Some Results on Greedy Embeddings in Metric Spaces

Tom Leighton
Massachusetts Institute of Technology
Math Department and CSAIL
Cambridge, MA 02139
ftl@math.mit.edu

Ankur Moitra [*]
Massachusetts Institute of Technology
EECS Department and CSAIL
Cambridge, MA 02139
moitra@mit.edu

Abstract

Geographic Routing *is a family of routing algorithms that uses geographic point locations as addresses for the purposes of routing. Such routing algorithms have proven to be both simple to implement and heuristically effective when applied to wireless sensor networks.* Greedy Routing *is a natural abstraction of this model in which nodes are assigned virtual coordinates in a metric space, and these coordinates are used to perform point-to-point routing.*

Here we resolve a conjecture of Papadimitriou and Ratajczak that every 3-*connected planar graph admits a greedy embedding into the Euclidean plane. This immediately implies that all* 3-*connected graphs that exclude* $K_{3,3}$ *as a minor admit a greedy embedding into the Euclidean plane. Additionally, we provide the first non-trivial examples of graphs that admit no such embedding. These structural results provide efficiently verifiable certificates that a graph admits a greedy embedding or that a graph admits no greedy embedding into the Euclidean plane.*

1 Introduction

1.1 Background

The study of routing has a long and rich history. But for many important classes of routing problems, routing schemes that are both simple and provably effective have so far been elusive. In particular, scalable wireless sensor networks require point-to-point communication but such *ad-hoc* networks admit no global hierarchical addressing scheme and there are still no broadly accepted, scalable point-to-point routing schemes despite numerous proposals. *Geographic Routing* is a family of routing algorithms that uses geographic point locations as addresses for the purposes of routing. Such routing algorithms have proven to be both simple to implement and heuristically effective when applied to wireless sensor networks.

Recent work on routing protocols for wireless sensor networks [2], [8] has focused particular attention on a class of 'greedy' algorithms wherein a packet at a node u that is destined for a node v is simply forwarded to any neighbor u' of u for which $d(u', v) < d(u, v)$ where $d(x, y)$ is the Euclidean distance between the locations of x and y in the plane. For such an algorithm to guarantee delivery it must be the case that for every u and v, such a u' exists (i.e. that wherever a packet is in the network, there is always a next hop that gets the packet closer in Euclidean distance to its ultimate destination).

Rao et al. [13] proposed a natural abstraction of this model in which nodes are assigned virtual coordinates in a metric space, and these coordinates are used to perform point-to-point routing.

Definition 1. A graph $G = (V, E)$ is said to have a *greedy embedding* into a metric space (X, d) if there is a function $f : V \rightarrow X$ such that for every pair of distinct nodes $u, v \in V$, there exists a neighbor u' of u in G such that $d(f(u'), f(v)) < d(f(u), f(v))$.

Papadimitriou and Ratajczak [11] considered the case where (X, d) is the Euclidean plane and gave simple examples of graphs which have a greedy embedding (e.g., Hamiltonian graphs) and graphs that admit no greedy embedding into the Euclidean plane (e.g., $K_{r,6r+1}$). Papadimitriou and Ratajczak conjectured that all 3-connected planar graphs admit a greedy embedding into the Euclidean plane.

Kleinberg [9] considered the case in which (X, d) is the hyperbolic plane and showed that *every* tree (and consequently every graph) has a greedy embedding in the hyperbolic plane. However random geometric graph processes in the Euclidean plane are one of the fundamental models for reasoning about wireless ad-hoc networks [4], [7], [12]. And it remains a central question as to whether these generative models result in graphs that can be embedded in a metric space that is fundamentally the same as the metric

space in which these graphs were generated.

Dhandapani [3] recently proved that all triangulated 3-connected planar graphs have greedy embeddings in the Euclidean plane, a relaxation of the Papadimitriou-Ratajczak conjecture. His proof made use of Schnyder Realizers, and used the geometric properties of Schnyder Drawings to find a greedy embedding.

1.2 Our Results

Here we resolve a conjecture of Papadimitriou and Ratajczak that every 3-connected planar graph admits a greedy embedding into the Euclidean plane. In fact, we construct a greedy embedding into the Euclidean plane for all circuit graphs (which generalize 3-connected planar graphs). This immediately implies that all 3-connected graphs that exclude $K_{3,3}$ as a minor admit a greedy embedding into the Euclidean plane.

Additionally, we provide the first non-trivial examples of graphs that cannot be greedily embedded into the Euclidean plane. These structural results provide efficiently verifiable certificates that a graph admits a greedy embedding or that a graph admits no greedy embedding into the Euclidean plane.

Perhaps of independent interest, we make use of a decomposition theorem due to Gao and Richter [6]. This is, to the best of our knowledge, this decomposition theorem's first use in theoretical computer science. We use this theorem to find a spanning subgraph that can be greedily embedded into the Euclidean plane. And we believe that this technique can be generally applicable for finding particular types of spanning subgraphs in circuit graphs as needed.

2   A Greedy Embedding for Circuit Graphs

Circuit graphs are a relaxation of 3-connected planar graphs. In this section, we prove that all circuit graphs contain a spanning *Christmas cactus graph* and provide a polynomial time algorithm to find such a spanning subgraph. We then construct a greedy embedding into the Euclidean plane for all Christmas cactus graphs. This proves a conjecture due to Papadimitriou and Ratajczak that every 3-connected planar graph admits a greedy embedding into the Euclidean plane. As a corollary, any 3-connected graph that excludes $K_{3,3}$ as a minor admits a greedy embedding into the Euclidean plane.

2.1   Christmas Cactus Graphs

A *cactus graph* is a graph for which every edge is part of at most one cycle. In what follows, we will be interested in a special type of cactus graph that we call a *Christmas cactus graph*.

Definition 2. A *Christmas cactus graph* $G = (V, E)$ is a connected cactus graph for which the removal of any node $v \in V$ disconnects G into at most 2 components.

It is well known that a cactus graph can be constructed from a tree by replacing edges with cycles of arbitrary size. Similarly, a Christmas cactus graph can be constructed from a tree by replacing nodes with cycles and contracting edges that are not in cycles. In particular we will make use of the following method for constructing a Christmas cactus graph:

Step 1. Let T be a tree and let U be any subset of nodes of T that contains every node with degree at least 3 in T.

Step 2. (Repeat) For each node $u \in U$, replace u with a cycle C_u of arbitrary length so that all neighbors of u (in the current graph) are connected by an edge to a distinct node in C_u.

Step 3. Let F be any set of edges in the graph resulting from *Step 2* which are not contained in a cycle.

Step 4. Contract out the edges in F.

For example, these steps are shown in Figure 1.

Simple cycles in a Christmas cactus graph are not necessarily node-disjoint because a path connecting two node-disjoint cycles can be contracted. However, for any two simple cycles C_1, C_2, $|C_1 \cap C_2| \le 1$ and for any three simple cycles C_1, C_2, C_3, $|C_1 \cap C_2 \cap C_3| = 0$.

2.2   Circuit Graphs

Barnette [1] introduced the class of graphs known as circuit graphs, which he defined to be graphs obtained by deleting a vertex from a 3-connected planar graph. Gao and Richter [6] proved rich structural theorems about the class of circuit graphs, and used these results to inductively prove that all circuit graphs contain a spanning closed 2-walk. We also make use of the structural properties of circuit graphs, but to find a spanning Christmas cactus subgraph. The class of circuit graphs is a relaxation of 3-connected planar graphs, but provides a more convenient class of graphs on which to construct inductive proofs. Gao and Richter equivalently define a circuit graph as:

Definition 3. A *circuit graph* is an ordered pair (G, C) such that:

1. G is 2-connected, and C is a polygon in G.
2. There is a non-crossing embedding of G in the plane s.t. C bounds an infinite face.
3. If (H, K) is a 2-separation[1] of G, then $C \nsubseteq H$, $C \nsubseteq K$.

A 3-connected planar graph is a circuit graph because G can be embedded using Tutte's rubber band embedding and

---

[1] A *k-separation* of a graph $H = (V, E)$ is a pair H_1, H_2 of edge disjoint subgraphs of H, each with at least $k + 1$ vertices, s.t. $H = H_1 \cup H_2$ and $|V(H_1) \cap V(H_2)| = k$.
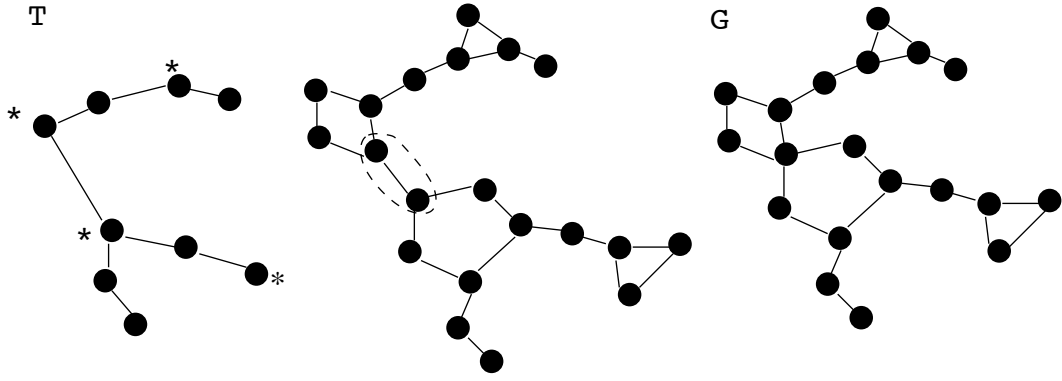
Figure 1. A Christmas cactus graph G generated by the tree T. The nodes in U are denoted with a $*$ and the edge in F is circled.
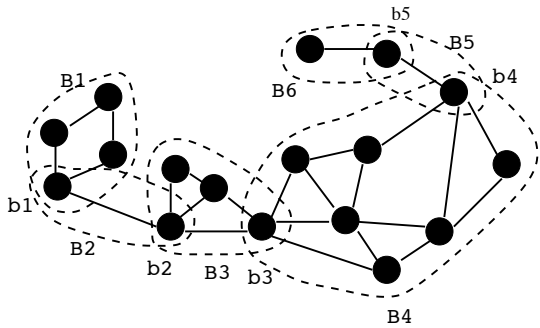

Figure 2. A plane chain of blocks.

any 2-separation (H, K) s.t. $C \subset H$ would imply that G is not 3-connected. We next review some key structural properties of circuit graphs:

Lemma 1. [6] Let (G, C) be a circuit graph embedded in the plane (s.t. no edges are crossing), and let C_1 be any polygon in G. Then the subgraph G_1 containing C_1 and all nodes and edges inside C_1 (in the plane embedding) is a circuit graph.

Definition 4. [6] A connected graph G is a *chain of blocks* if each block of G contains at most two cut vertices and each cut vertex [2] lies in exactly two blocks. Then a chain of blocks can be written as $B_1, b_1, B_2, ..b_{k-1}, B_k$ such that the common vertex of blocks B_i and B_{i+1} is b_i. Then a *plane chain of blocks* is a chain of blocks and a plane embedding s.t. for all j, $\cup_{i \neq j} B_i$ is in the infinite face of B_j.

Because each cut vertex is in exactly two blocks, then each b_i must be distinct. A block is called trivial if the block is just the edge b_i, b_{i+1}. Then the seminal work of Gao

and Richter gives the following structural result for circuit graphs:

Theorem 1. [6] Let (G, C) be a circuit graph, and let $x, y \in C$ be distinct. Then there exists a partition of $V(G) - V(C)$ into $V_1, V_2, ..., V_m$ and distinct vertices $v_1, v_2, ..., v_m \in V(C) - \{x, y\}$ s.t. the graph induced by $V_i \cup \{v_i\}$ is a plane chain of blocks $B_{i,1}, b_{i,1}, ..., b_{i,k-1}, B_{i,k}$ s.t. $v_i \in V(B_{i,1}) - b_{i,1}$ and each nontrivial block $B_{i,j}$ has an outer polygon $C_{i,j}$ s.t. $(B_{i,j}, C_{i,j})$ is a circuit graph [3].

Gao and Richter use this structural result to find a spanning closed 2-walk that visits x, y only once, in any circuit graph by induction. In what follows, we use this structural result to find a spanning subgraph in any circuit graph that can be greedily embedded in the plane. We will use $\delta_G(x)$ to denote the degree of x in G. The proof of this theorem given by Gao and Richter is constructive, and can be used to find such a decomposition in polynomial time, given the planar drawing of the circuit graph.

Theorem 2. *For any circuit graph (G, C) and distinct $x, y \in C$, there exists a Christmas cactus graph $T(G)$ that spans G s.t. x, y are jointly in a cycle and $\delta_{T(G)}(x) = \delta_{T(G)}(y) = 2$. And such a subgraph can be found in polynomial time.*

Proof: The proof is by induction on the number of vertices in the circuit graph (G, C). Let x, y be distinct and $x, y \in V(C)$. Applying Theorem 1, there exists a partition of $V(G) - V(C)$ into $V_1, V_2, ..., V_m$ and distinct vertices $v_1, v_2, ..., v_m \in V(C) - \{x, y\}$ s.t. the graph induced by

---

[2] A *cut vertex* is a vertex that when removed from G, disconnects the graph.

[3] Gao and Richter actually state that $B_{i,j}$ is a block, but from the construction of this block in the proof of the theorem, it is clear that $B_{i,j}$ is a subgraph containing all nodes and edges inside (and including) a polygon $C_{i,j}$ in G. This implies that $(B_{i,j}, C_{i,j})$ is a circuit graph and Gao and Richter explicitly state this when actually invoking the structural theorem to prove that all circuit graphs contain a closed, spanning 2 walk.

$V_i \cup \{v_i\}$ is a plane chain of blocks $B_{i,1}, b_{i,1}, ..., b_{i,k-1}, B_{i,k}$, $v_i \in V(B_{i,1}) - b_{i,1}$ and each nontrivial block $B_{i,j}$ has a outer polygon $C_{i,j}$ s.t. $(B_{i,j}, C_{i,j})$ is a circuit graph.

From the definition of a plane chain of blocks, each cut vertex $b_{i,k}$ must be distinct. The chain of blocks $B_{i,1}, b_{i,1}, ..., b_{i,k-1}, B_{i,k}$ is a plane chain of blocks w.r.t. the original plane embedding. v_i is on the infinite face in G, and this implies that $v_i \in C_{i,1}$ because v_i must be in the infinite face in $(B_{i,1}, C_{i,1})$. Choose $b_{i,0} = v_i$ and $b_{i,k}$ to be any vertex $\in B_{i,k} - b_{i,k-1}$ that is contained in the infinite face $C_{i,k}$. If $B_{i,k}$ is a trivial block, then just choose $b_{i,k}$ to be the remaining endpoint. Directly from the structural theorem, $v_i \in B_{i,1} - b_{i,1}$. Then each $b_{i,j}$ is distinct, and contained in both infinite faces $C_{i,j}$ and $C_{i,j+1}$.

By induction each (non-trivial) $B_{i,j}$ has a Christmas cactus spanning subgraph s.t. $b_{i,j-1}, b_{i,j}$ are jointly on a simple cycle and $\delta_{T(B_{i,j})}(b_{i,j-1}), \delta_{T(B_{i,j})}(b_{i,j}) = 2$. Then join the spanning Christmas cactus graph of $B_{i,j}$ to the spanning Christmas cactus graph of $B_{i,j+1}$ by joining $b_{i,j}$ in each spanning Christmas cactus graph by an edge, and contracting the edge.

If the neighboring block in the plane chain is a trivial block, then just connect a non-trivial block to the next non-trivial block by a path of the trivial blocks. If $B_{i,1}$ is not trivial, then the result is a Christmas cactus spanning subgraph for the plane chain of blocks $B_{i,1}, b_{i,1}, ..., b_{i,k-1}, B_{i,k}$ s.t. v_i is on a cycle and has $\delta_{T(B_{i,1...k})}(v_i) = 2$. If $B_{i,1}$ is trivial, then the result is a Christmas cactus spanning subgraph for the plane chain of blocks $B_{i,1}, b_{i,1}, ..., b_{i,k-1}, B_{i,k}$ s.t. v_i is not on a cycle and $\delta_{T(B_{i,1...k})}(v_i) = 1$.

Then consider the base cycle $V(C)$, and join the spanning Christmas cactus graph of each induced plane chain of blocks $V_i \cup \{v_i\}$ to the node v_i on the cycle by an edge, and contract the edge.

The result is a spanning Christmas cactus graph of G s.t. x, y are jointly on a cycle and $\delta_{T(G)}(x), \delta_{T(G)}(y) = 2$ and the theorem is true by induction. This construction also yields a polynomial time algorithm because each decomposition is polynomial time constructible and x, y do not appear in the decomposition so the number of decompositions that must be computed is bounded by $\frac{n}{2}$. $\square$

2.3 Constructing a Greedy Embedding

In this section, we construct a greedy embedding of any Christmas cactus graph in the Euclidean plane.

Let G be an arbitrary Christmas cactus graph, and let F be the set of edges in G that are not contained in a simple cycle. For the purposes of this construction, all edges in F will be considered to be simple cycles (on two nodes). Then every edge in G is contained in exactly one simple cycle.
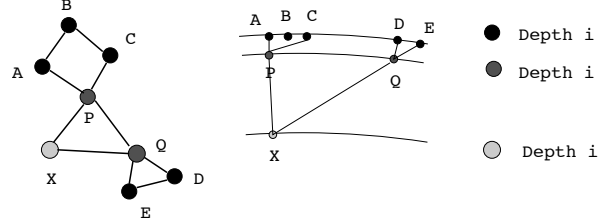
Figure 3. Embedding a Christmas cactus graph on concentric semi-circles.

Definition 5. A *depth tree T* w.r.t. G is a tree that contains a node for each simple cycle in G, where nodes in T are connected iff $|V(C_1) \cap V(C_2)| = 1$.

Select an arbitrary node of T to serve as the "root" and define the depth of a cycle in the graph G as the depth of the corresponding node in T. Then define the depth of any node in G to be the minimum depth of any cycle containing that node. G will be embedded on concentric semi-circles of radius $1 = R_0 < R_1 < R_2...$ s.t. all nodes at depth i will be embedded on the semi-circle of radius R_i. Let the center of all the semi-circles be the origin.

For any cycle $C = (p, x_1, ..., x_m)$ at depth $k > 0$ there will be a unique node on the cycle that is at depth $k - 1$ and all remaining nodes will be at depth k. Assume that for the cycle C, the unique node on the cycle that is at depth $k - 1$ is p. Then p will be embedded on the semi-circle at radius R_{k-1}, and node x_1 will be placed at the intersection of the semi-circle of radius R_k and the ray that contains p and the origin. The remaining nodes (if any) $x_2, ..., x_m$ will be embedded on the semi-circle of radius R_k s.t. the nodes $x_1, x_2, ..., x_m$ appear in clock-wise order along the semi-circle of radius R_k at a distance to be specified shortly.

Definition 6. A node $u \in G$ is a *descendant* of the cycle C at depth k if after removing all edges in C from G, node u is not in the component that also contains node p - the unique node in C at depth $k - 1$.

A node u in the cycle C at depth k, that is not the unique node in C at depth $k - 1$, is also a descendant of the cycle C by this definition.

The embedding will proceed in phases, and at the end of phase i all nodes at depth $\leq i$ will be placed. To simplify the analysis, after each phase (and subphase) we will preserve the greedy property that the subgraph induced by all currently placed nodes along with the current embedding, must exhibit the greedy routing property. Formally, if at the end of a subphase, the set of nodes in G already placed is P and the subgraph induced by P is G_P then for all $s, t \in P$ there exists a node u (adjacent to s in G_P) s.t. $d(u, t) < d(s, t)$.
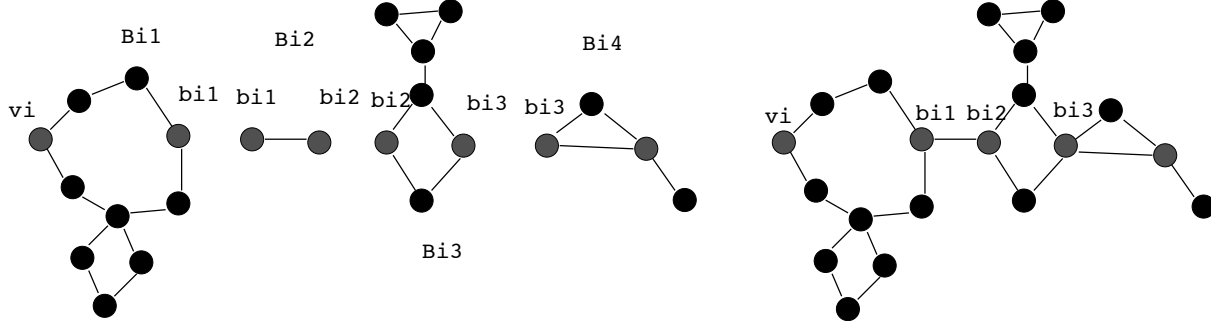
Figure 4. Connecting spanning Christmas cactus graphs in a plane chain of blocks.

We will use a geometric lemma to establish the properties needed for this embedding scheme. Consider the coordinates (assume $\epsilon > 0$ and $0 \le \alpha, \beta \le \pi$):

$$
\begin{aligned}
c &= (0, 1 + z) \\
b &= (-\sin\beta, \cos\beta) \\
a &= (-(1+\epsilon)\sin(\beta - \alpha), (1+\epsilon)\cos(\beta - \alpha))
\end{aligned}
$$

subject to the constraints:

$$
\begin{aligned}
0 &< \alpha, \beta \le \frac{\pi}{2} \\
0 &< \epsilon \le \frac{1 - \cos\beta}{6} \\
0 &\le z \le \epsilon \\
\sin\alpha &\le \frac{\epsilon(1 - \cos\beta)}{2(1+\epsilon)}
\end{aligned}
$$

Claim 1. $\frac{d}{dz}(d(a,c)^2 - d(b,c)^2) < 0$

Proof:

$$
\begin{aligned}
&\frac{d}{dz}(d(a,c)^2 - d(b,c)^2) \\
=\ & 2(1 + z - (1+\epsilon)\cos(\beta - \alpha)) - 2(1 + z - \cos\beta) \\
=\ & 2(\cos\beta - (1+\epsilon)\cos(\beta - \alpha)) \\
=\ & 2(\cos\beta - (1+\epsilon)(\cos\beta\cos\alpha + \sin\beta\sin\alpha)) \\
\le\ & 2\cos\beta(1 - (1+\epsilon)\cos\alpha) \\
<\ & 0
\end{aligned}
$$

where the last inequality follows because $\sin\alpha \le \frac{\epsilon}{1+\epsilon}$ and $\cos\alpha \ge 1 - \frac{\epsilon^2}{(1+\epsilon)^2} = \frac{1+2\epsilon}{(1+\epsilon)^2} > \frac{1}{1+\epsilon}$ □

Hence $d(a,c)^2 - d(b,c)^2$ is minimized for $z = \epsilon$.

Claim 2. $d(a,c) - d(b,c) \ge \epsilon^2$

Proof: By claim 1,

$$
\begin{aligned}
&d(a,c)^2 - d(b,c)^2 \\
\ge\ & (1+\epsilon)^2 \sin^2(\beta - \alpha) - \sin^2\beta \\
& + (1+\epsilon)^2(1 - \cos(\beta - \alpha))^2 - (1 + \epsilon - \cos\beta)^2 \\
\ge\ & (1+\epsilon)^2 \sin^2(\beta - \alpha) - 2(1+\epsilon)(1 - \cos\beta) \\
& + (1+\epsilon)^2(1 - \cos(\beta - \alpha))^2 - \epsilon^2 \\
=\ & 2(1+\epsilon)^2(1 - \cos(\beta - \alpha)) - 2(1+\epsilon)(1 - \cos\beta) - \epsilon^2 \\
=\ & -\epsilon^2 + 2(1+\epsilon)(\epsilon + \cos\beta - (1+\epsilon)\cos(\beta - \alpha)) \\
\ge\ & -\epsilon^2 + 2(1+\epsilon)(\epsilon + \cos\beta - (1+\epsilon)(\cos\beta + \sin\alpha)) \\
=\ & -\epsilon^2 + 2(1+\epsilon)(\epsilon(1 - \cos\beta) - (1+\epsilon)\sin\alpha) \\
\ge\ & -\epsilon^2 + (1+\epsilon)\epsilon(1 - \cos\beta) \\
\ge\ & 5\epsilon^2
\end{aligned}
$$

Then

$$
\begin{aligned}
d(a,c) - d(b,c) &= \frac{d(a,c)^2 - d(b,c)^2}{d(a,c) + d(b,c)} \\
&\ge \frac{1}{5}(d(a,c)^2 - d(b,c)^2) \\
&\ge \epsilon^2
\end{aligned}
$$

since $d(a,c) + d(b,c) \le 5$. □

For notational convenience, given an embedding $f : V \to R^2$ define the angle $\angle a, b, c$ on nodes $a, b, c, \in V(G)$ as the angle formed by the rays $(f(b), f(a)$ and $(f(b), f(c))$.

Theorem 3. For any Christmas cactus graph G, there exists a greedy embedding of G into the Euclidean plane.

Proof: Assume that every edge is in a simple cycle by considering any edge not in a simple cycle as a 2-cycle. Construct the depth tree T w.r.t. G, and root T at an arbitrary node. Trace out a semi-circle of radius 1 centered at the origin. Suppose that the cycle C in G at depth 0 contains m nodes, $C = (1, 2, ..., m)$. Then divide the semi-circle of radius R_0 into m equal angle sectors and place node i at the

beginning of the ith sector. The arc subtended by $1, 2, ..., m$ is strictly smaller than the perimeter of the semi-circle of radius $R_0 = 1$ because no node is placed at the end of the mth sector.

For any triple $(a, a+1, c)$ such that $c > a$ the angle $\angle a, a+1, c$ is strictly larger than $\frac{\pi}{2}$ and $d(a+1, c) < d(a, c)$. Similarly for any triple $(a, c-1, c)$ such that $c > a$ then the angle $\angle a, c-1, c$ is strictly larger than $\frac{\pi}{2}$ and $d(a, c-1) < d(a, c)$. Hence this embedding is greedy.

This establishes the base case for the inductive construction. Now assume that all nodes at depth $\leq i$ have been placed and that the induced subgraph on these nodes, G_i, along with the embedding on concentric semi-circles (as described earlier) is greedy.

Definition 7. If all nodes in G_i have been embedded s.t. this embedding exhibits the greedy property, then for all $s, t \in G_i$ there exists $u \in G_i$ s.t. $(u, s) \in E(G_i)$ and $d(u, t) < d(s, t)$. Fix $n_{s,t} = u$ and define $\delta(G_i) = \min_{s,t} d(s, t) - d(t, n_{s,t})$.

Then draw a ball B_u of radius $\delta(G_i)/3$ around each node $u \in G_i$. Clearly if a node t at depth $i+1$ s.t. $(u, t) \in E(G)$ is placed in B_u, then for any node $s \in G_i, \neq t$ the neighbor $n_{s,t}$ that is strictly closer to t will also be strictly closer to u. And if $s = t \in G_i$ then the neighbor u will be strictly closer to u.

Definition 8. Let $\beta(G_i)$ be defined as the minimum (non-zero) angle over all s, t at depth $\leq i$ from s to the origin, to t in the current embedding.

Assume that all nodes at depth $\leq i$ have been placed, and that the subgraph induced by these nodes along with the embedding exhibits the greedy property. We must embed all cycles at depth $i+1$, and preserve the greedy property.

Subphase:

For each cycle C at depth $i+1$, $C = (p, x_1, ..., x_m)$ let p be the unique node in the cycle at depth i. Call x_1 the *representative node* for the cycle C (choose an orientation of C at random, and choose the next node after p). Let the radius of the outermost semi-circle in the current embedding be R_i and define $\delta(G_i)$ and $\beta(G_i)$ as before w.r.t. the current embedding. Also, define $\epsilon = \min(\frac{\delta(G_i)}{3}, R_i \frac{1 - \cos \frac{2}{3}\beta(G_i)}{6})$.

Place each representative node x_1 at the intersection of the semi-circle of radius $R_{i+1} = R_i + \epsilon$ and the ray containing both the origin and p. Let P be the set of currently placed nodes (all nodes at depth $\leq i$ and one representative node for each cycle at depth $i+1$). We will show that the subgraph induced by these nodes, along with the current embedding, exhibits the greedy property:

For any nodes $s, t \in G_i$ there is trivially a node u (adjacent to s in G_i) in the current embedding s.t. $d(u, t) < d(s, t)$ because the nodes in G_i along with the embedding of these nodes exhibited the greedy property (and the embedding of nodes in G_i has not been changed) by induction. For any

nodes s, t s.t. $s \in G_i$ and t is a representative node for a cycle C at depth $i+1$, then t is in the ball B_p of radius $\frac{\delta(G_i)}{3}$ centered around the node p in C that is the unique node in C at depth i. Then (t, p) is an edge in G, and all $s \in G_i$ have a (already placed) neighbor that is strictly closer to t.

Lastly, consider routing from a node s that is a representative node on a cycle C at depth $i+1$ to any node that has already been placed. Again, let p be the unique node in C that is at depth i. By construction, the perpendicular bisector to the segment sp contains all nodes currently placed, except s, on the same side as p:

Trivially, any node t at depth $\leq i$ will be on the same side of the perpendicular bisector to sp as the node p because the perpendicular bisector is a parallel shift of the tangent to the semi-circle of radius R_i at the point p. And for any node t that is placed on the semi-circle at radius R_{i+1}, the angle from t to the origin to s will be at least $\beta(G_i)$, and from the geometric lemma (choosing $\alpha = 0$) then this node will also be on the p side of the perpendicular bisector to sp. And thus p is strictly closer to t than s is to t for any node t at depth $i+1$ that has been placed in this subphase.

All cases are covered: the subgraph on the currently embedded nodes, along with the current embedding is greedy because for all s, t there is an already placed neighbor of s that is strictly closer to t.

Subphase:

For a cycle $C = (p, x_1, ..., x_m)$ at depth $i+1$, only the unique node p at depth i and the representative node x_1 have been embedded so far. The embedding must now be extended to include all nodes at depth $i+1$ and this is done by placing nodes $x_2, ..., x_m$ in clockwise order around the semi-circle of radius R_{i+1} starting from the location of node x_1.

Let G_i^1 be the subgraph induced by all nodes already placed. Define $\delta(G_i^1)$ as before. Note that this difference is now defined over all s, t that have already been placed, which includes all nodes that are at depth $\leq i$ and all nodes that are depth $i+1$ that are representative nodes for a cycle at depth $i+1$. Also define $\epsilon = R_{i+1} - R_i$, and note that $\beta(G_i^1) = \beta(G_i)$.

Then place nodes $x_2, ..., x_m$ on the semi-circle of radius R_{i+1} at even intervals starting from the (already fixed) location of x_1 s.t. the angle (in radians) from x_1 to the origin to x_m is $\alpha \leq \min(\frac{\delta(G_i^1)}{3}, \frac{\beta(G_i)}{3})$ and s.t. $\sin \alpha \leq \frac{\epsilon(1 - \cos \frac{2}{3}\beta(G_i))}{2(1+\epsilon)}$. Place all nodes on a cycle at depth $i+1$ that have not already been placed, according to this rule. The induced subgraph on all placed nodes after this subphase, along with the embedding will exhibit the greedy property:

Now all nodes at depth $\leq i+1$ have been placed. Consider all pairs $s, t \in G_{i+1}$. Again, if $s, t \in G_i^1$ then there will still be a neighbor of s that is strictly closer.

If $s \in G_i^1$, and t is placed in this subphase then t is at most distance $\frac{\delta(G_i^1)}{3}$ from the representative node u on the

same cycle. $s \neq u$ will have a neighbor that is strictly closer to this representative node u, and this same neighbor will also be strictly closer to t. If $s = u$, then s and t will be connected by a path on the semi-circle of radius R_{i+1} and each successive node on this path will be strictly closer to t.

The only remaining case is when s is a node that is placed in this subphase:

Any node that is not in the set $x_2, x_3, ..., x_m$ cannot be strictly contained in the sector from x_1 to the origin to x_m because we chose $\alpha \leq \frac{\beta(G_i)}{3}$. Then if s is not x_1 or x_m, s will have a neighbor that is strictly closer to t, choosing the next node on the path $x_1, x_2, ..., x_m$ radially in the direction of t.

The case in which $s = x_1$ has already been covered because the node $x_1 \in G_i^1$. Then suppose $s = x_m$: If t is in the set $x_1, ..., x_m$, then choosing the next node on the path $x_1, x_2, ..., x_m$ radially in the direction of t and s will have a neighbor strictly closer to t. And if t is not in the set $x_1, ..., x_m$: $\alpha \leq \frac{1}{3}\beta(G_i)$ and this implies that the angle from p to the origin to the node t is at least $\frac{2}{3}\beta(G_i)$. From the geometric lemma, $d(x_m, t) - d(p, t) > 0$ and p is strictly closer to t.

Then all cases are covered, and the subgraph on the currently embedded nodes, along with the current embedding is greedy because for all s, t there is an already placed node that is strictly closer to t.

This completes the inductive construction, because all nodes at depth $\leq i + 1$ have been placed. And this also completes the proof that all Christmas cactus graphs can be greedily embedded in the Euclidean plane. □

Corollary 1. Any 3-connected graph G that excludes $K_{3,3}$ as a minor admits a greedy embedding into the Euclidean plane.

3 Greedy Embeddings for Trees

In this section, we provide the first non-trivial examples of graphs that admit no greedy embedding into the Euclidean plane. We prove a combinatorial condition that guarantees non-embeddability and this condition provides a certificate for non-embeddability that can be verified in linear time. We use this result to construct graphs that can be greedily embedded into the Euclidean plane, but for which no spanning tree admits such an embedding.

3.1 Irreducible Triples

Definition 9. An *irreducible triple* is a triple $\{b, c, d\}$ of nodes in a graph G such that $deg(b) = 3$ and $(b, c), (b, d) \in E(G)$ and removing either (b, c) or (b, d) disconnects the graph. The *parent* of an irreducible triple $\{b, c, d\}$ is the unique node $a \notin \{b, c, d\}$ such that $(a, b) \in E(G)$.

Definition 10. Two irreducible-triples $\{b, c, d\}$ and $\{x, y, z\}$ are said to be *independent* if $\{b, c, d\} \cap \{x, y, z\} = \emptyset$ and if deleting edges $(b, c), (b, d), (x, y)$, and (x, z) leaves b and x connected. A set of irreducible triples is *mutually independent* if the irreducible triples are pair-wise independent.

Lemma 2. For any set of 3 or more mutually independent irreducible triples, let a be the parent of an irreducible triple in the set. Then for all irreducible triples $\{x, y, z\}$ in the set, $\{a\} \cap \{x, y, z\} = \emptyset$ (including the triple for which a is the parent).

Proof: Clearly any parent a of an irreducible triple $\{b, c, d\}$ cannot intersect $\{b, c, d\}$ directly from the definition of parent. Suppose there is a set of 3 or more mutually independent triples in a graph G, and that the parent a of an irreducible triple $\{b, c, d\}$ is contained in another irreducible triple $\{x, y, z\}$. Suppose that $a = y$. Then deleting the edge (x, y) disconnects G, but b is still connected to y. This implies that b is not still connected to x. This contradicts the definition of independence.

Suppose that $a = x$. Then $\Gamma(b) = \{x, c, d\}$ and $\Gamma(x) = \{b, y, z\}$ because x is the parent of the triple $\{b, c, d\}$ and b is the parent of the triple $\{x, y, z\}$. Then consider a third irreducible triple in the set of 3 or more mutually independent irreducible triples, $\{l, m, n\}$.

From the definition of an irreducible triple, deleting the edge (b, c) must partition G into components C_1, C_2. Deleting the edge (b, d) also partitioned G into two components. Note that (b, d) cannot connect C_1 and C_2 that resulted from deleting (b, c). Then deleting (b, d) after deleting (b, c) must partition G into three components, one of which contains b, one of which contains c, one of which contains d. Continuing the argument deleting edges $(b, c), (b, d), (x, y)$ and (x, z) partitions G into five components, one of which contains b and x, one of which contains c, one of which contains d, one of which contains y and one of which contains z. The component that contains b and x contains only the nodes b and x, because $deg(b) = deg(x) = 3$ before deleting two of the edges incident to b and two of the edges incident to x.

Node l must be contained in a different component than b and x. Let this component be the component that contains y. Then $\{l, m, n\}$ would not be independent from $\{x, y, z\}$ because deleting the edge (x, y) from G would leave l and x in different components, and this contradicts the definition of independence.

Thus if there is a set of 3 or more mutually independent irreducible triples, then the parent a of any irreducible triple $\{b, c, d\}$ in the set cannot be contained in any other triple in the set. □

Lemma 3. For any set of 3 or more mutually independent triples, let $\{b, c, d\}$ and $\{x, y, z\}$ be two irreducible triples in

the set and let a, w be the respective parents of these irreducible triples (note that a and w are not guaranteed to be distinct). Then any simple path from y to c in the graph G must be of the form $(y, x), (x, w), ..., (a, b), (b, c)$. Any simple path from x to c must be of the form $(x, w), ..., (a, b), (b, c)$. And any simple path from x to b must be of the form $(x, w), ..., (a, b)$.

**Proof:** The proof immediately follows from the previous lemma. □

We will implicitly use the path lemma throughout the proof that any graph G containing a set of 6 or more mutually independent irreducible triples cannot be greedily embedded. Let $S = \cup \{b, c, d\}$ be the set of all nodes in any irreducible triple in the set. Let $\{b, c, d\}$ be a particular irreducible triple in the set and let a be the parent of this triple. Suppose G admits a greedy embedding $f : V \to R^2$. Consider the halfspace H_b that is bounded by the perpendicular bisector to $f(b), f(c)$ that contains $f(b)$. This halfspace must contain $S / \{c\}$ from the path lemma, because the only simple paths from c to a node $t \in S / \{c\}$ must begin by traversing the edge (c, b). Similarly, the halfspace H_a that is bounded by the perpendicular bisector to $f(a), f(b)$ and contains $f(a)$ must contain $S / \{b, c, d\}$ again from the path lemma.

Lemma 4. Let G be a graph that admits a greedy embedding $f : V \to R^2$ into the Euclidean plane and let $(p, x) \in E(G)$ be an edge s.t. deleting (p, x) disconnects G. Let C_x be the component containing x that results from deleting the edge (p, x), and let z be an arbitrary node $\notin C_x$. Then $\{x\} = W_z = \arg\min_{w \in C_x} \|f(w) - f(z)\|_2$

Proof: Suppose that $W_z \neq \{x\}$, and there is a node $w \in \arg\min_{w \in C_x} \|f(w) - f(z)\|_2$ and $w \neq x$. All the neighbors of w are in C_x and no node in C_x is strictly closer to z. Then there is no neighbor of w that is strictly closer to z, and f is not a greedy embedding. □

Lemma 5. Let G be a graph that admits a greedy embedding $f : V \to R^2$ into the Euclidean plane, and that contains an irreducible triple $\{b, c, d\}$. Then any greedy embedding into the Euclidean plane must map the nodes b, c, d to points in R^2 s.t. the angle $\angle f(c) f(b) f(d) > \frac{\pi}{3}$

Proof: Suppose that the angle $\angle f(c) f(b) f(d)$ is $\leq \frac{\pi}{3}$. Then by the law of sines, the side (c, d) cannot be the strictly largest side in the triangle (c, b, d). Let (b, d) be the largest side in the triangle. Node $d \notin C_c$ and c must be the closest node in C_c (the component that results from deleting the edge (b, c)) to d. And when routing from node c to node d, node b must be selected for the next hop. However $d(b, d) \geq d(c, d)$ and this embedding cannot be greedy. □
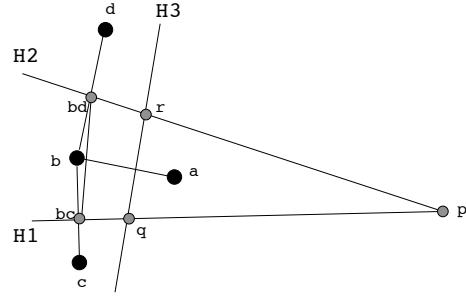
Figure 5. Embedding a quadruple $\{a, b, c, d\}$ **such that** $f(a)$ **is contained in a side of the angle** $\angle f(c) f(b) f(d)$ **that is** $< \pi$.

Claim 3. Any graph that contains two independent irreducible triples $\{b, c, d\}$ and $\{x, y, z\}$ - where $\Gamma(b) = \{a, c, d\}, \Gamma(x) = \{w, y, z\}$ - cannot be greedily embedded such that $f(a)$ is contained in a side of the angle $\angle f(c) f(b) f(d)$ that is $\leq \pi$ and $f(w)$ is contained in a side of the angle $\angle f(y) f(x) f(z)$ that is $\leq \pi$.

Proof: Assume that both $\angle f(c) f(b) f(d)$ and $\angle f(y) f(x) f(z) \neq \pi$. Consider a greedy embedding of the quadruple $\{a, b, c, d\}$ depicted in Figure 5.

If the embedding is greedy, then there must be a path from a to c s.t. the distances to the destination node, c, are strictly decreasing along this path. There must also be such a path from a to d. Any such path contains b as an intermediary node, and this implies that $d(f(b), f(c)) < d(f(a), f(c))$ and $d(f(b), f(d)) < d(f(a), f(d))$. This implies that $f(c)$ and $f(d)$ must be contained on the b side of the line H_3. As a result, the segment (bc, bd) must be contained on the b side of H_3 because the line segment is contained in the convex hull of the points $f(c), f(d), f(b)$.

Using a similar argument, all nodes in G not in $\{b\} \cup C_c \cup C_d$ must be strictly contained in the triangle (p, q, r), because all nodes in G not in $\{b\} \cup C_c \cup C_d$ must be strictly on the b side of H_1, strictly on the b side of H_2 and strictly on the a side of H_3 respectively. Because the segment (bc, bd) is contained on the b side of H_3, we can relax this constraint to the requirement that all nodes in G not in $\{b\} \cup C_c \cup C_d$ must be strictly contained in the triangle (bc, bd, p).

An identical argument holds for the quadruple $\{w, x, y, z\}$, and all nodes in G not in $\{x\} \cup C_y \cup C_z$ must be strictly contained in the triangle (xy, xz, o).

Consider the point bc. This lies in the convex hull of $f(b), f(c)$ and any convex body (specifically the triangle (xy, xz, o)) that strictly contains $f(b)$ and $f(c)$ must strictly contain bc. Similarly the point bd must be strictly contained in the triangle (xy, xz, o). This implies that the segment (bc, bd) must be strictly contained in the triangle (xy, xz, o).
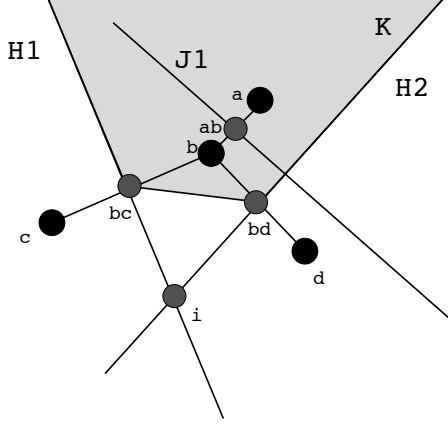
Figure 6. Embedding a quadruple $\{a,b,c,d\}$ **such that** $f(a)$ **is contained in a side of the angle** $\angle f(c)f(b)f(d)$ **that is** $> \pi$.

An identical argument holds for the triangle (bc, bd, p) and this triangle must strictly contain the segment (xy, xz).

However, this yields a contradiction because there are two triangles T_1 and T_2 such that T_1 must strictly contain the base of T_2 and T_2 must strictly contain the base of T_1. An almost identical argument holds when $\angle f(c)f(b)f(d) = \pi$ or $\angle f(y)f(x)f(z) = \pi$. □

Claim 4. If a graph G is greedily embedded and contains an irreducible triple $\{b,c,d\}$ - where $\Gamma(b) = \{a,c,d\}$ - that is embedded such that $f(a)$ is contained in a side of the angle $\angle f(c)f(b)f(d)$ that is $> \pi$, then let i be the point of intersection of the perpendicular bisector to the segment $(f(b), f(c))$ and the perpendicular bisector to the segment $(f(b), f(d))$. All nodes not in $C_c \cup C_d$ are mapped outside the interior of the quadrilateral $(f(c), f(b), f(d), i)$.

Proof: Consider Figure 6. Clearly, we must only prove that all nodes not in $C_c \cup C_d$ are mapped outside the quadrilateral $(bc, f(b), bd, i)$ because the line H_1 must contain all nodes not in C_c on the b side, and the line H_2 must contain all nodes not in C_d on the b side.

Consider the halfspace J_1 defined as the a side of the perpendicular bisector to $f(a), f(b)$. Node a is not mapped into the quadrilateral $(bc, f(b), bd, i)$ by assumption, and any node in G not in $\{b\} \cup C_c \cup C_d$ must be contained in the halfspace J_1. If the intersection of J_1 with the quadrilateral $(bc, f(b), bd, i)$ is empty, then clearly the claim is proven. The halfspace J_1 has an empty intersection with the triangle $(f(b), bc, bd)$ because the points $f(b), f(c), f(d)$ must be on the b side of the perpendicular bisector to $(f(a), f(b))$.

Consider the triangle (bc, bd, i). Assume that J_1 intersects the triangle (bc, bd, i). Then J_1 must contain at least

one of the points bc, bd, i. J_1 cannot contain bc or bd because $f(b), f(c)$, and $f(d)$ must all be closer to $f(b)$ than to $f(a)$. As a result, if J_1 intersects the triangle (bc, bd, i) then J_1 must contain the point i.

A line can intersect another line more than once only if the two lines are identical. Consider the line L bounding the halfspace J_1. Suppose this line is identical to H_1. This can only happen if $f(a) = f(c)$ and this would imply that the embedding is not greedy because node a will not have a neighbor that is strictly closer to $f(c)$. This implies that the line L can intersect H_1 and H_2 at most once each.

J_1 contains the point i, but not either of the points bc or bd. This implies that the line L intersects both segments (bc, i) and (bd, i). This line can be cut into two rays, leaving from the point ab in opposite directions. Both rays begin at the point ab inside the shaded region K, and cannot leave this region through the segment (bc, bd) because both end points of this segment are not contained in J_1.

Then one of the rays must exit the region K through a side bounded by either the line H_1 or the line H_2. This yields a contradiction because the line L will intersect either the line H_1 or the line H_2 twice. Thus J_1 cannot contain i and the claim is proven. □

Note that $\angle bcibd < \frac{2\pi}{3}$ because $\angle bc f(b)bd > \frac{\pi}{3}$ and $\angle f(b)bci = \angle f(b)bdi = \frac{\pi}{2}$. When a point x is contained in the sector $\angle bcibd$ we will say the intersection point i contains x.

Lemma 6. Any graph G containing 6 or more mutually independent irreducible triples cannot be greedily embedded in the Euclidean plane.

Proof: Suppose that a graph G contains 6 mutually independent irreducible triples. Then there are two cases to consider:

Suppose that the irreducible triples are embedded such that for each quadruple $\{a,b,c,d\}$ - where $\{b,c,d\}$ is an irreducible triple and $\Gamma(b) = \{a,c,d\}$ - $f(a)$ is contained in a side of the angle $\angle f(c)f(b)f(d)$ that is $> \pi$. Then for each quadruple $\{a,b,c,d\}$ define the points bc, bd, i as in Figure 6.

No node in G is mapped to a point in the triangle (bc, bd, i) from the previous claim. Then consider another quadruple $\{w,x,y,z\}$ where $\{x,y,z\}$ is an irreducible triple and $\Gamma(x) = \{w,y,z\}$. Re-using the argument used in the previous claim, if the perpendicular bisector L to the segment $(f(x), f(y))$ does not contain i on the x side, then L must intersect either H_1 or H_2 twice. But L is a perpendicular bisector to two points that are contained on the same side of H_1 and on the same side of H_2, and L cannot be identical to H_1 or H_2.

Then defining the intersection point i for each quadruple $\{a,b,c,d\}$ as in Figure 6, each intersection point must contain all other intersection points in an angle that is $< \frac{2\pi}{3}$.
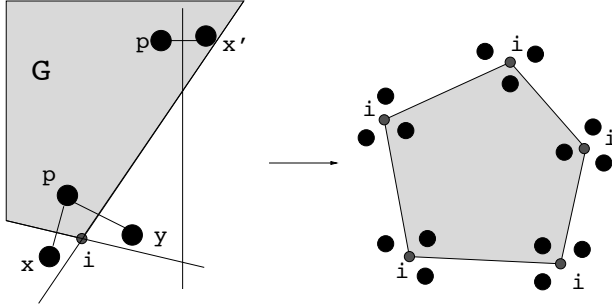
Figure 7. Embedding quadruples.

Define polygon on these intersection points - all angles in the polygon are $< \frac{2\pi}{3}$. And from elementary geometry any polygon on $n \geq 6$ nodes must contain an angle that is at least $\frac{2\pi}{3}$. Then there can be at most 5 intersection points. This yields a contradiction, because there are 6 intersection points.

Suppose that one quadruple $\{a, b, c, d\}$ is embedded such that $f(a)$ is contained on the side of the angle $\angle f(c)f(b)f(d)$ that is $\leq \pi$. Consider the points bc, bd, and define intersection points for all remaining irreducible triples. Then each intersection point for the remaining 5 irreducible triples must contain bc and bd because these intersection points must contain $f(b), f(c)$ and $f(d)$. We can apply the argument used above to the perpendicular bisectors H_1, H_2 and H_3 and this implies that the angles bounded by H_1, H_3 and H_2, H_3 must contain each intersection point defined for the remaining 5 irreducible triples.

The angles bounded by H_1, H_3 and H_2, H_3 sum to at most π, and using the 5 intersection points and the points bc, bd we have a polygon on 7 nodes such that the angles sum to at most $\frac{13}{3}\pi$, which yields a contradiction because the sum of the angles in a 7-gon is 5π from elementary geometry. □

Corollary 2. The complete binary tree B_{31} with 31 nodes cannot be greedily embedded into the Euclidean plane.

Proof: The complete binary tree B_{31} contains 6 mutually independent irreducible triples. □

Theorem 4. There exist graphs that can be greedily embedded into the Euclidean plane, but for which no spanning tree can be greedily embedded into the Euclidean plane.

Proof: Let G be the cycle graph on n nodes, and for each node i add a 4-cycle $(w_i, x_i), (x_i, y_i), (y_i, z_i), (z_i, w_i)$ and an extra node p_i, such that w_i is joined by an edge to i, and p_i is joined to the node y_i. Any spanning tree of this graph contains n mutually independent irreducible triples, however, this graph is a Christmas cactus graph and can be greedily embedded into the Euclidean plane. □

4 Acknowledgment

References

[1] D. Barnette. Trees in polyhedral graphs. *Canada J. Math*, 18:731–736, 1966.

[2] P. Bose, P. Morin, I. Stojmenovic, and J. Urrutia. Routing with guaranteed delivery in ad hoc wireless networks. *Wireless Networks*, 7:609–616, 2001.

[3] R. Dhandapani. Greedy drawings of triangulations. *Symposium on Discrete Algorithms*, pages 102–111, 2008.

[4] A. Frieze, J. Kleinberg, R. Ravi, and W. Debany. Line-of-sight networks. *Symposium on Discrete Algorithms*, pages 968–977, 2007.

[5] Z. Gao. 2-connected coverings of bounded degree in 3-connected graphs. *J. Graph Theory*, 20:327–338, 1995.

[6] Z. Gao and R. Richter. 2-walks in circuit graphs. *J. Comb. Theory Series B*, 62:259–267, 1994.

[7] A. Goel, S. Rai, and B. Krishnamachari. Sharp thresholds for monotone properties in random geometric graphs. *ACM Symposium on Theory of Computing*, pages 580–586, 2004.

[8] B. Karp and H. T. Kung. Gpsr: Greedy perimeter stateless routing for wireless networks. *Proceedings of the 6th annual International Conference on Mobile Computing and Networking*, pages 243–254, 2000.

[9] R. Kleinberg. Geographic routing in hyperbolic space. *INFOCOM 2007, 26th IEEE Conference on Computer Communications*, pages 1902–1909, 2007.

[10] F. Kuhn, R. Wattenhofer, Y. Zhang, and A. Zolliger. Geometric ad-hoc routing: Of theory and practice. *Annual ACM Symposium on Principles of Distributed Computing*, pages 63–72, 2003.

[11] C. Papadimitriou and D. Ratajczak. On a conjecture related to geometric routing. *Theoretical Computer Science*, 244:3–14, 2005.

[12] M. Penrose. *Random Geometric Graphs in Oxford Studies in Probability*, volume 5. Oxford University Press, 2003.

[13] A. Rao, C. Papadimitriou, S. Shenker, and I. Stoica. Geographic routing without location information. *Proceedings of the 9th annual International Conference on Mobile Computing and Networking*, pages 96–108, 2003.

[14] C. Thomassen. A theorem on paths in planar graphs. *J. Graph Theory*, 7:169–176, 1983.