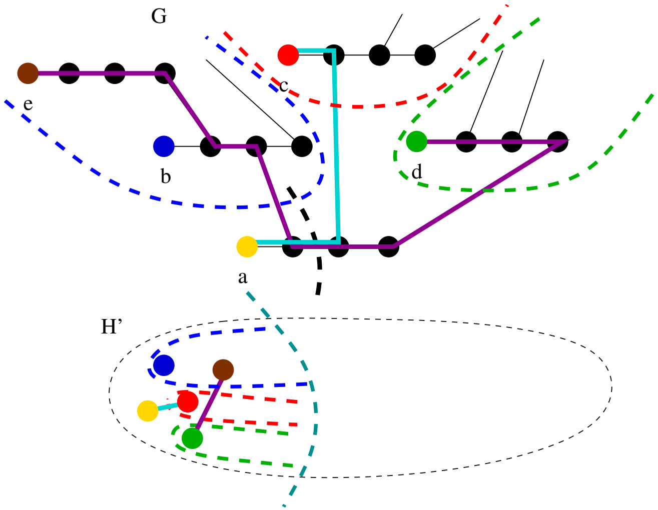

Recent Work

Recent work on both **upper bounds** and **lower bounds**

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1. [Englert, Gupta, Krauthgamer, Raecke, Talgam, Talwar]
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... super-constant lower bounds for cut sparsification, and **constructive** results that match our **existential** results

Open Questions

Question

Can the approximation for 0*-extension be improved?*

Open Questions

Question

Can the approximation for 0-extension be improved?

Question

What if the semi-metric is ℓ_1?

(immediately implies improvements to cut-sparsification)

Open Questions

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Question

Is flow-sparsification easier than the 0-extension problem?

Questions?

Thanks!