Some Results on Greedy Embeddings in Metric Spaces

Ankur Moitra, Tom Leighton

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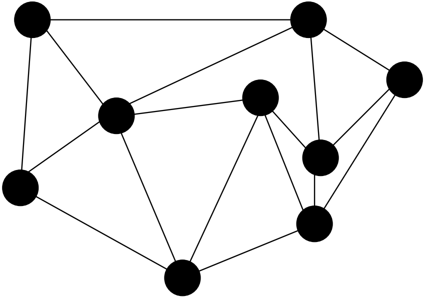
Greedy Embedding: Example



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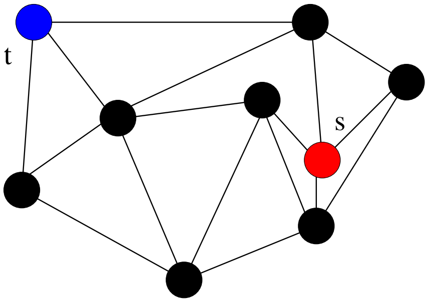

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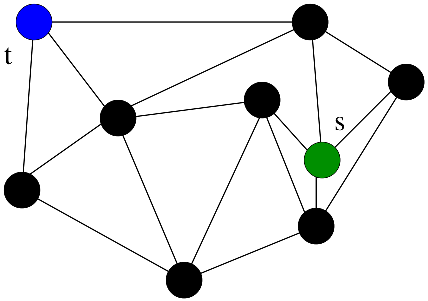

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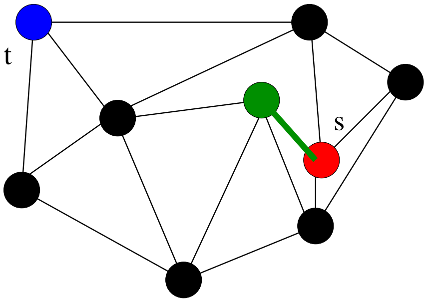

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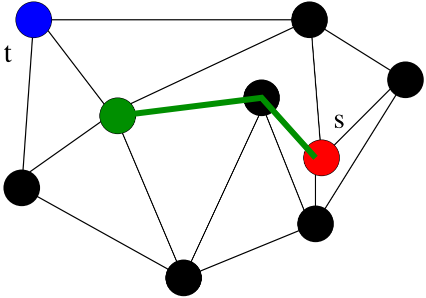

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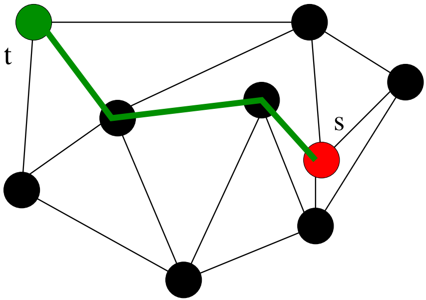

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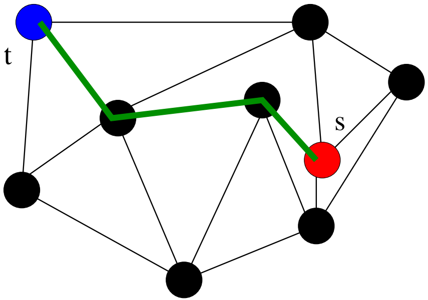

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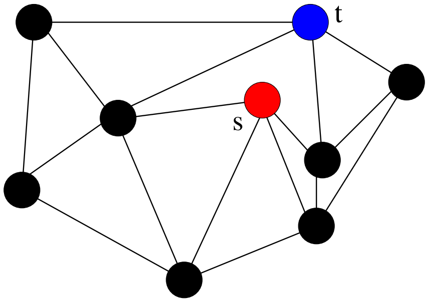

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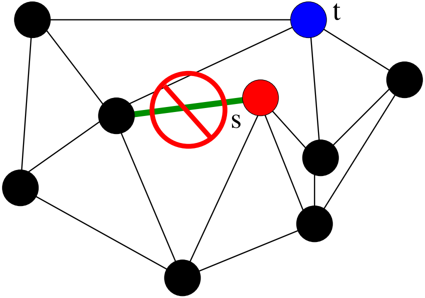

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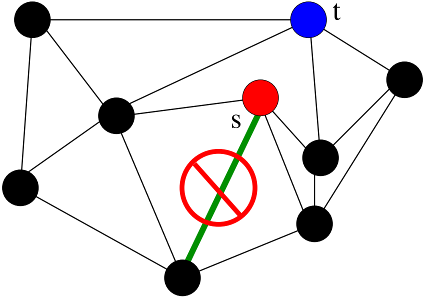

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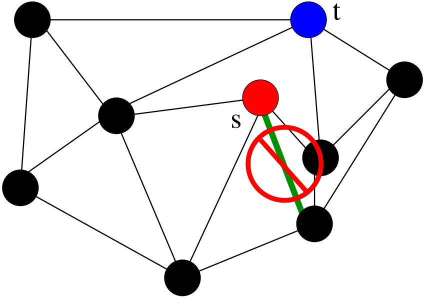

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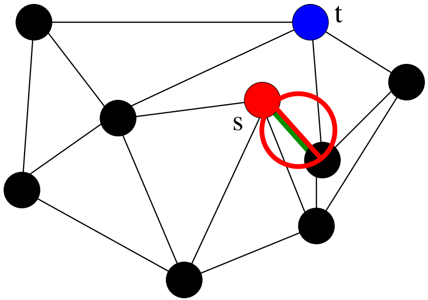

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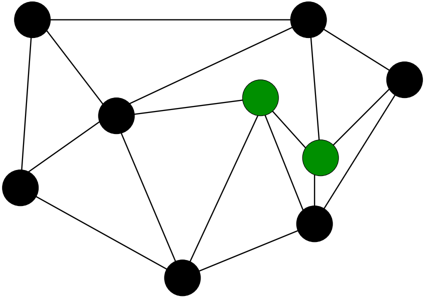

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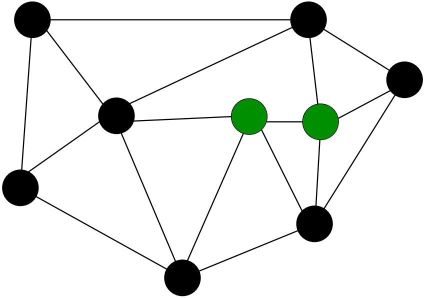

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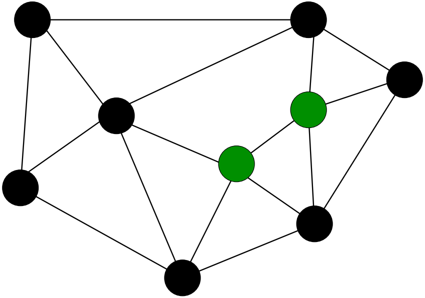

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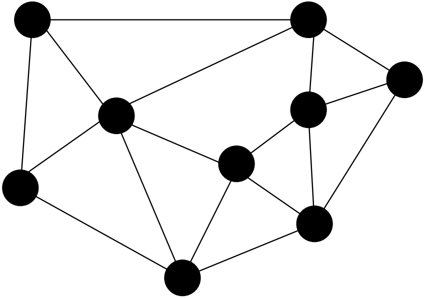

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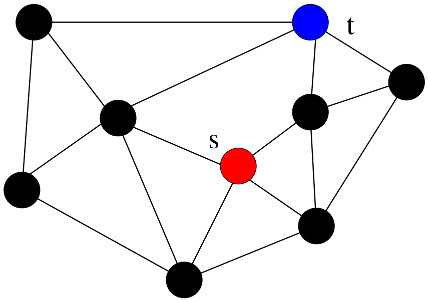

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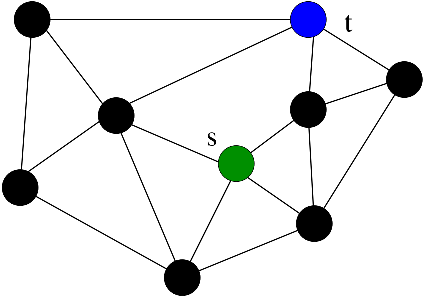

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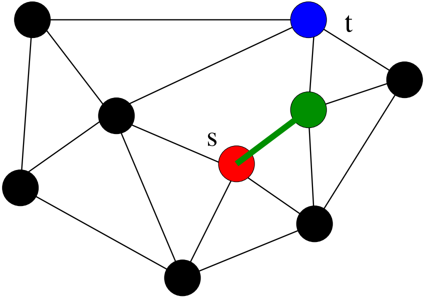

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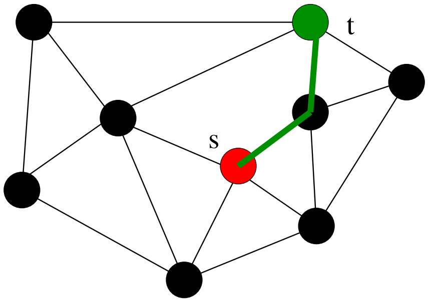

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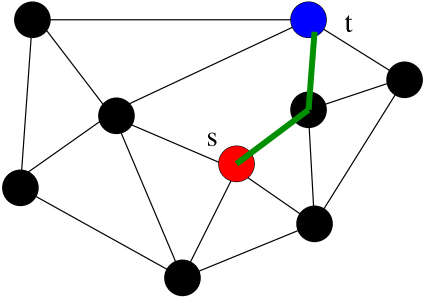

Greedy Routing

Reminder: (X, d) is a metric space, $f : V \rightarrow X$

Greedy Routing

Always forward packets to a neighbor that is strictly closer to the destination

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Always forward packets to a neighbor that is strictly closer to the destination

(distances are measured using the distance function d applied to the images of nodes in the metric space)

Greedy Embedding: Definition

Fact

For all (s,t) there exists a path connecting s to t in which distances to t are decreasing $\iff$ greedy routing never fails

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Definition

A graph G admits a greedy embedding into a metric space (X, d) if there is a function $f : V \to X$ s.t. greedy routing never fails

Results on Greedy Embeddings

Lemma (Papadimitriou, Ratajczak, 2005)

$K_{1,7}, K_{2,13}, ...K_{r,6r+1}$ *admit no greedy embedding into Euclidean plane*

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Conjecture (Papadimitriou, Ratajczak, 2005)

All 3-connected planar graph admits a greedy embedding into the Euclidean plane

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Theorem (Kleinberg, 2007)

All connected graphs admit a greedy embedding into the Hyperbolic plane

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Theorem (Dhandapani, 2008)

All 3-connected, triangulated planar graphs admit a greedy embedding into the Euclidean plane

Our Results

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1. The Papadimitriou-Ratajczak Conjecture is true!

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1. The Papadimitriou-Ratajczak Conjecture is true!
2. A combinatorial condition which is sufficient to guarantee no greedy embedding into Euclidean plane exists

Proof Outline

Theorem

All 3-connected planar graph admits a greedy embedding into the Euclidean plane

1. All 3-connected planar graphs contain a spanning Christmas Cactus graph
2. All Christmas Cactus graphs admit a greedy embedding into the Euclidean plane

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Cactus Graphs: Definition

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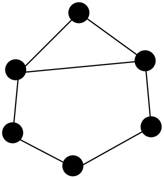
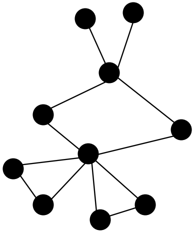
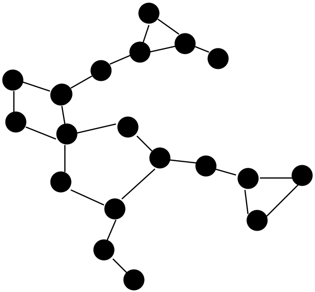
A **cactus graph** $G = (V, E)$ *is a connected graph for which each edge is in at most one simple cycle*



not cactus



cactus, not Christams cactus



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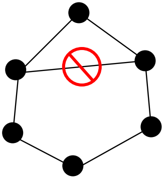

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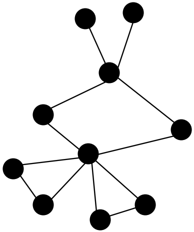

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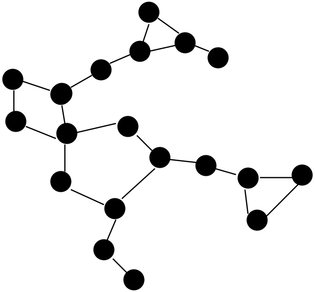
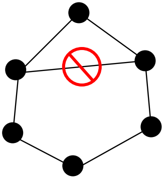


Christmas cactus

Christmas Cactus Graphs: Definition

Definition

A **Christmas cactus graph** $G = (V, E)$ *is a cactus graph for which the removal of any node* $v \in V$ *disconnects* G *into at most 2 components.*



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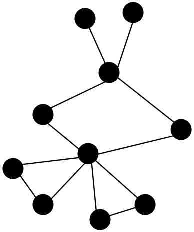

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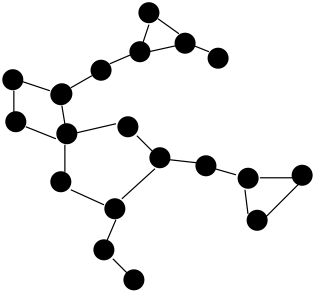

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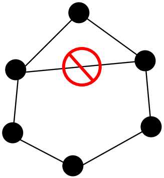

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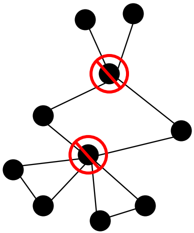

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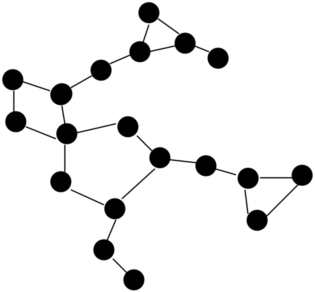

Christmas cactus

?? What is a Christmas cactus??

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Spanning Subgraphs in 3-Connected Planar Graphs

Theorem (this paper)

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$$\Downarrow$$

Theorem (Gao, Richter, 1994)

All 3-connected planar graphs contain a spanning 2-walk

Spanning Subgraphs in 3-Connected Planar Graphs

Theorem (this paper)

All 3-connected planar graphs contain a spanning Christmas cactus graph

$$\Downarrow$$

Theorem (Gao, Richter, 1994)

All 3-connected planar graphs contain a spanning 2-walk

$$\Downarrow$$

Theorem (Barnette, 1966)

All 3-connected planar graphs contain a spanning 3-tree

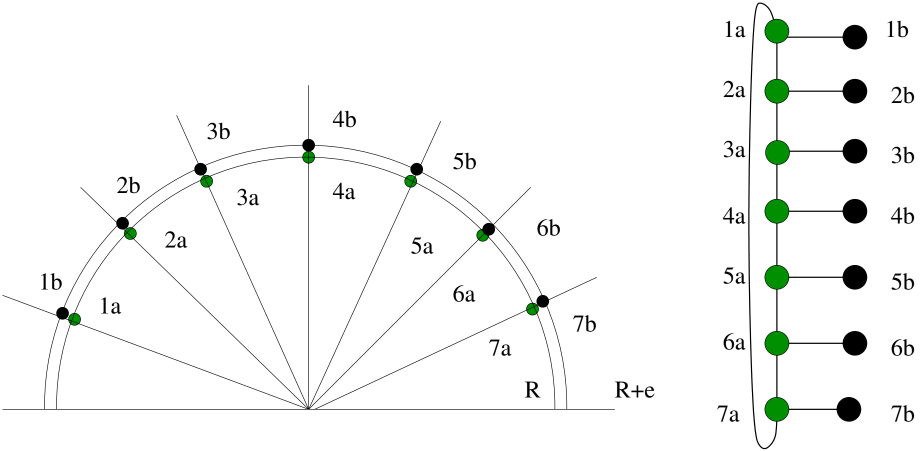
Proof Outline

Theorem

All 3-connected planar graph admits a greedy embedding into the Euclidean plane

1. ✓All 3-connected planar graphs contain a spanning Christmas Cactus graph
2. All Christmas Cactus graphs admit a greedy embedding into the Euclidean plane

Example I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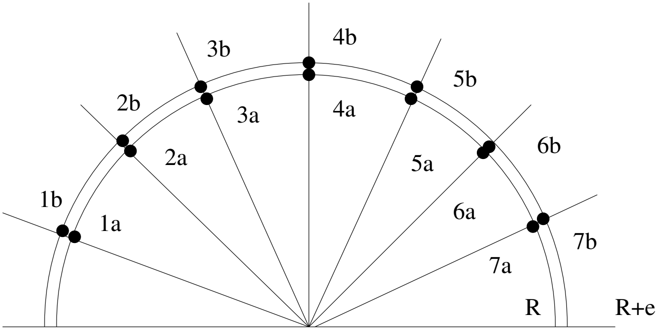

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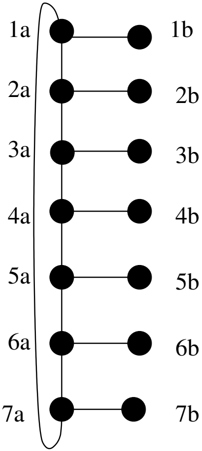

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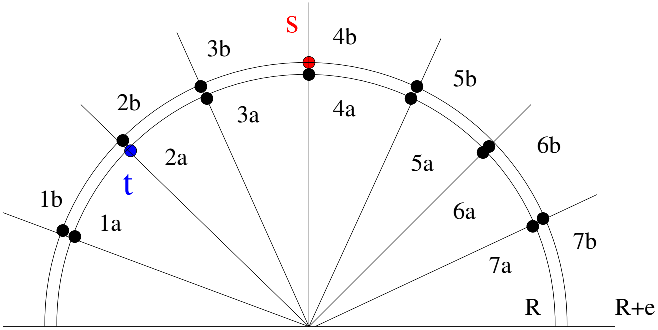

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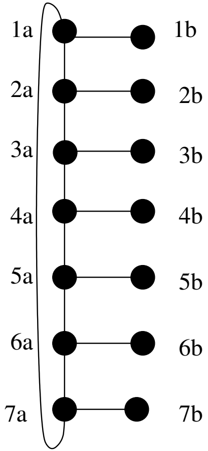

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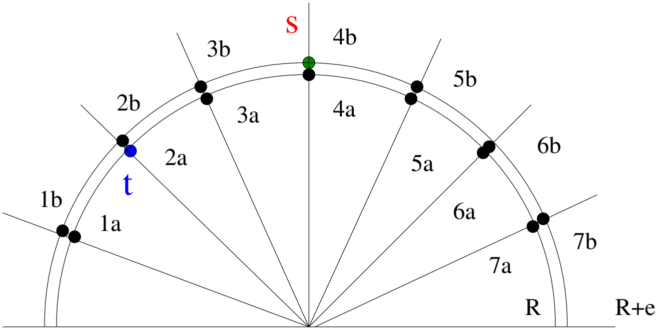
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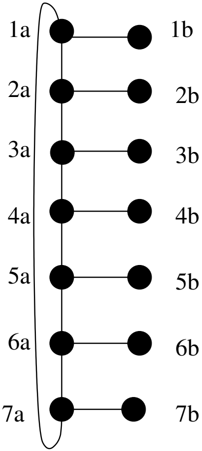
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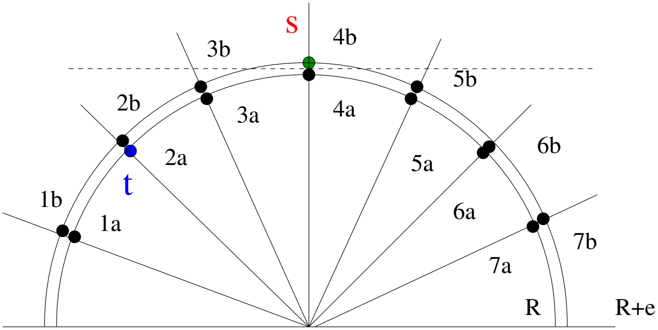

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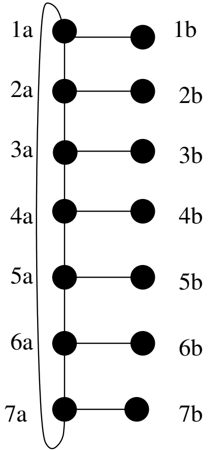

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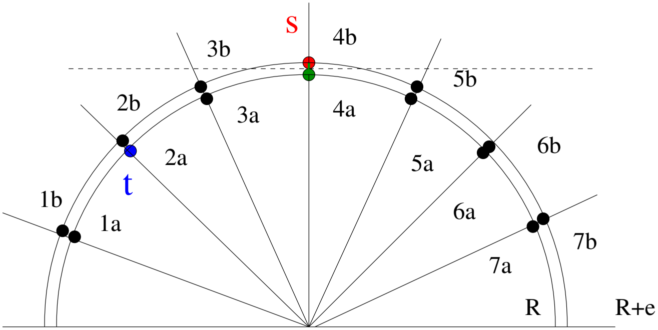

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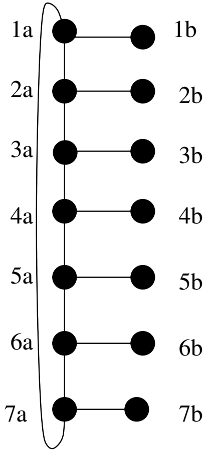

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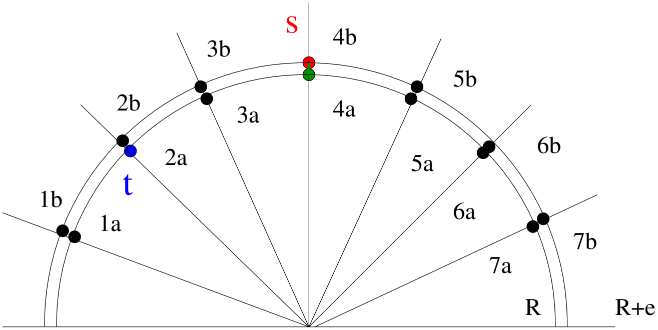

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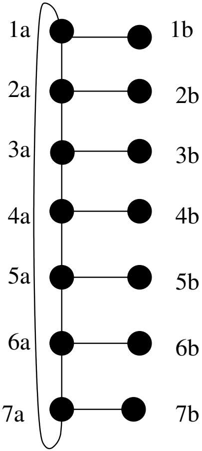

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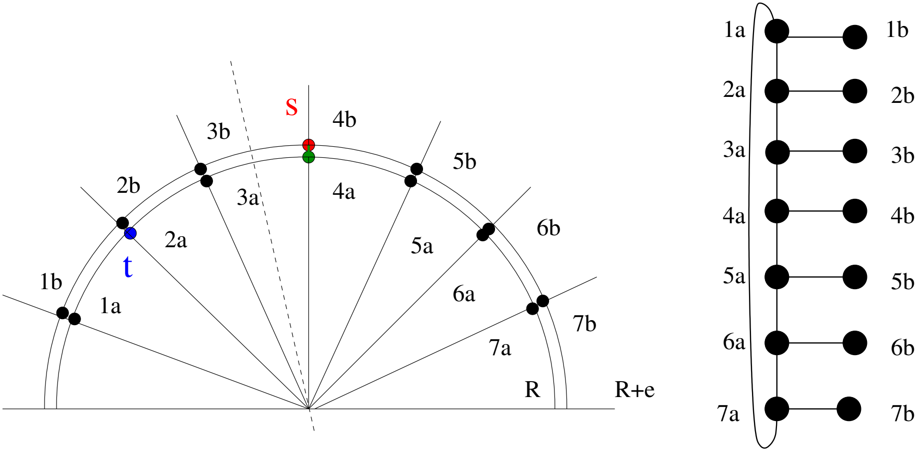
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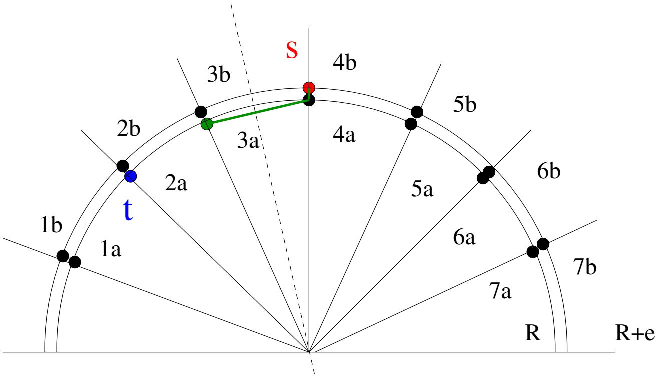
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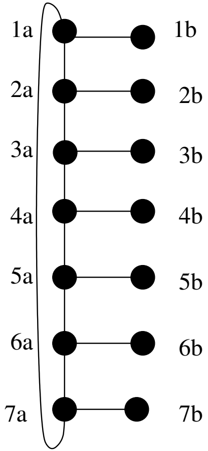
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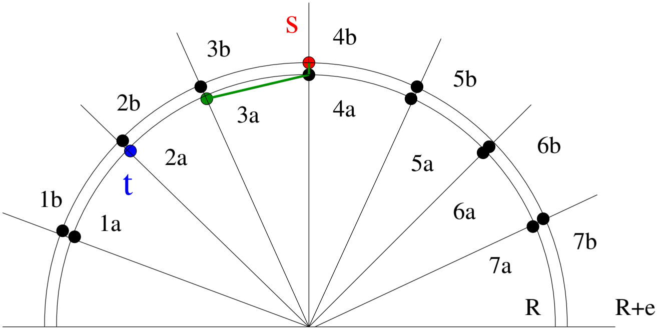

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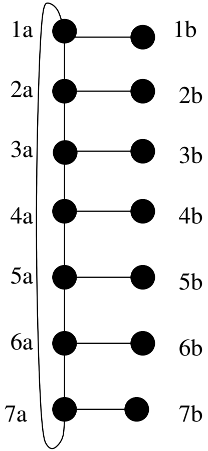

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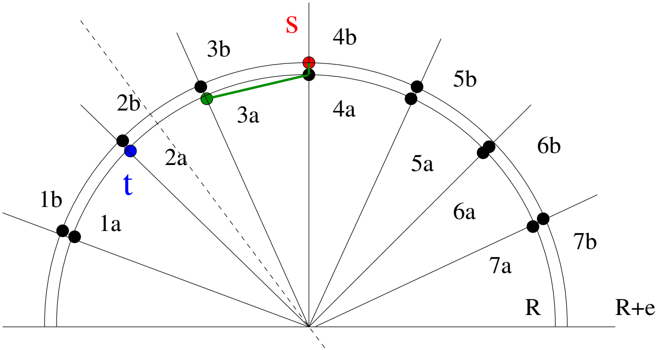
Example I



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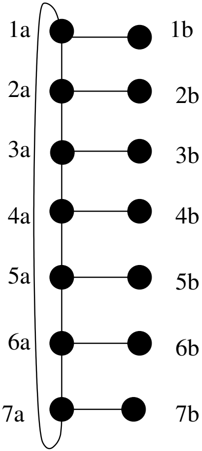

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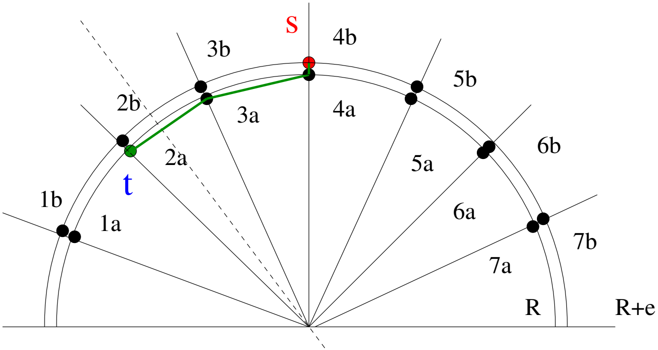
Example I



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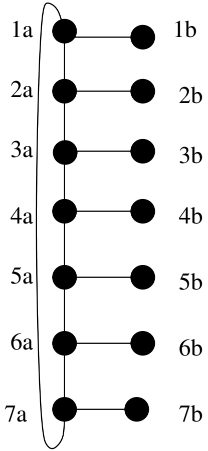

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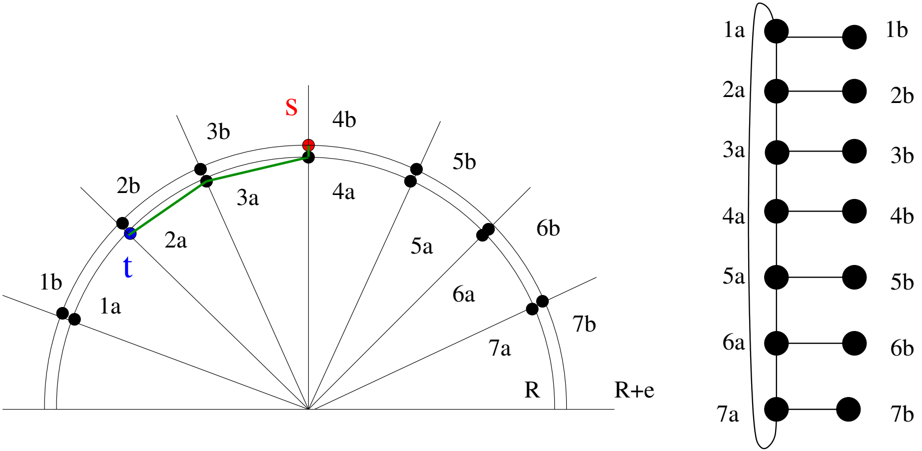
Example I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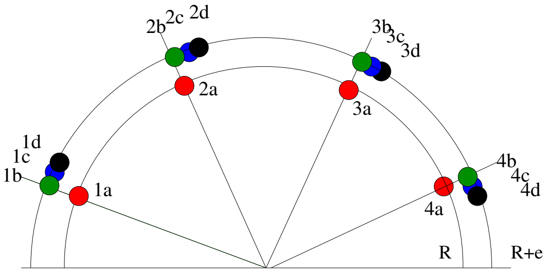
Example II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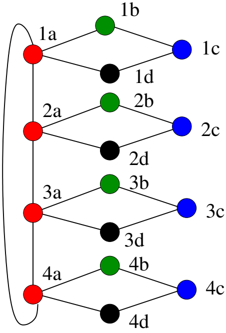
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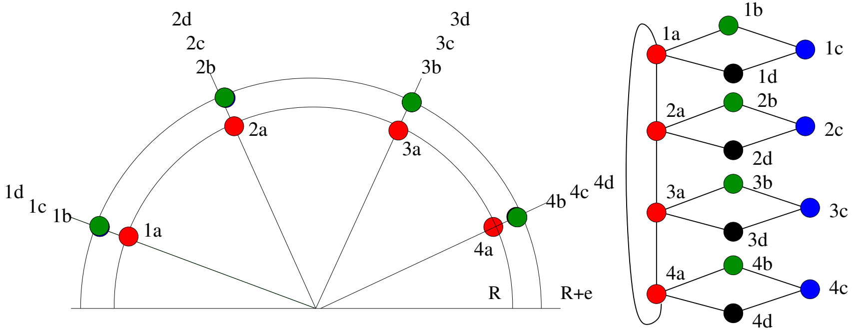

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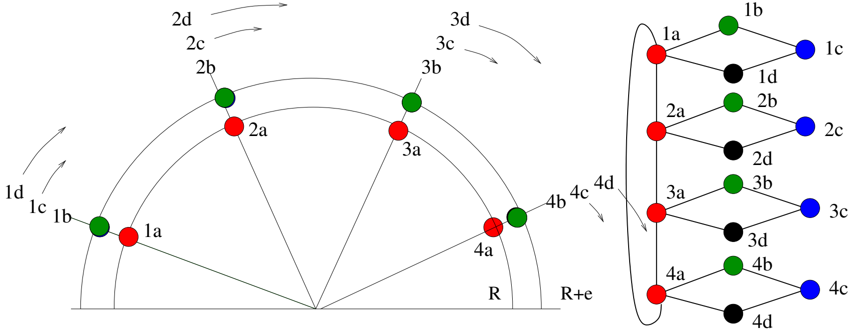

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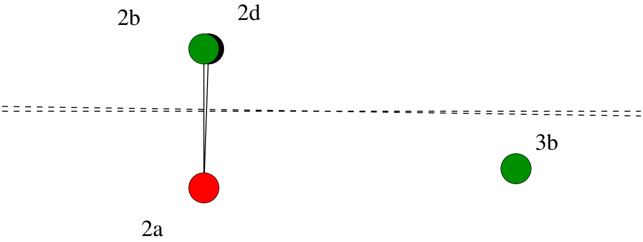
Example II



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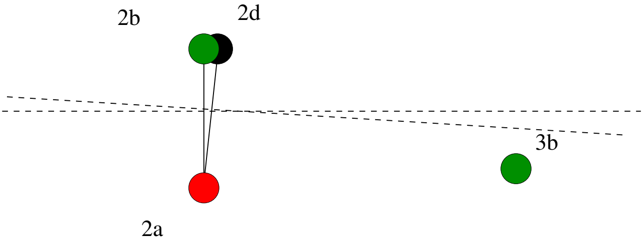

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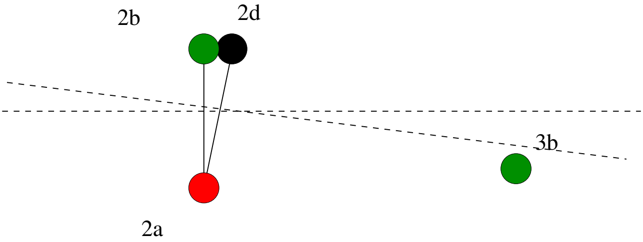

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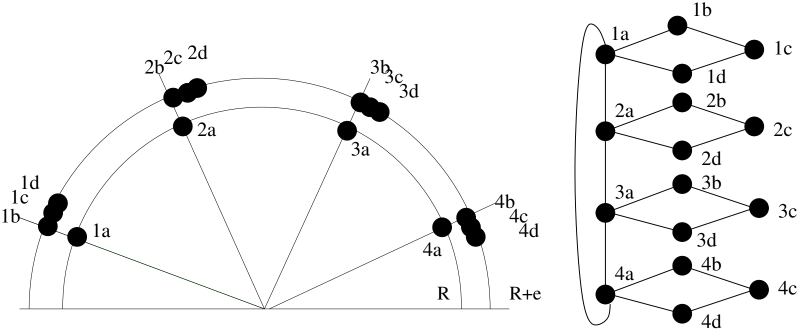

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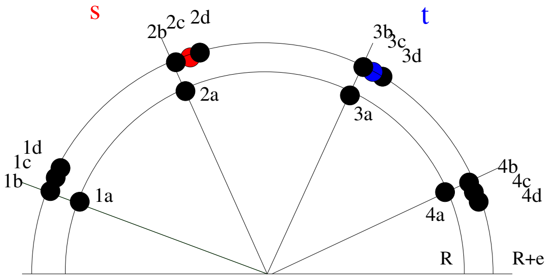
Example II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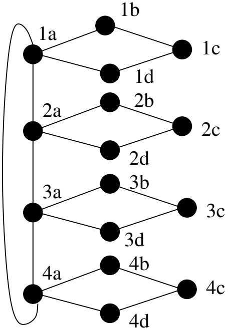
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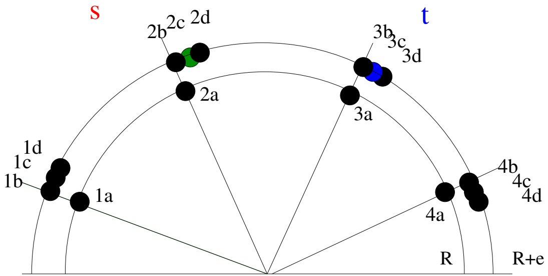

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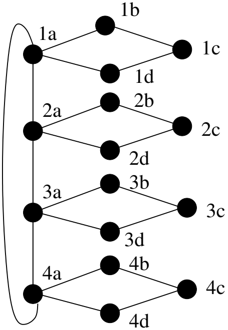

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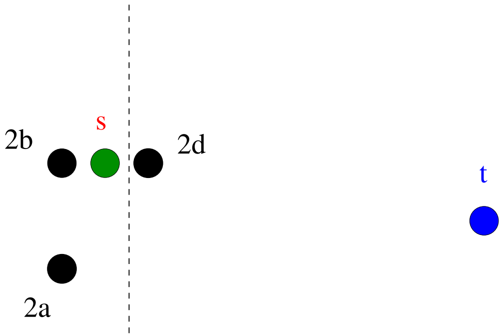

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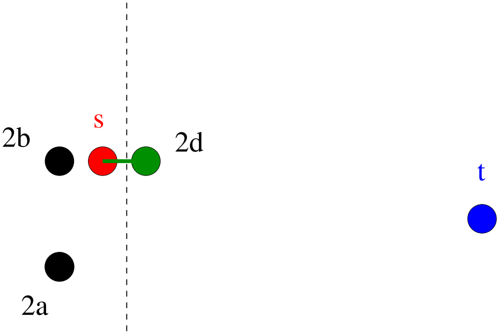

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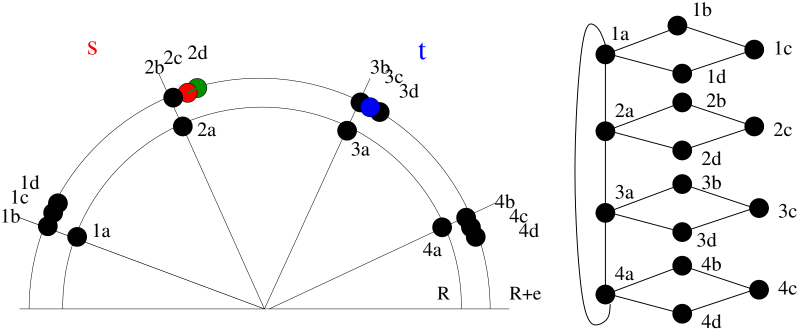
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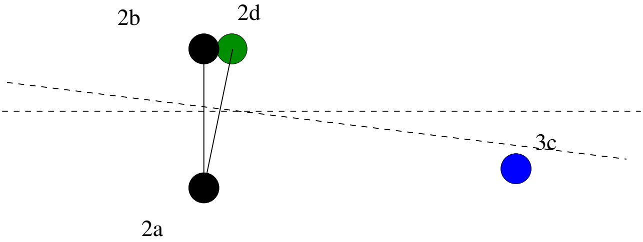

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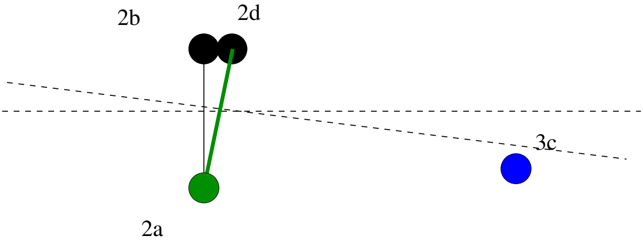

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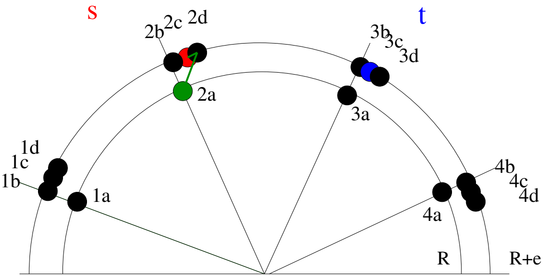

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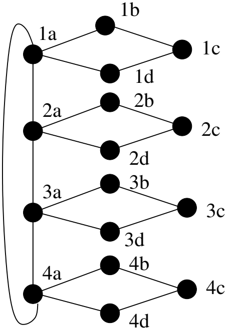

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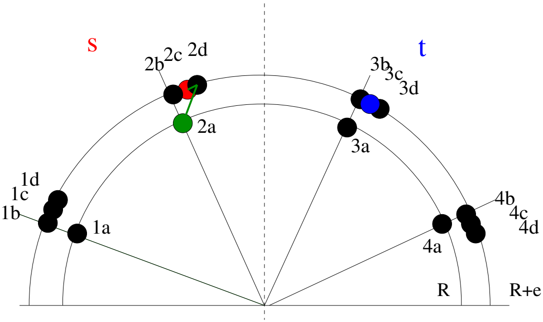
Example II



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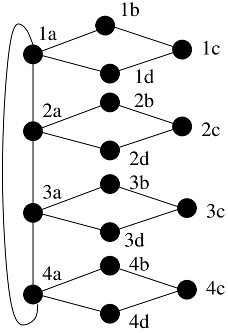

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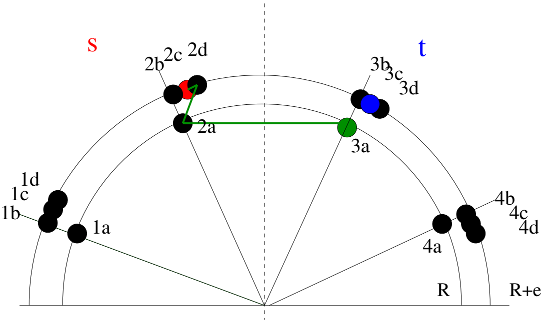
Example II



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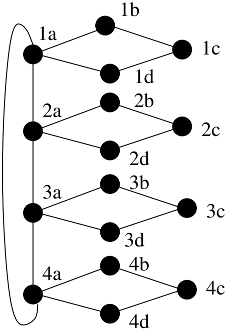

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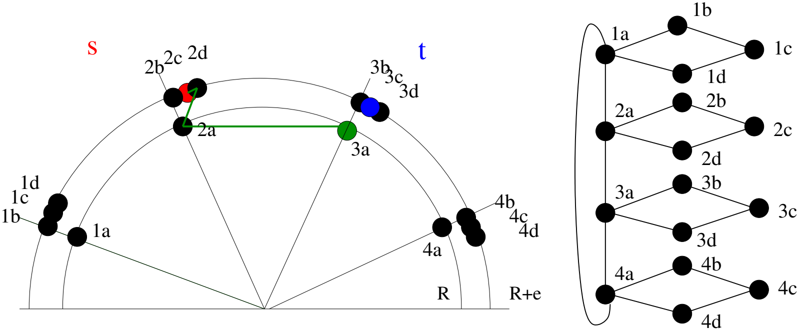
Example II



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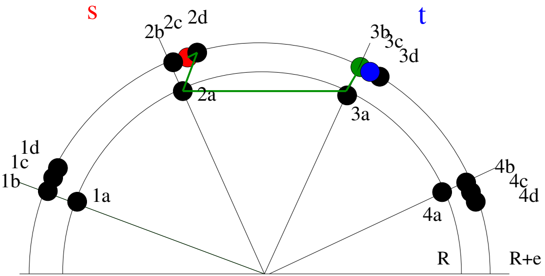
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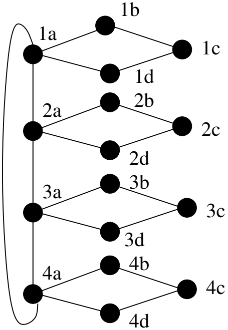
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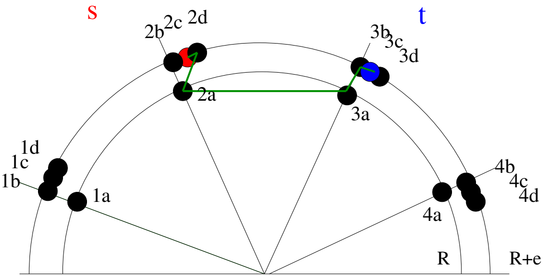
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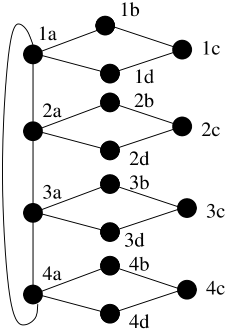
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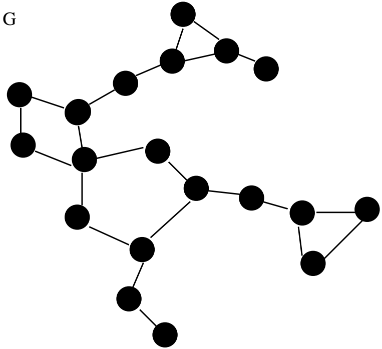
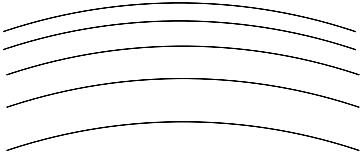
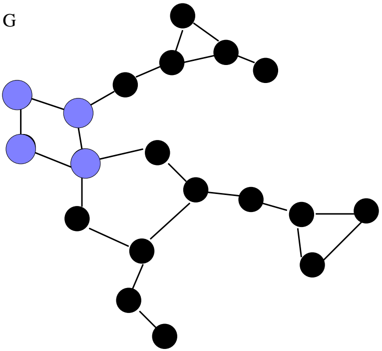
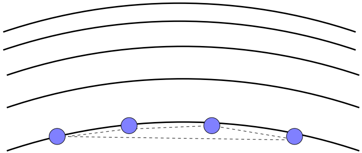
General Christmas Cactus Graphs



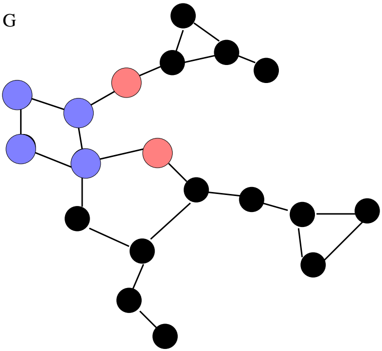


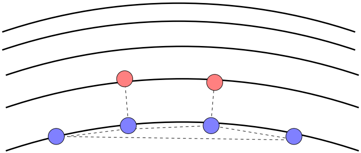
General Christmas Cactus Graphs



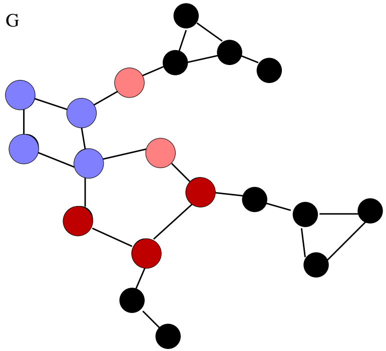


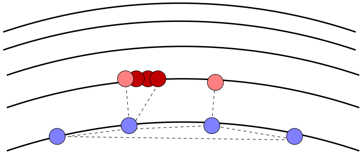
General Christmas Cactus Graphs



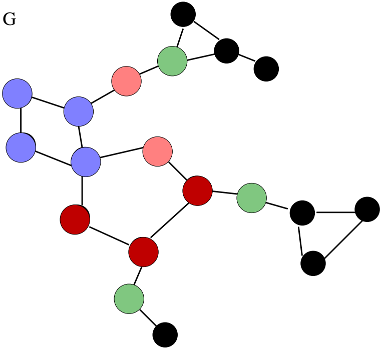


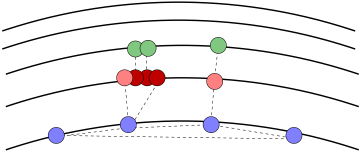
General Christmas Cactus Graphs





General Christmas Cactus Graphs





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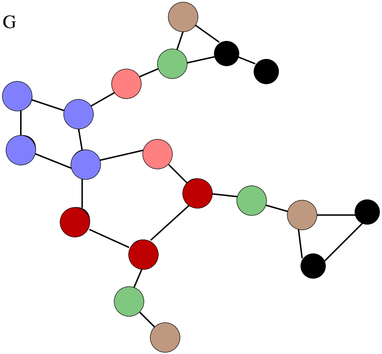

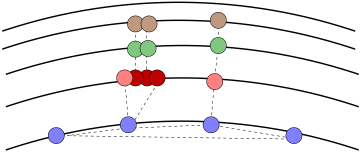

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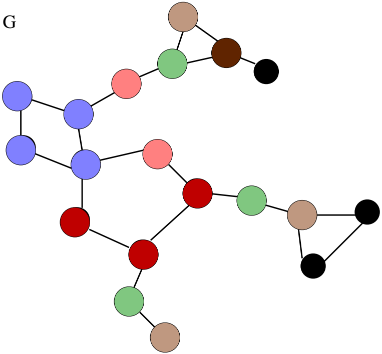

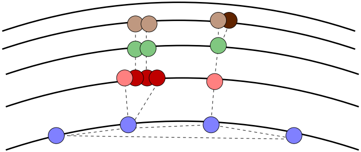

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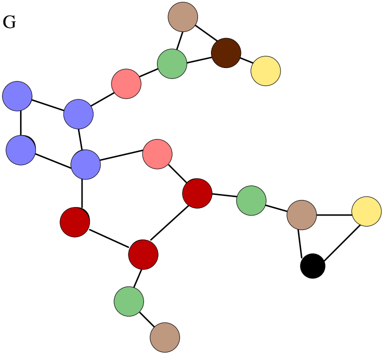
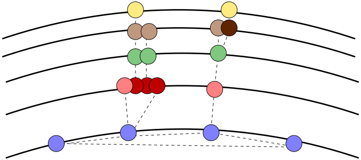
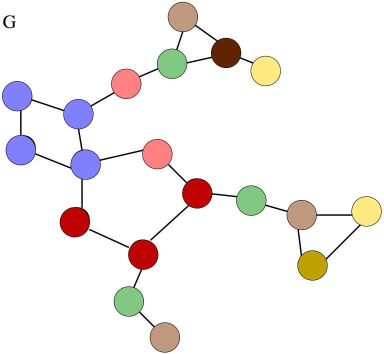
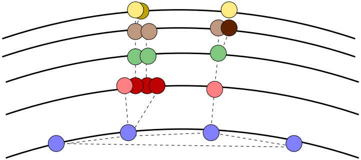




General Christmas Cactus Graphs





Proof Outline

Theorem

All 3-connected planar graph admits a greedy embedding into the Euclidean plane

1. ✓All 3-connected planar graphs contain a spanning Christmas Cactus graph
2. ✓All Christmas Cactus graphs admit a greedy embedding into the Euclidean plane

Open Questions

Our embedding requires *exponential* sized coordinates:

Open Questions

Our embedding requires *exponential* sized coordinates:

Conjecture

Any greedy embedding scheme for general Christmas cactus graphs requires exponential sized coordinates

Open Questions

Our embedding requires *exponential* sized coordinates:

Conjecture

Any greedy embedding scheme for general Christmas cactus graphs requires exponential sized coordinates

OR: Are there greedy embedding schemes for which coordinates are only polynomial sized?

Partial Results

Theorem

There exist graphs that admit a greedy embedding into the Euclidean line, but all greedy embeddings require exponential sized coordinates

Partial Results

Theorem

There exist graphs that admit a greedy embedding into the Euclidean line, but all greedy embeddings require exponential sized coordinates

Theorem

There are metric spaces s.t. all connected graphs can be greedily embedded, and average coordinate size is constant

Questions?

Thanks!