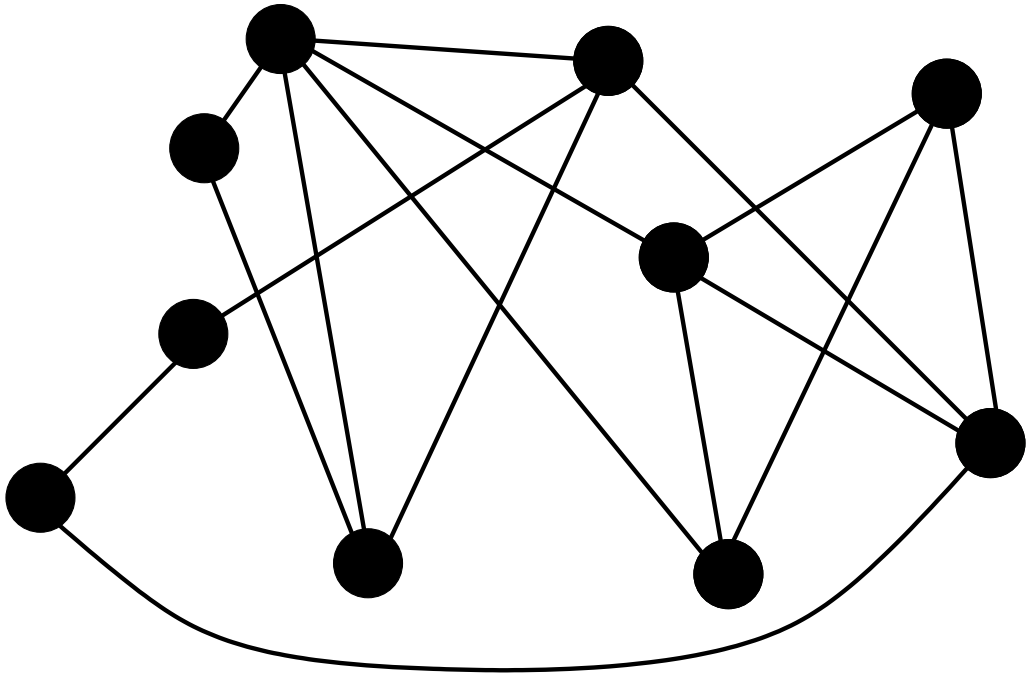
Approximation Algorithms for Multicommodity-Type Problems with Guarantees Independent of the Graph Size

Ankur Moitra, MIT

October 23, 2009

The Minimum Bisection Problem

Goal: Minimize cost of bisection



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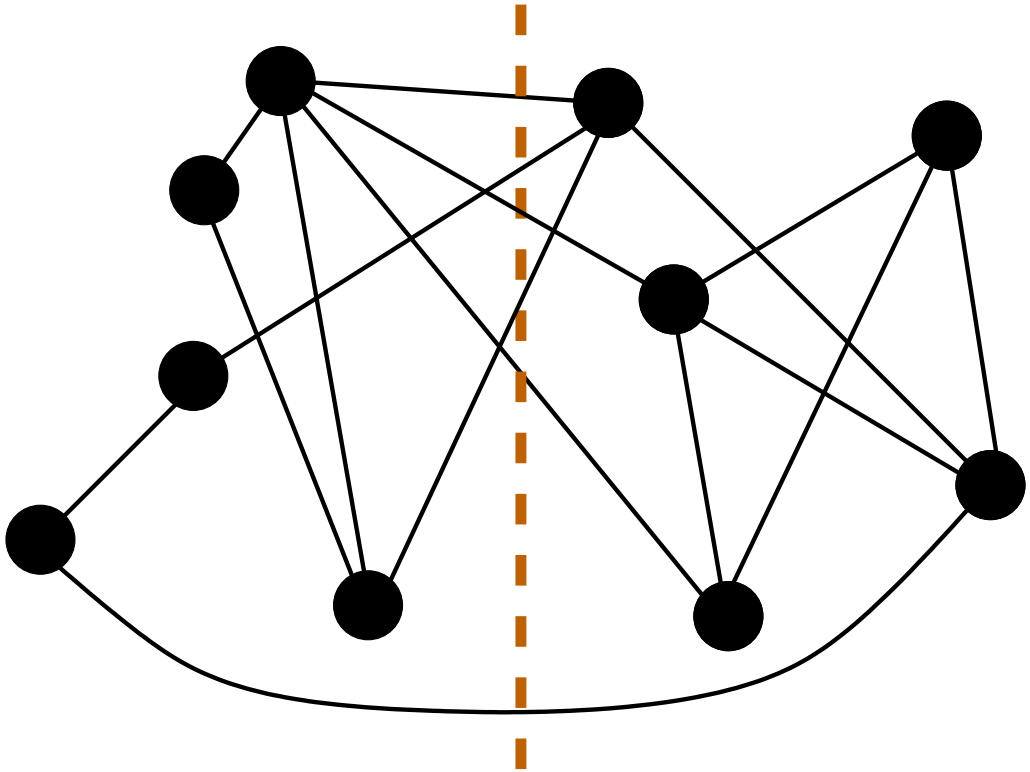

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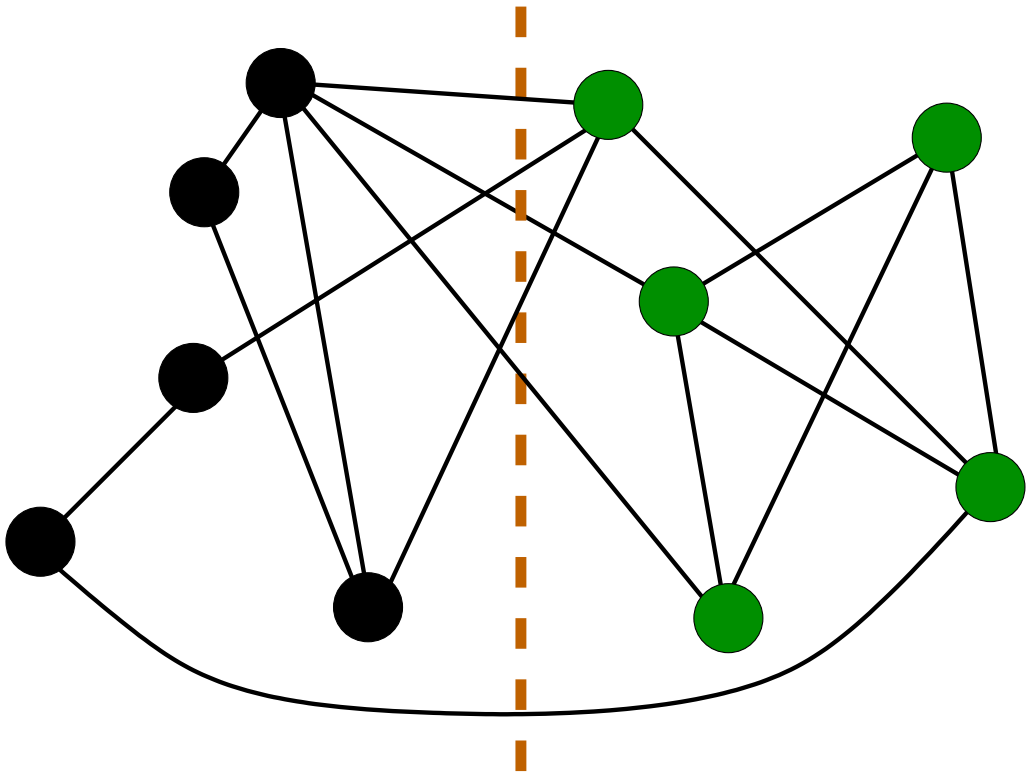

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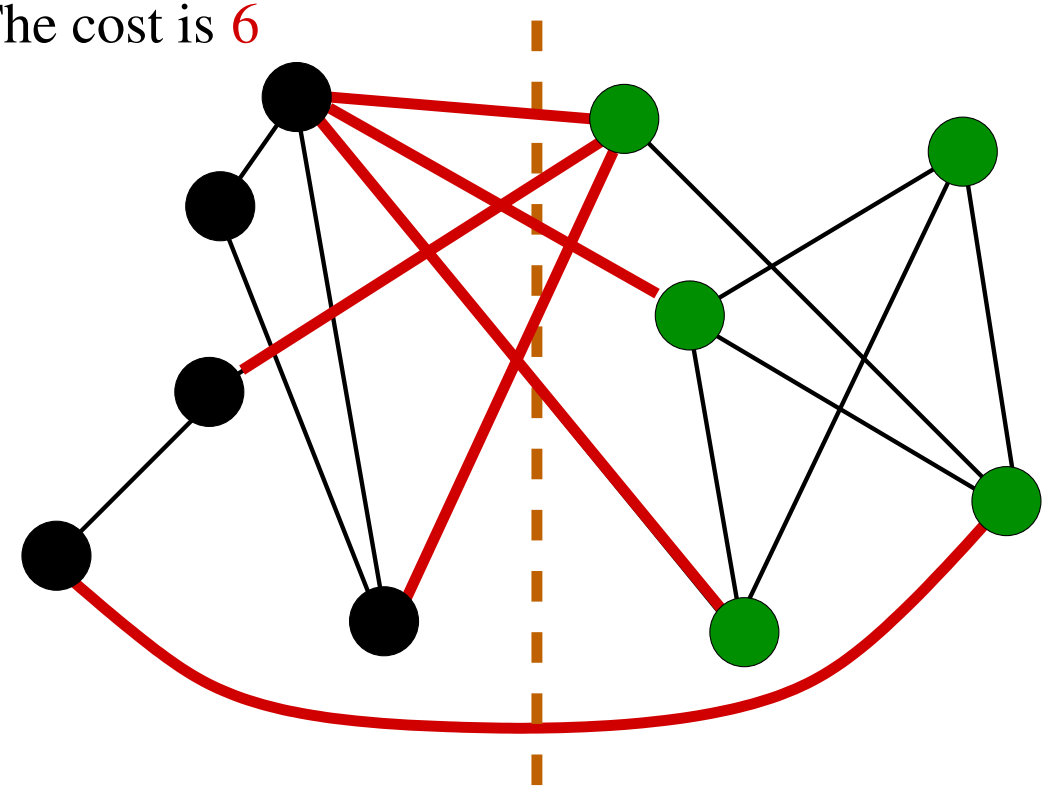

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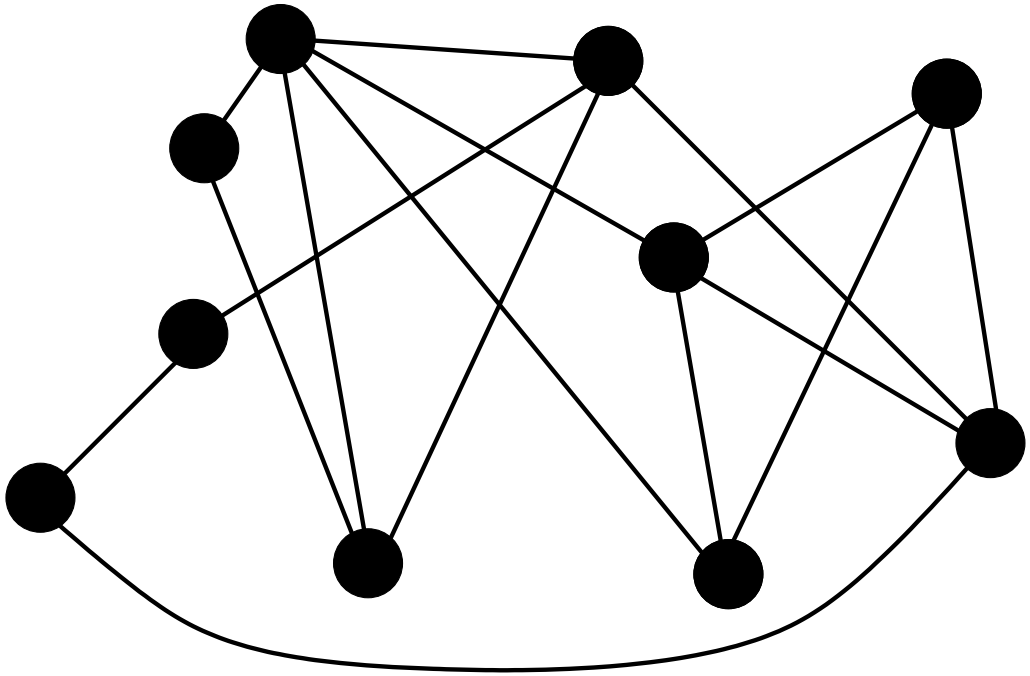

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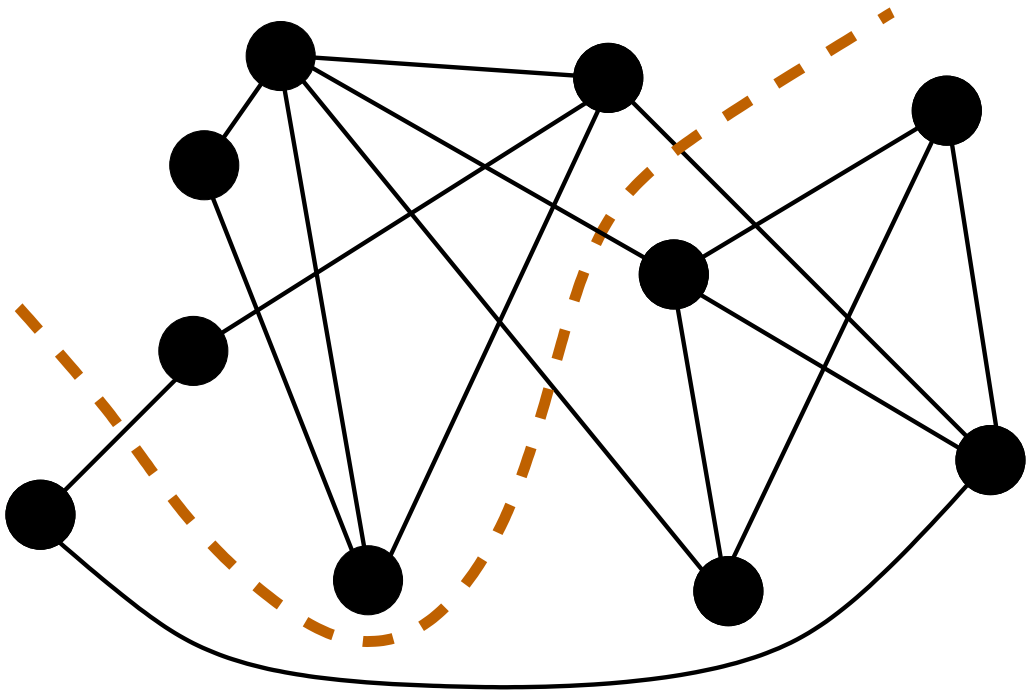

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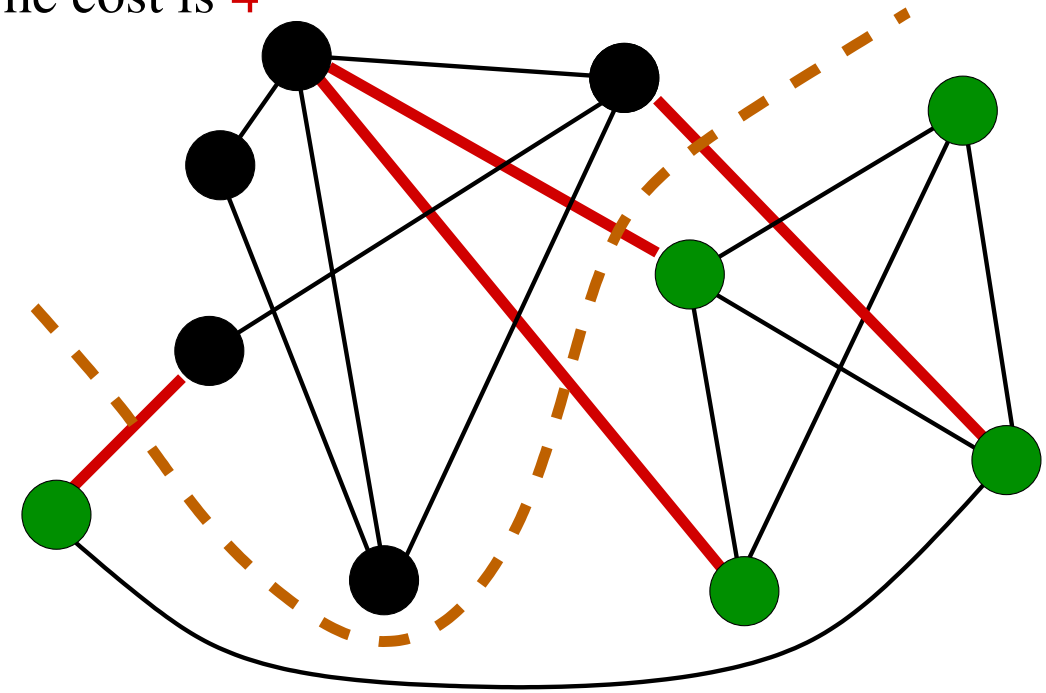

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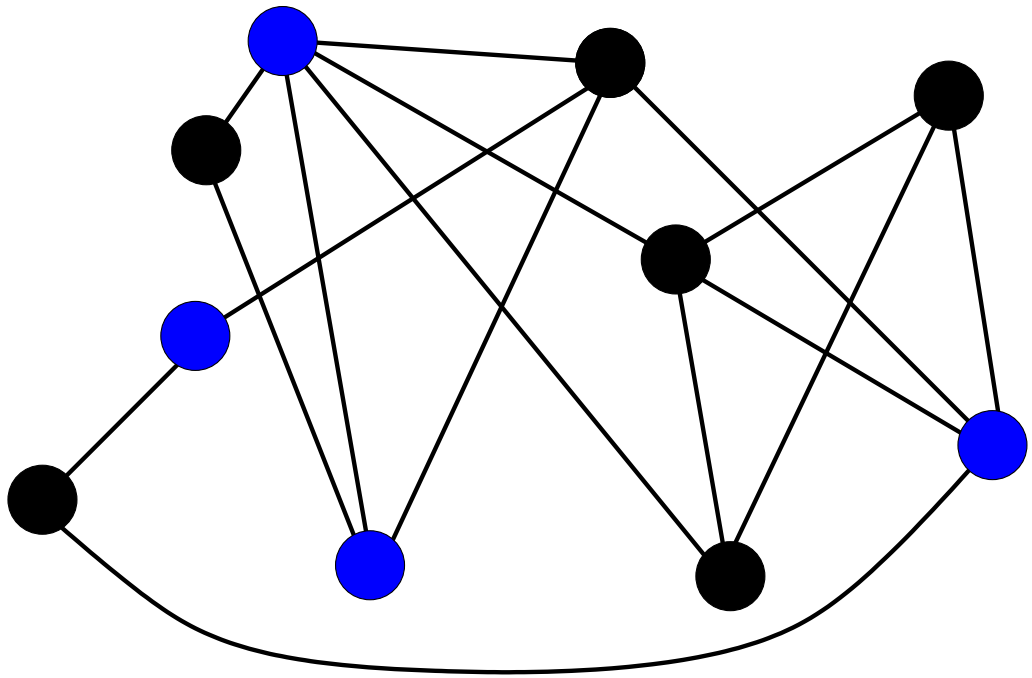

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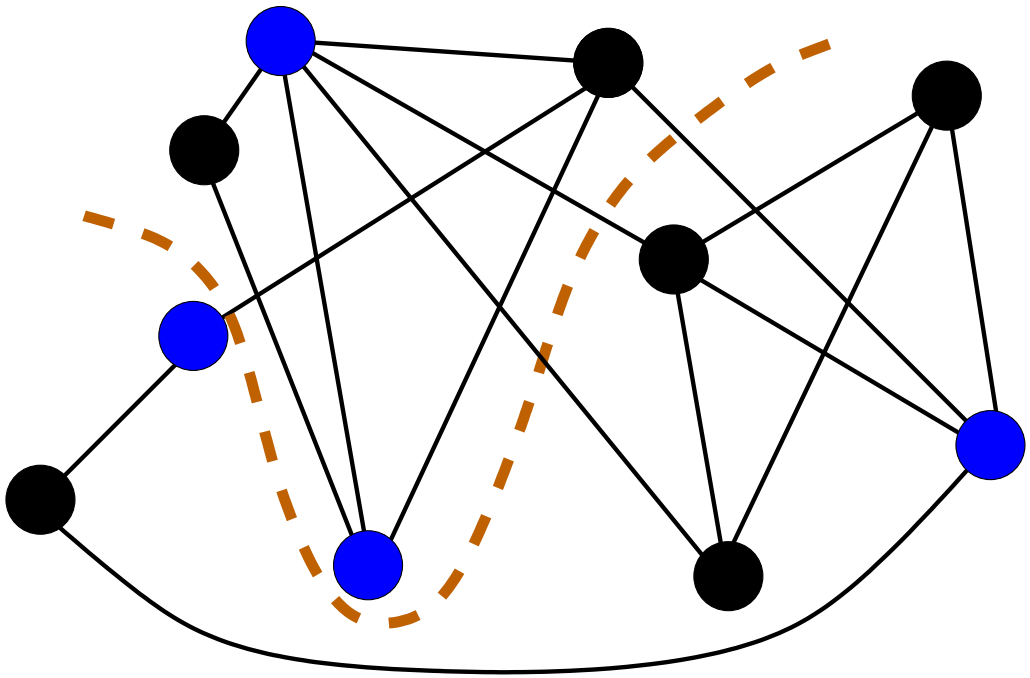

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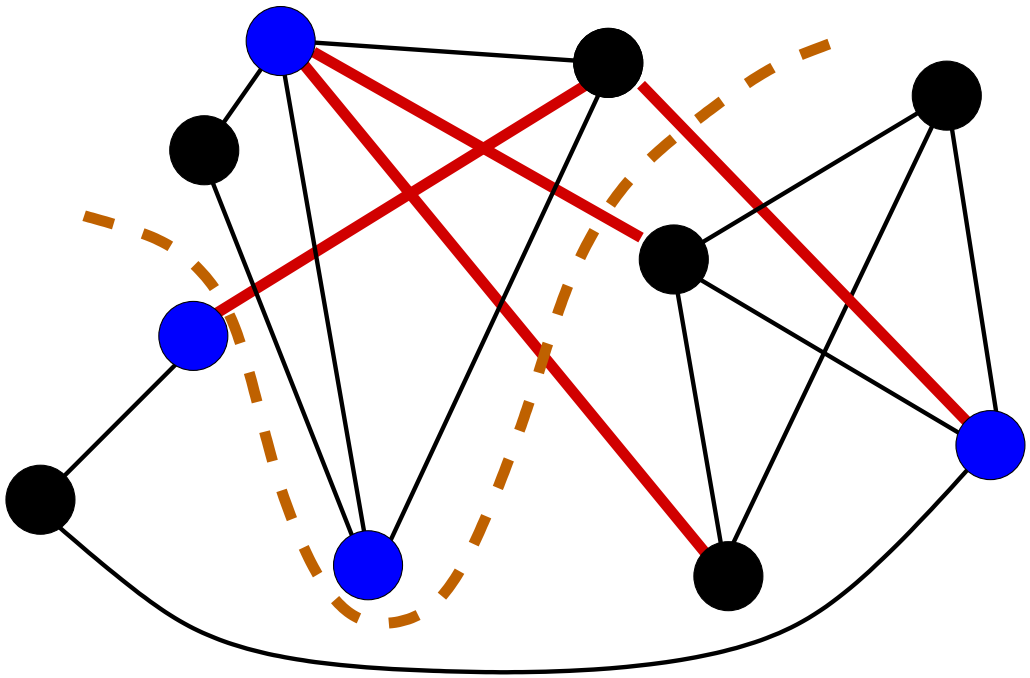

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3. $O(\frac{\log k}{\log \log k})$ 0-**extension** for k terminals
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Yes we can, and ...

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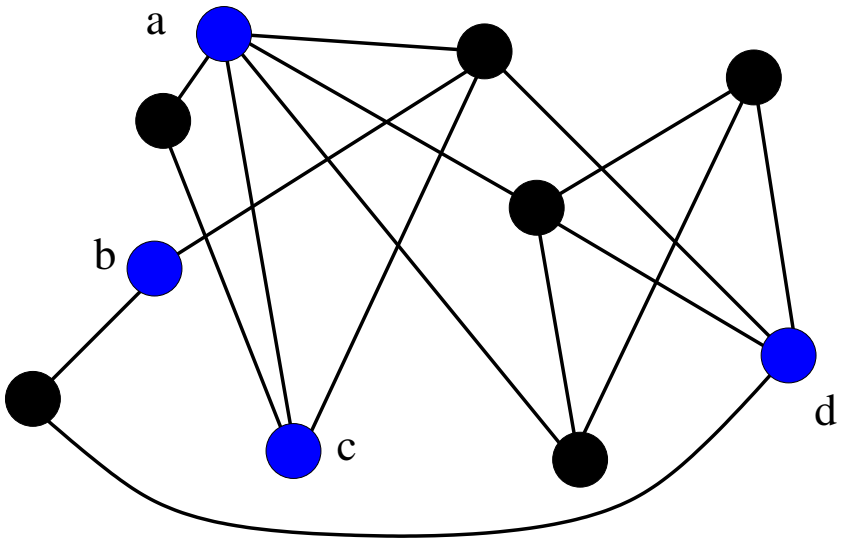
Our approach is to prove the existence of vertex sparsifiers that preserve cuts...

General Approach: Vertex Sparsifiers

Graph G=(V,E)



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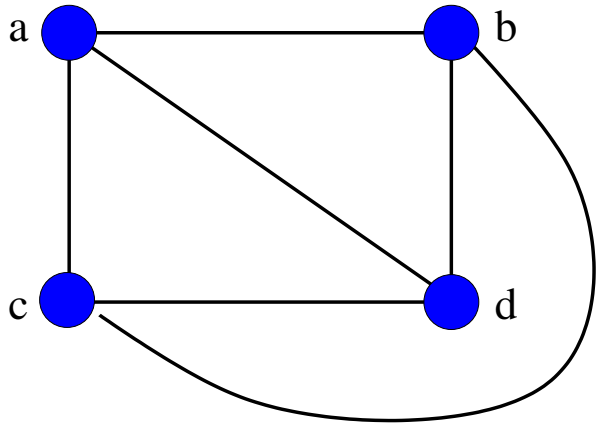

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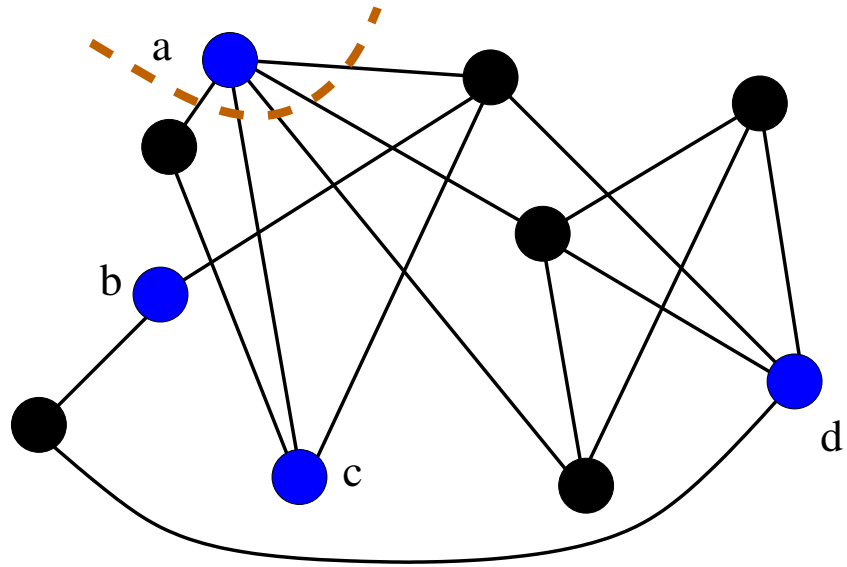

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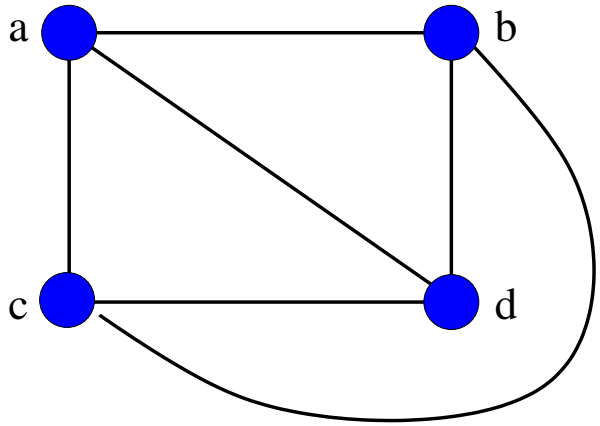

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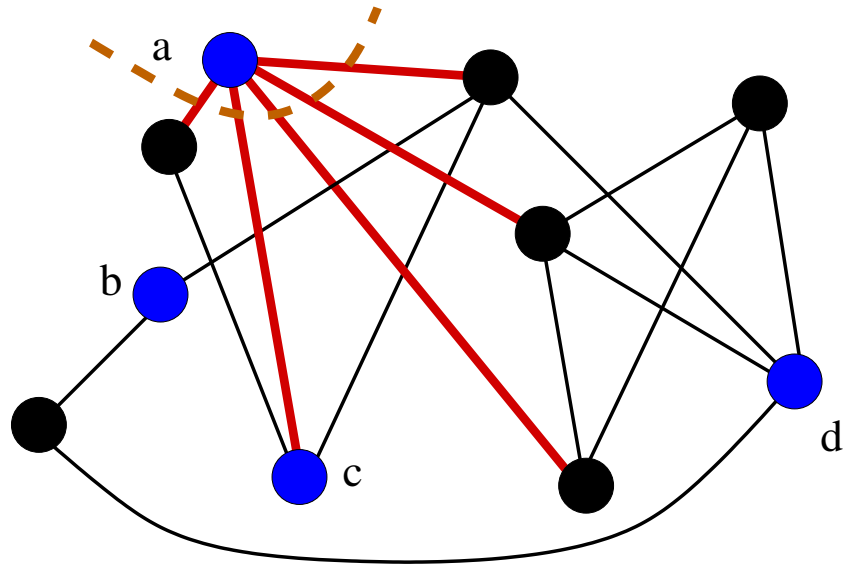

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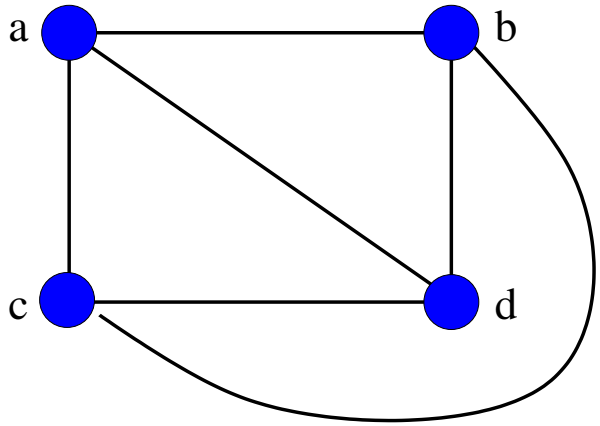

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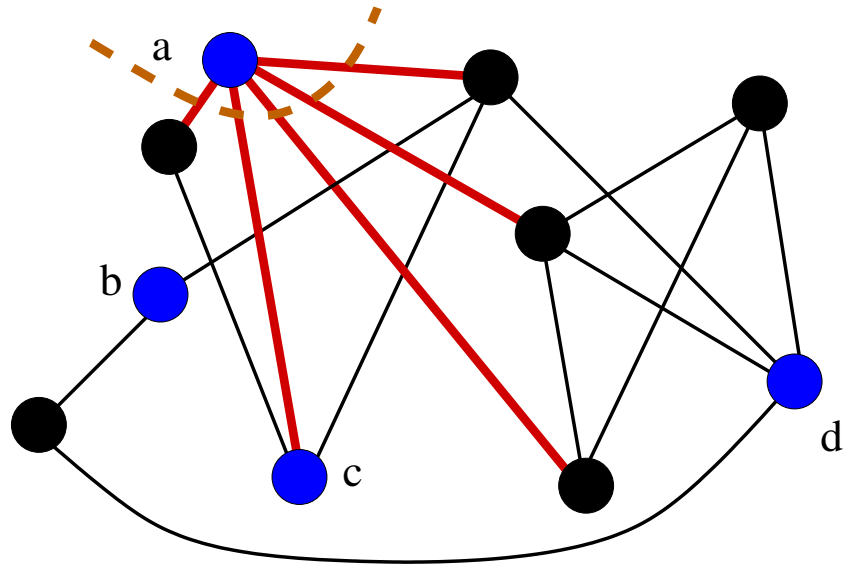
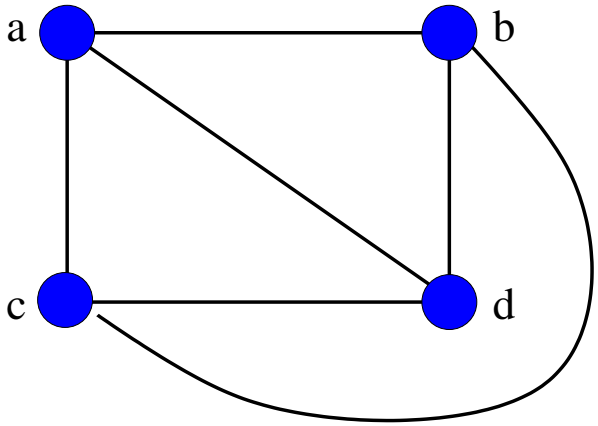


$h_K(a) = 5$

Sparsifier G'=(K,E')



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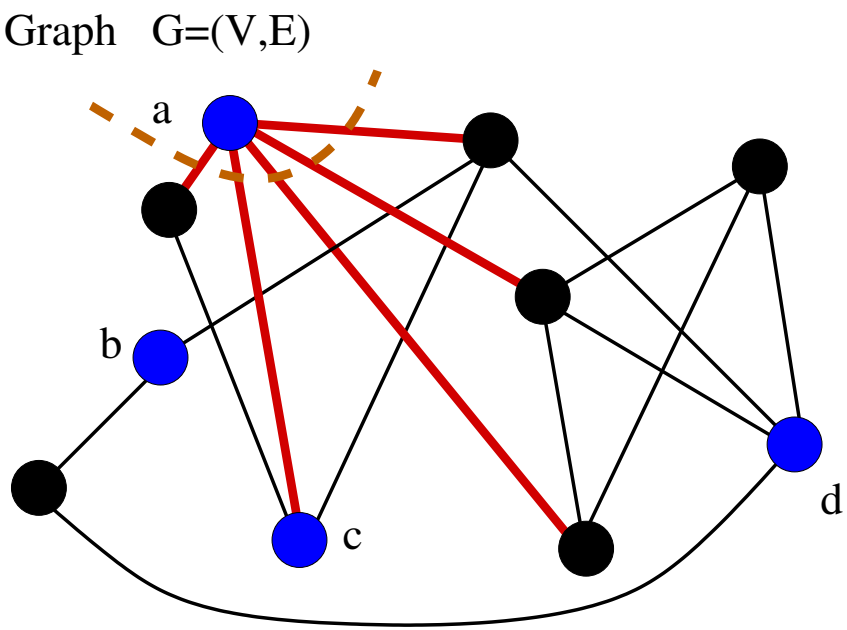

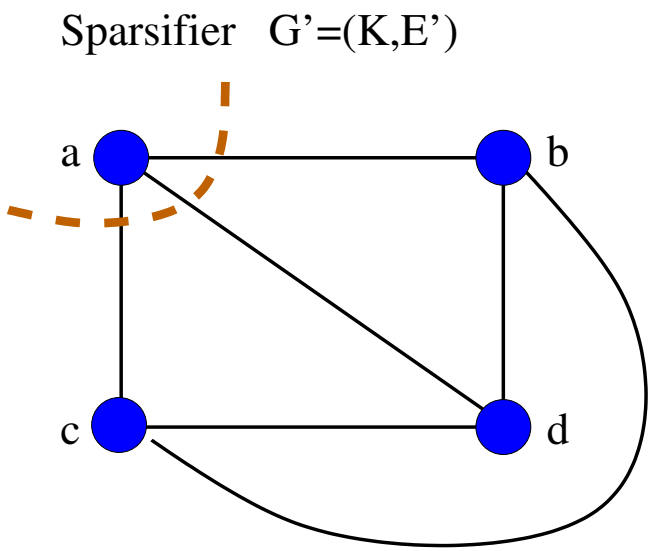

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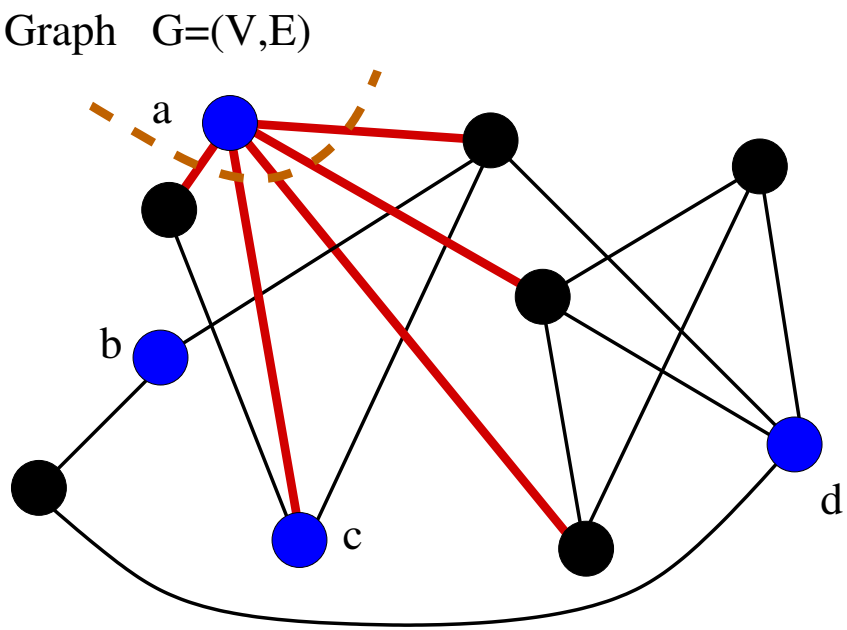

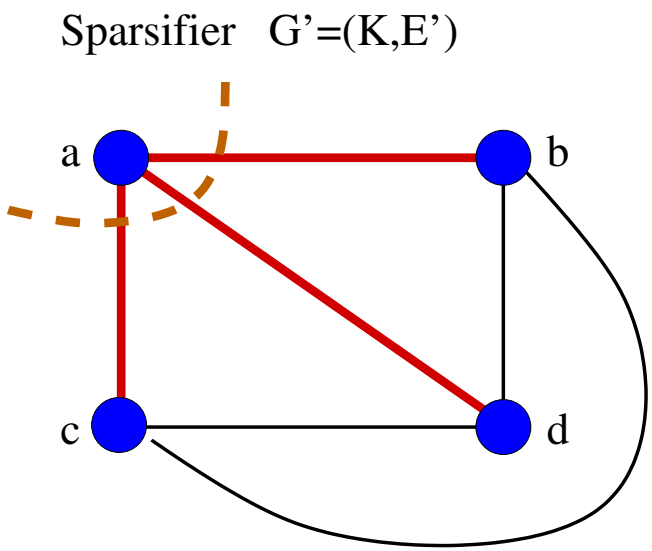

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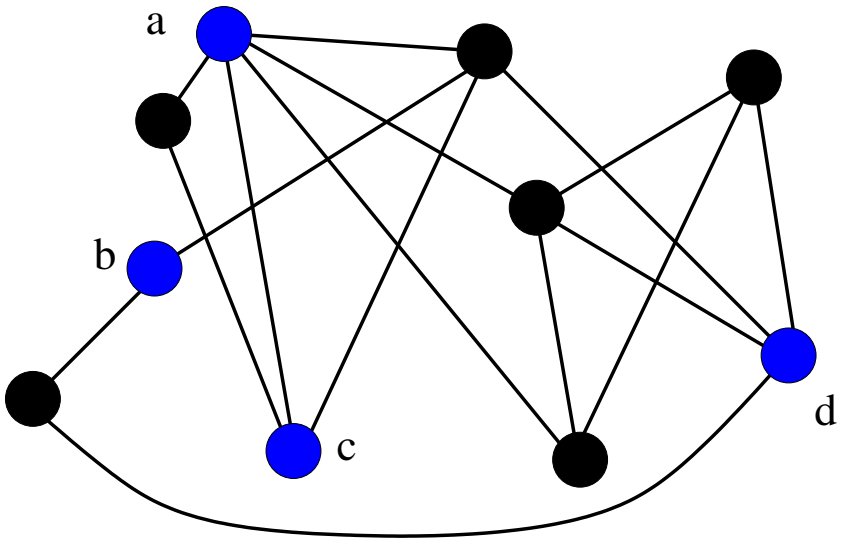

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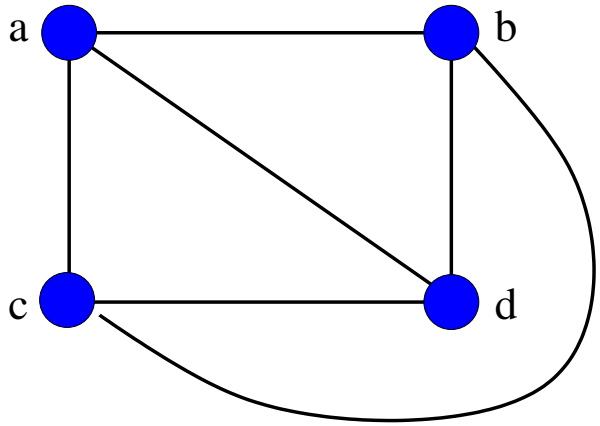

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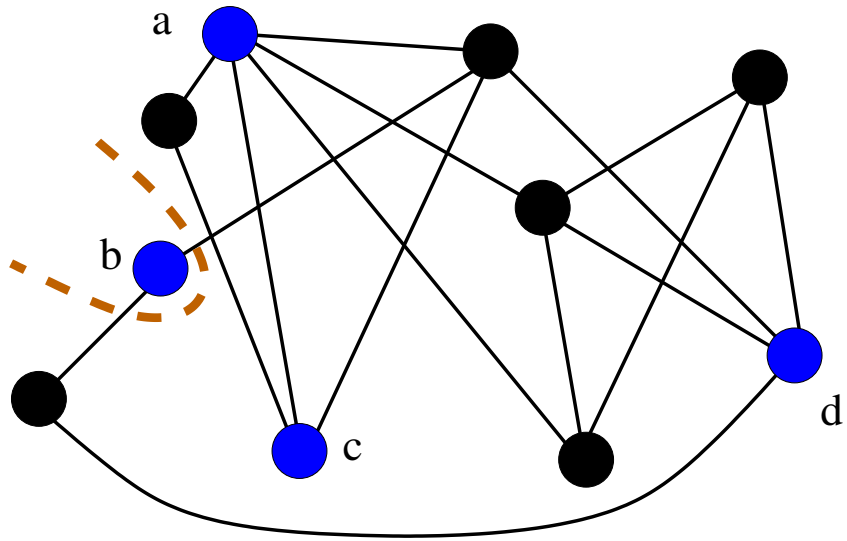

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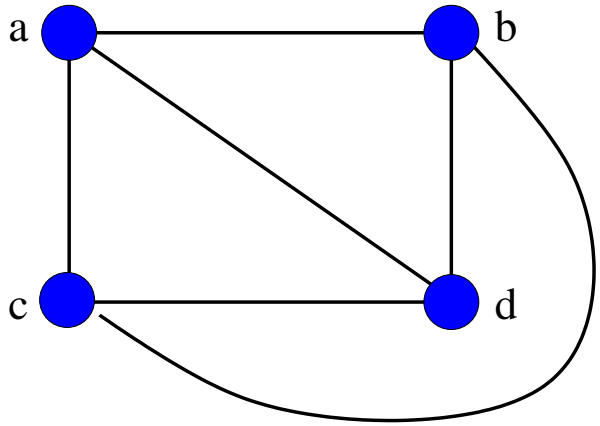

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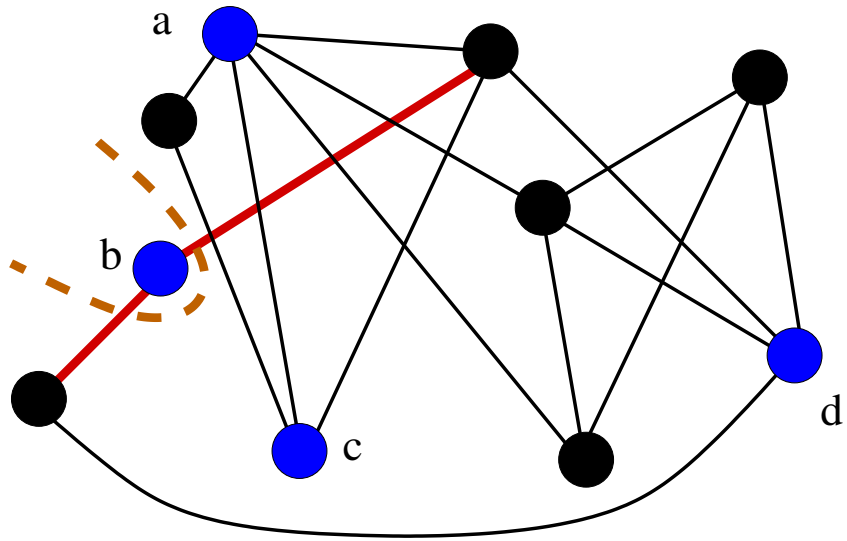

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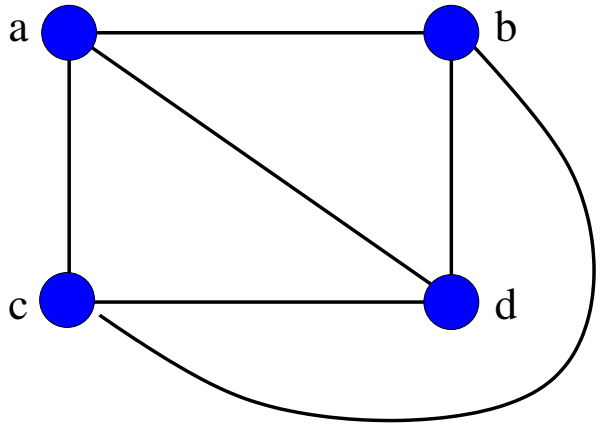

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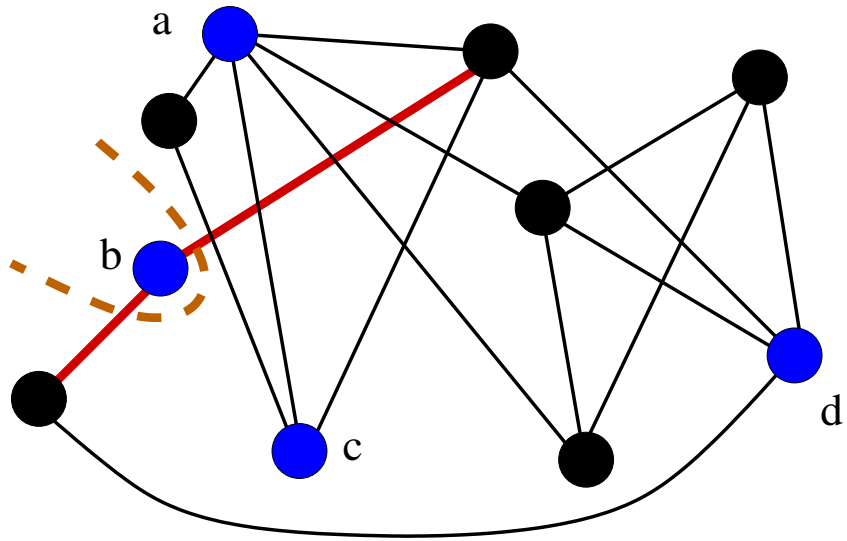

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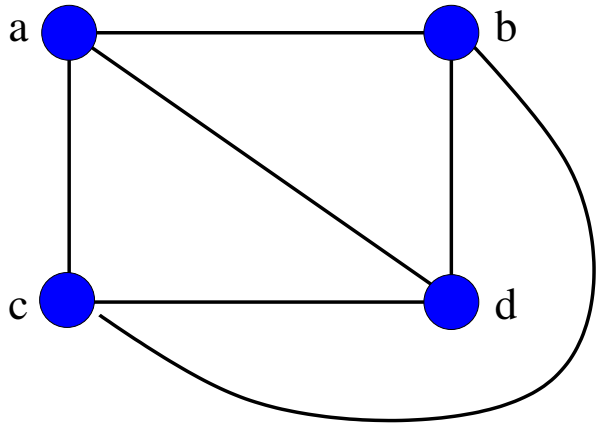

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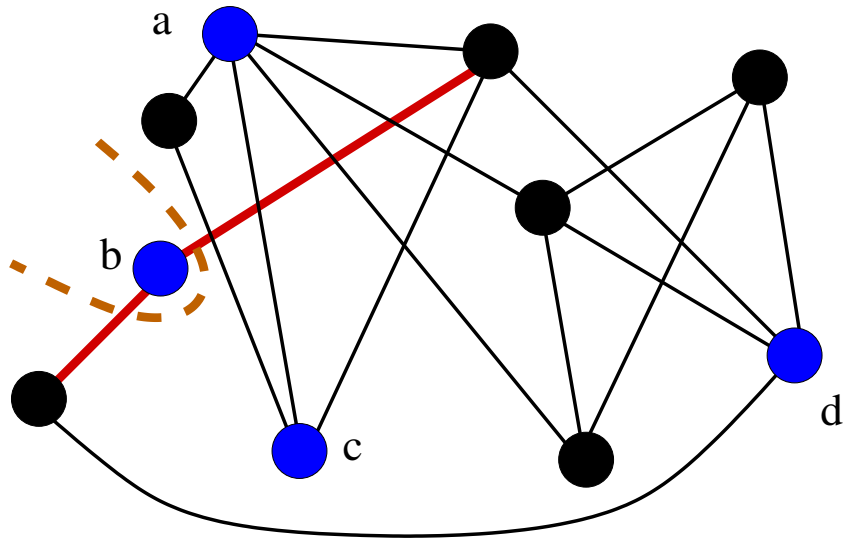

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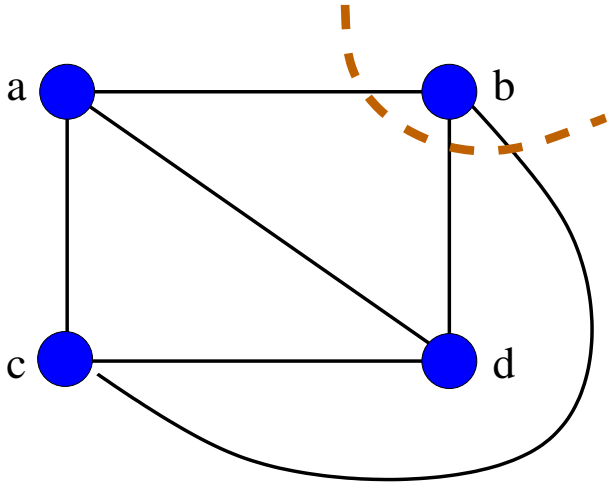

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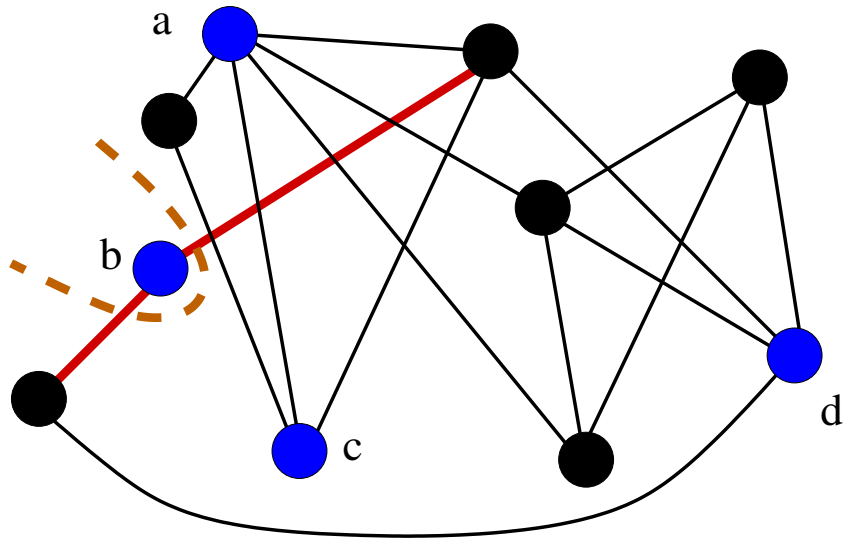

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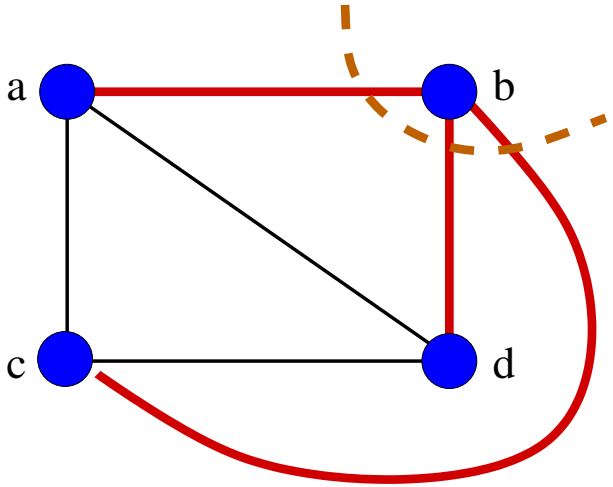

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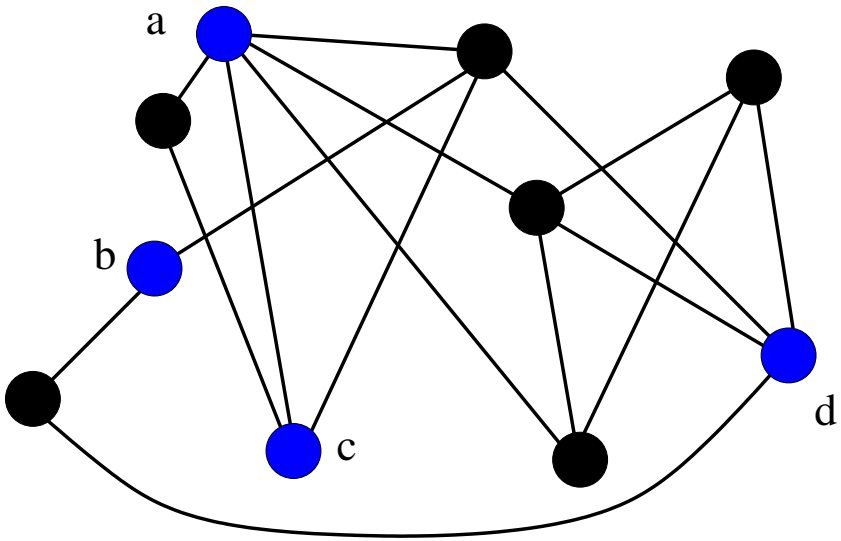

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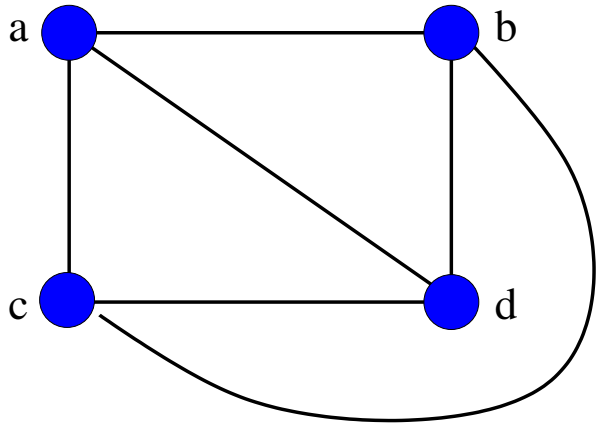

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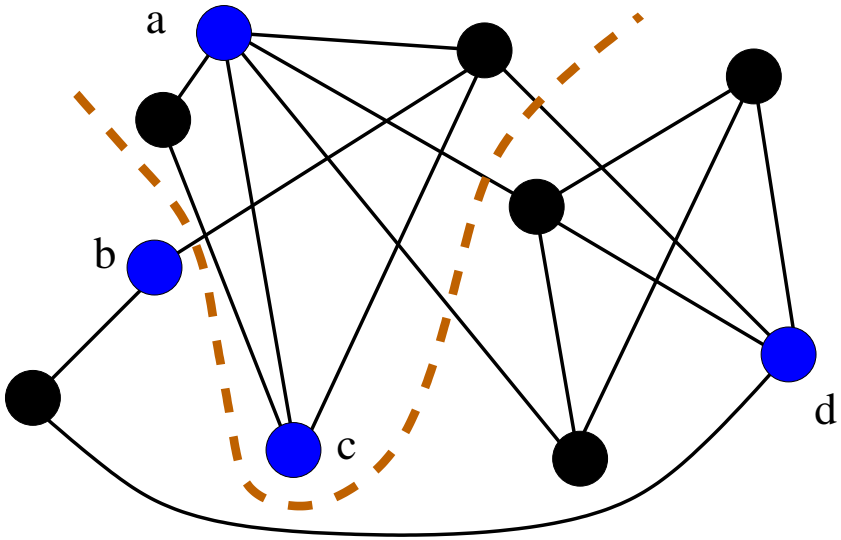

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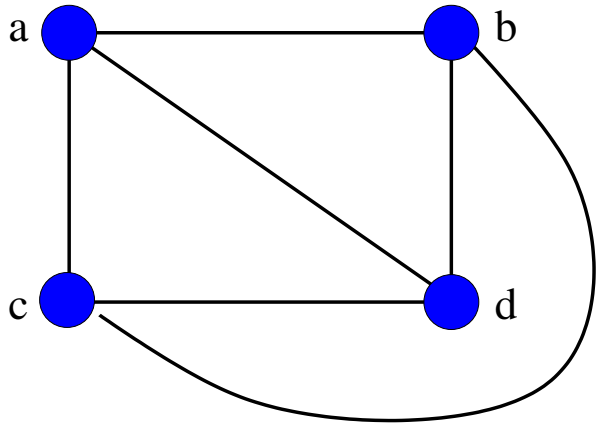

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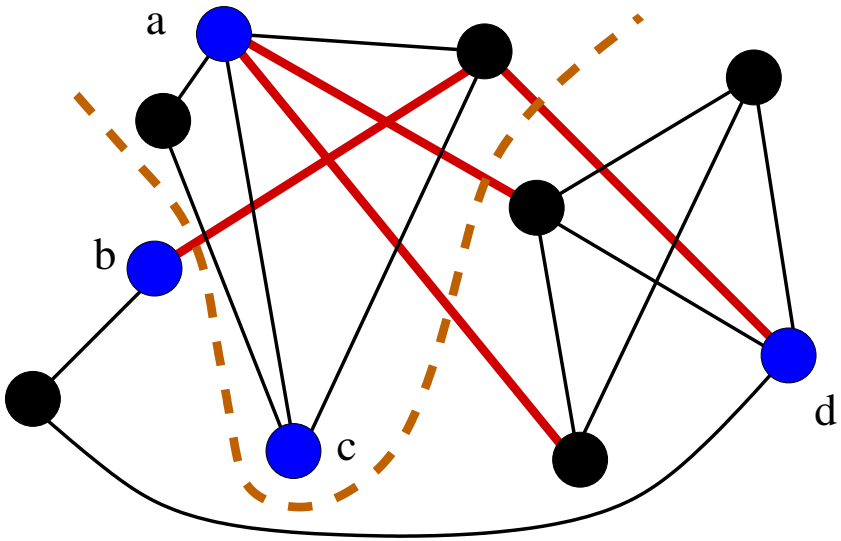
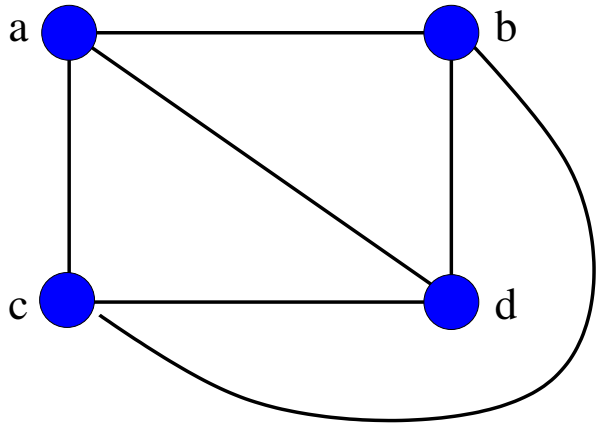


Sparsifier G'=(K,E')



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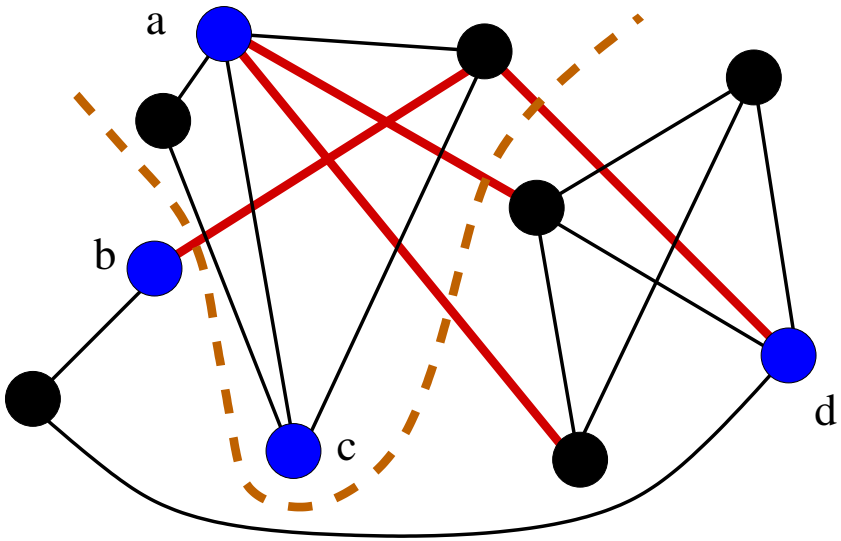

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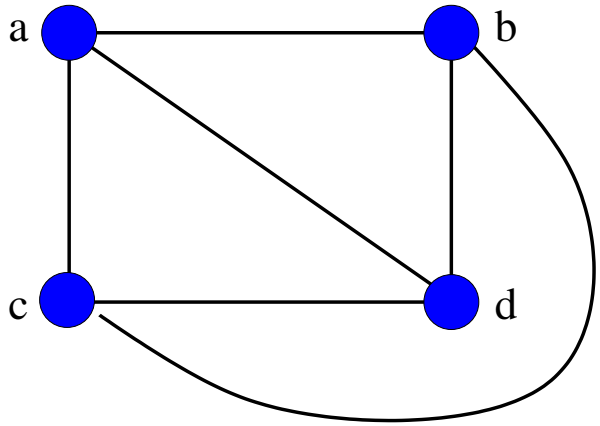

$h_K(ac) = 4$

General Approach: Vertex Sparsifiers

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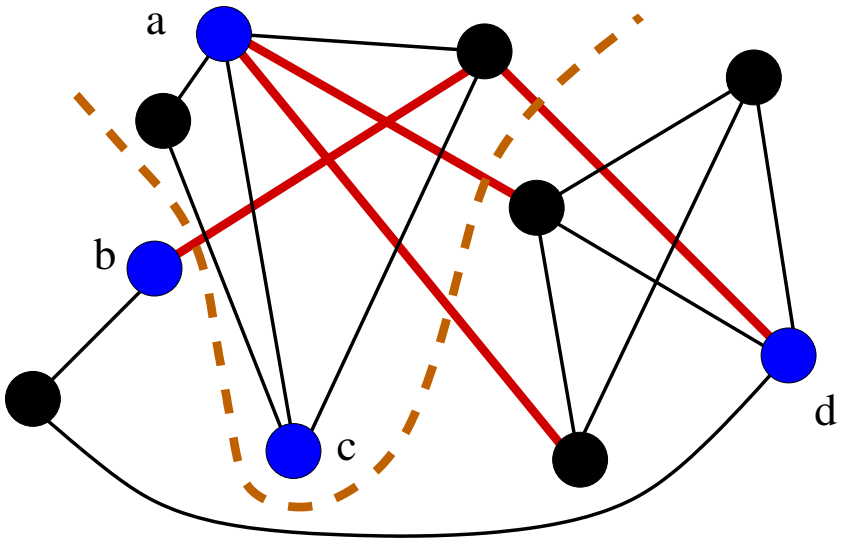

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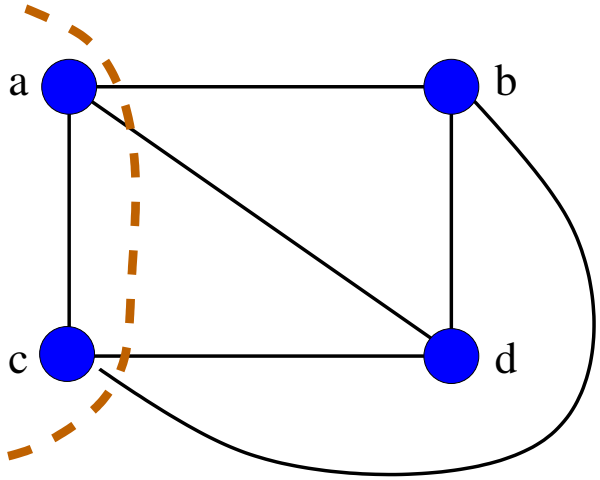

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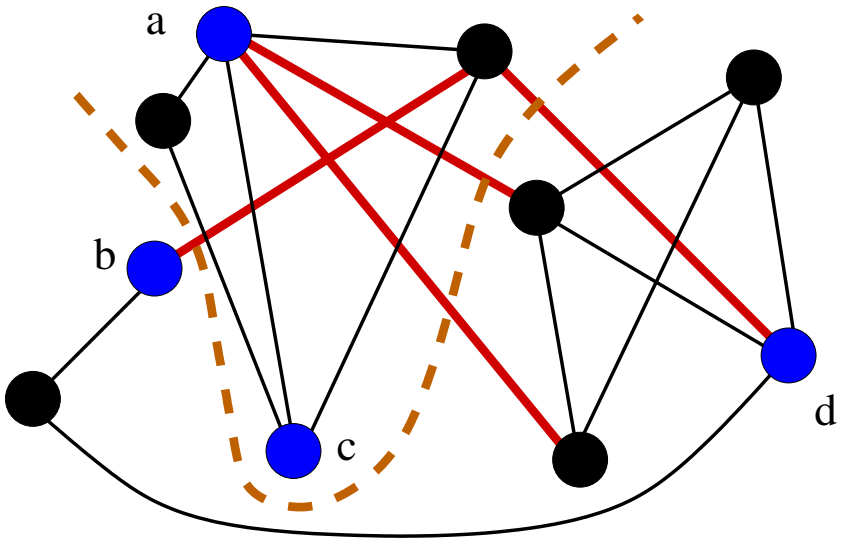

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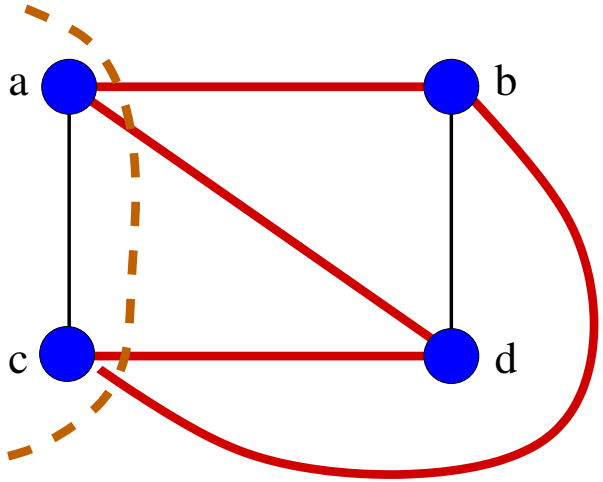

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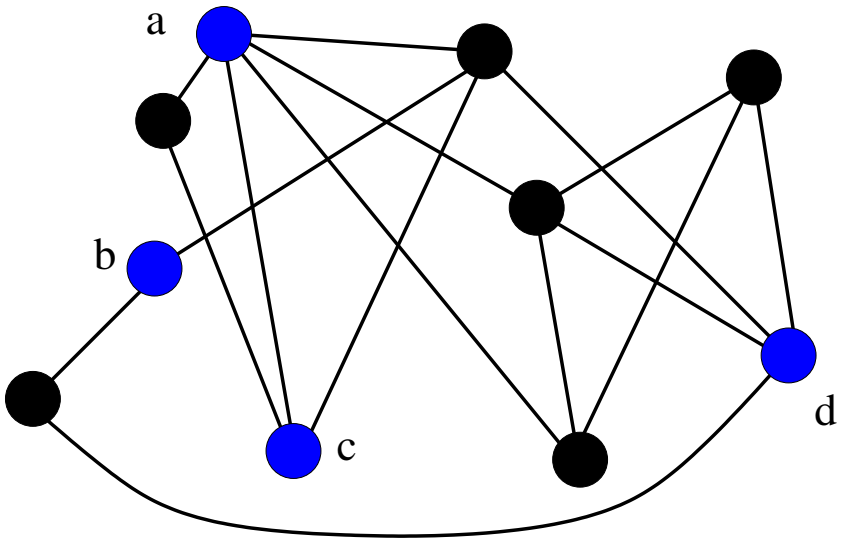

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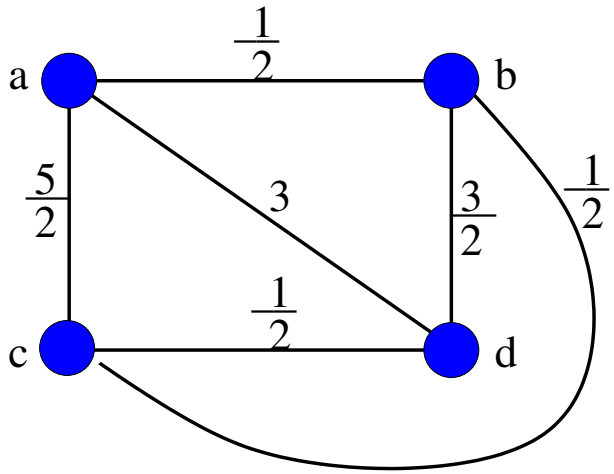

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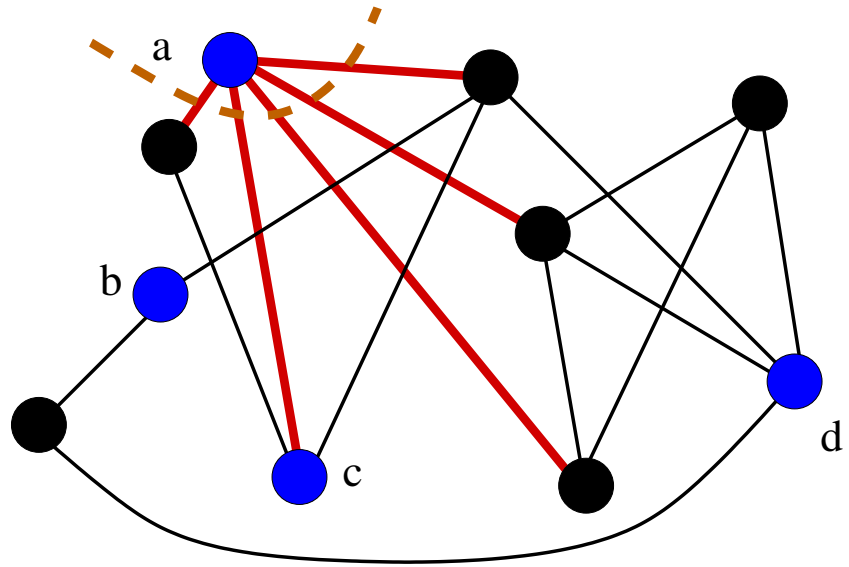
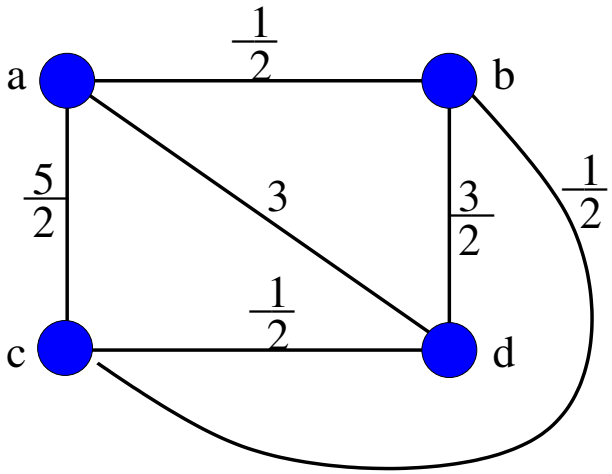


$h_K(a) = 5$

Sparsifier G'=(K,E')



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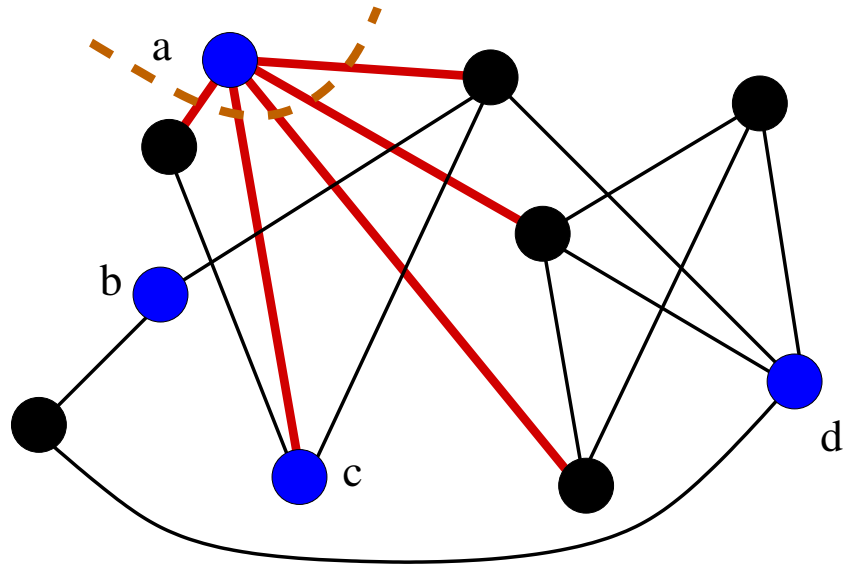

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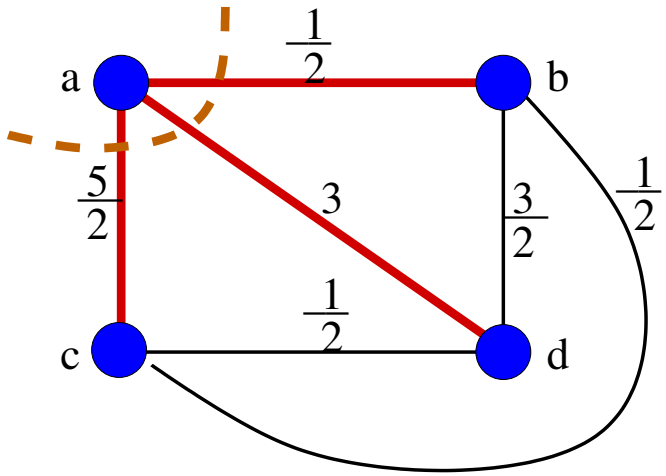
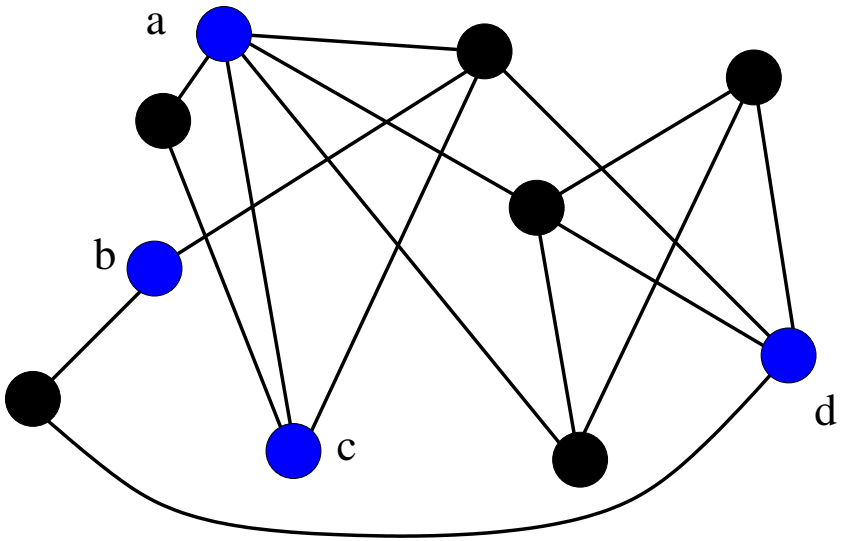


$h'(a) = 6$

General Approach: Vertex Sparsifiers

Graph G=(V,E)



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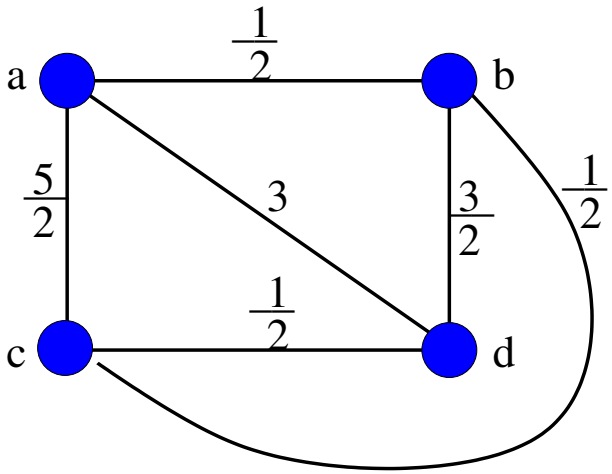

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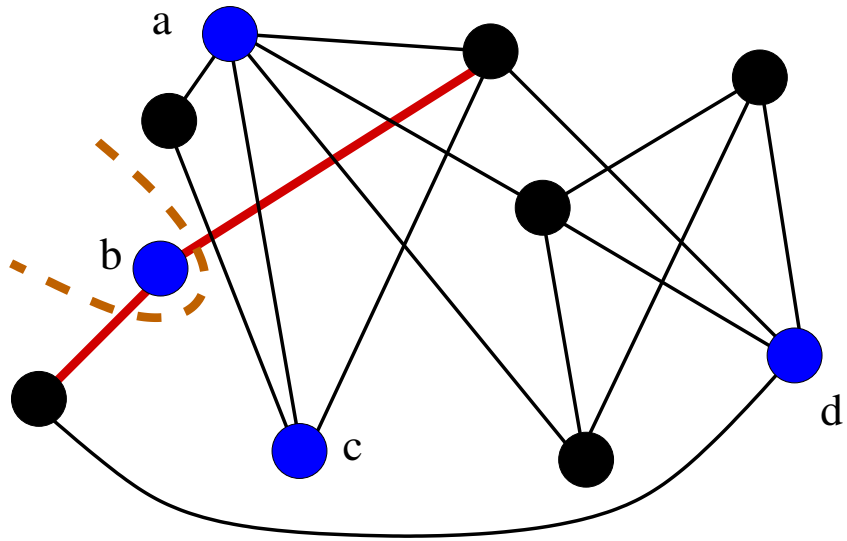
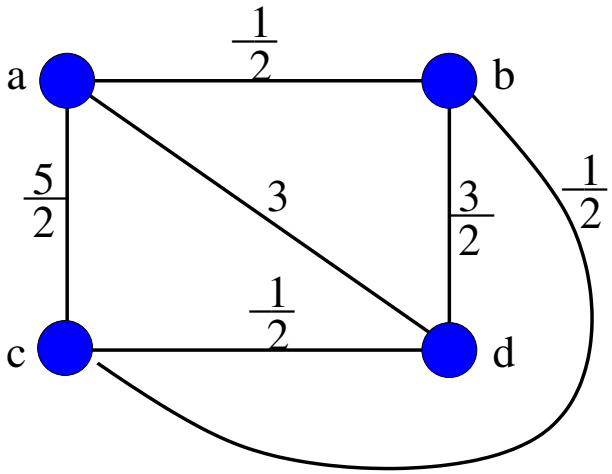


$h_K(a) = 5$

$h_K(b) = 2$

Sparsifier G'=(K,E')



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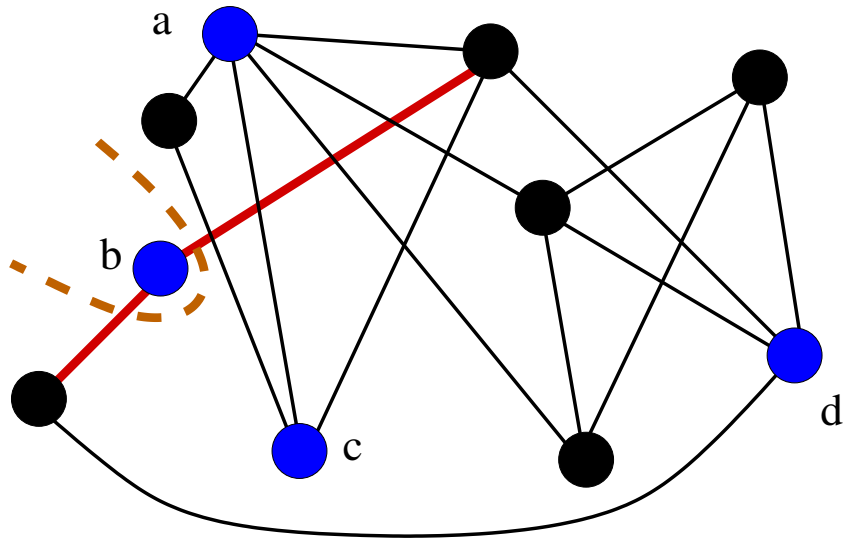
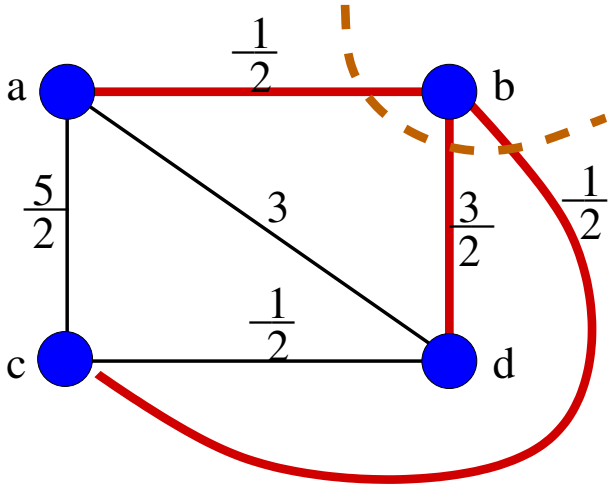


$h_K(a) = 5$

$h_K(b) = 2$

Sparsifier G'=(K,E')



$h'(a) = 6$

$h'(b) = 2.5$

General Approach: Vertex Sparsifiers

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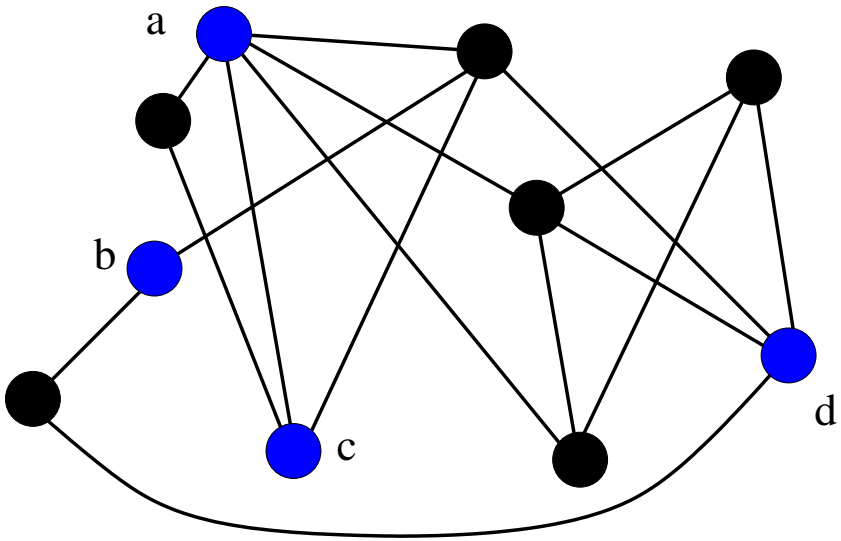
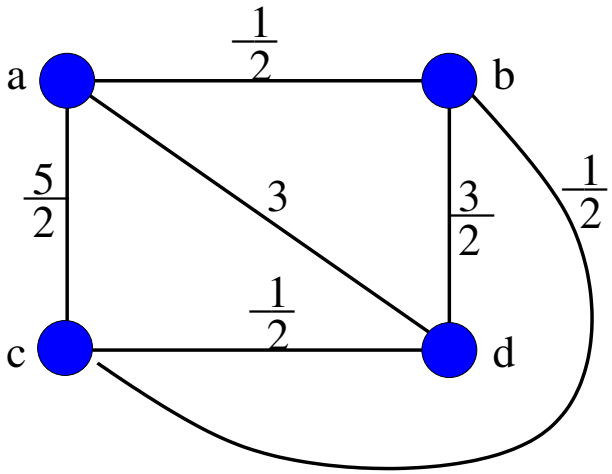


$h_K(a) = 5$
$h_K(b) = 2$

Sparsifier G'=(K,E')



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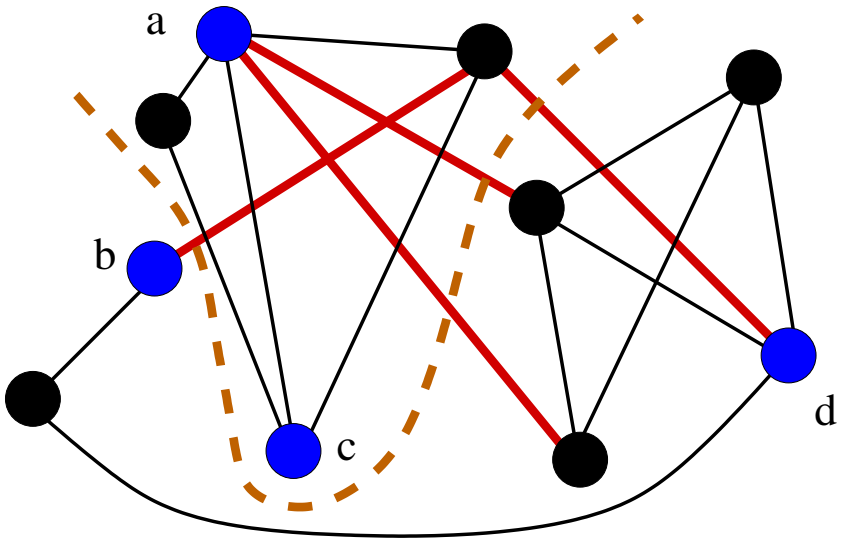
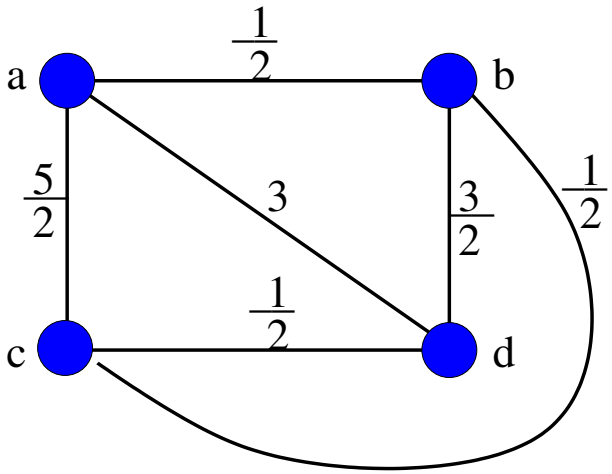


$h_K(a) = 5$

$h_K(b) = 2$

$h_K(ac) = 4$

Sparsifier G'=(K,E')



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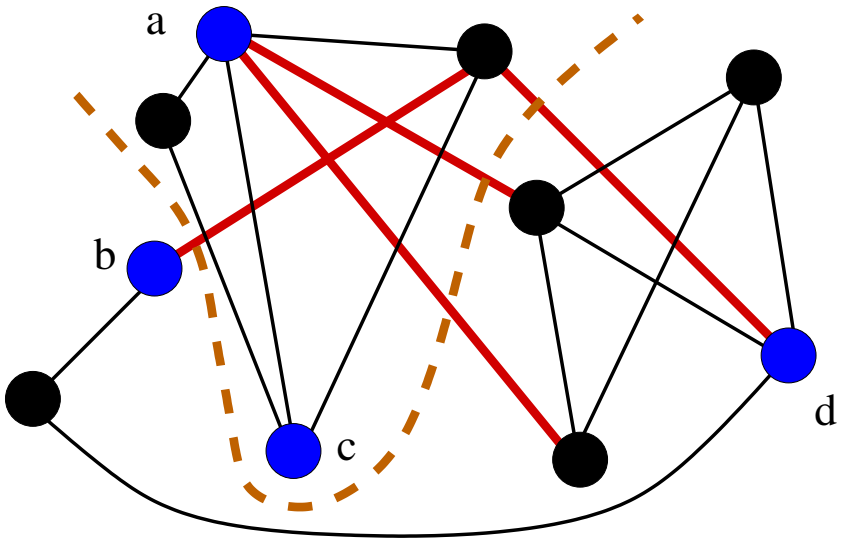

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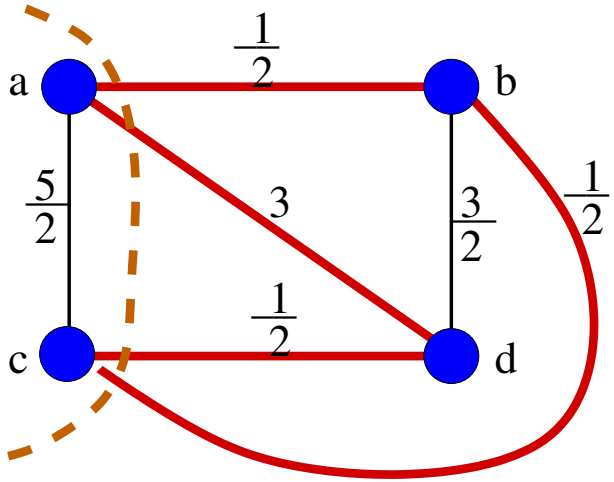

$h'(a) = 6$
$h'(b) = 2.5$

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General Approach: Vertex Sparsifiers

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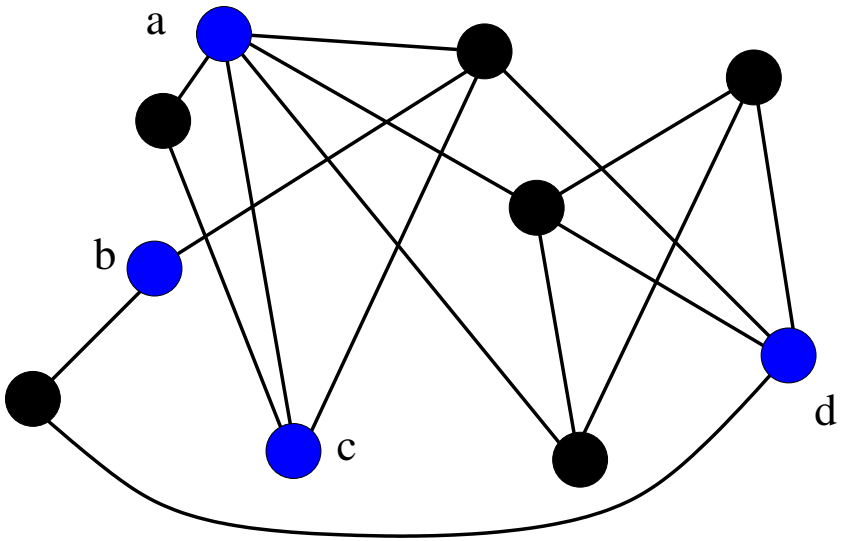

$h_K(a) = 5$ $h_K(d) = 4$ $h_K(ad) = 5$
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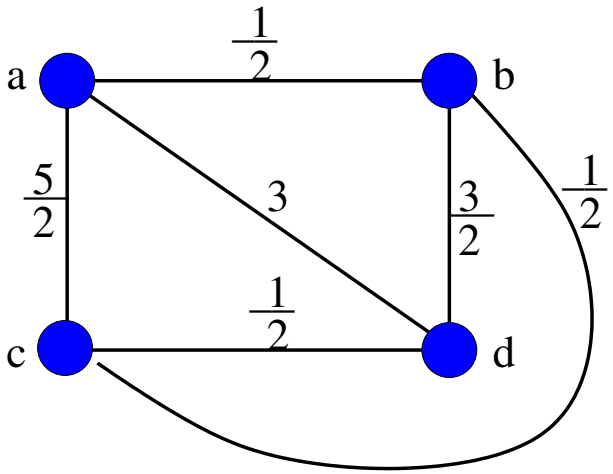
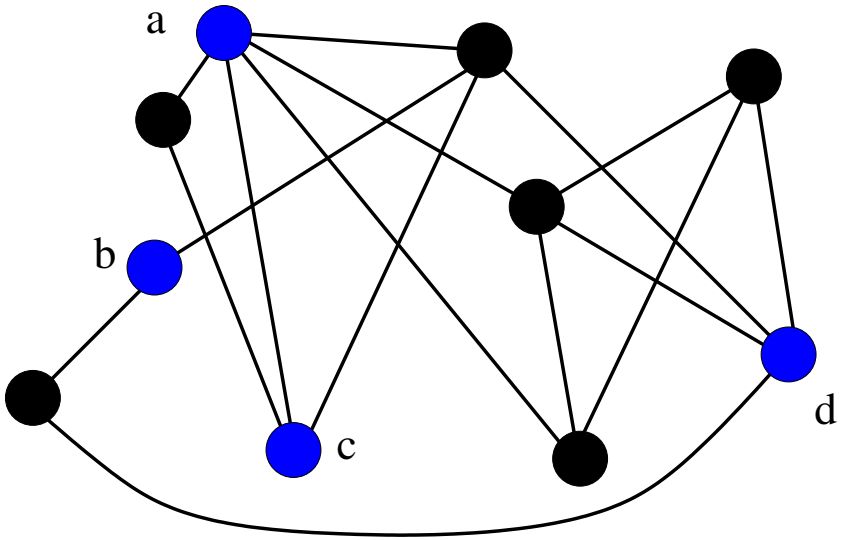
Sparsifier G'=(K,E')

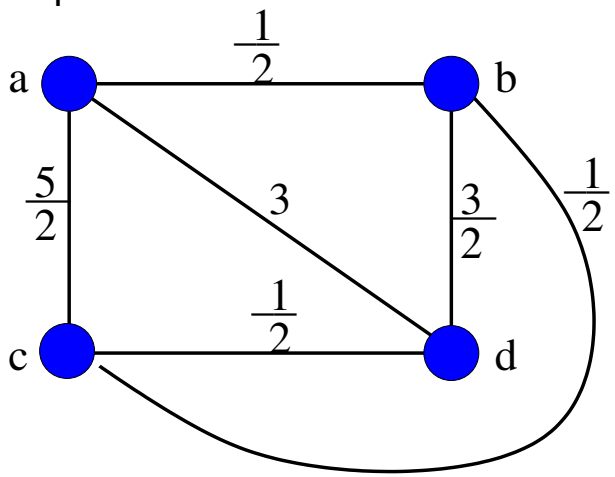


$h'(a) = 6$ $h'(d) = 5$ $h'(ad) = 5$
$h'(b) = 2.5$ $h'(ab) = 7.5$
$h'(c) = 3.5$ $h'(ac) = 4.5$

General Approach: Vertex Sparsifiers

The Quality of the Sparsifer is $\frac{5}{4}$





$h_K(a) = 5$ $h_K(d) = 4$ $h_K(ad) = 5$
$h_K(b) = 2$ $h_K(ab) = 7$
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$h'(a) = 6$ $h'(d) = 5$ $h'(ad) = 5$
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Vertex Sparsifiers, Informally

Definition

$G' = (K, E')$ is a **Vertex Sparsifier** for $G = (V, E)$ if all cuts in G' are at least as large as the corresponding min-cut in G.

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Definition

The **Quality** of a Vertex Sparsifier is the maximum ratio of a cut in G' to the corresponding min-cut in G.

Vertex Sparsifiers

Good Vertex Sparsifiers exist!

Vertex Sparsifiers

Good Vertex Sparsifiers exist!

And can be computed in polynomial time!

Vertex Sparsifiers

Good Vertex Sparsifiers exist!

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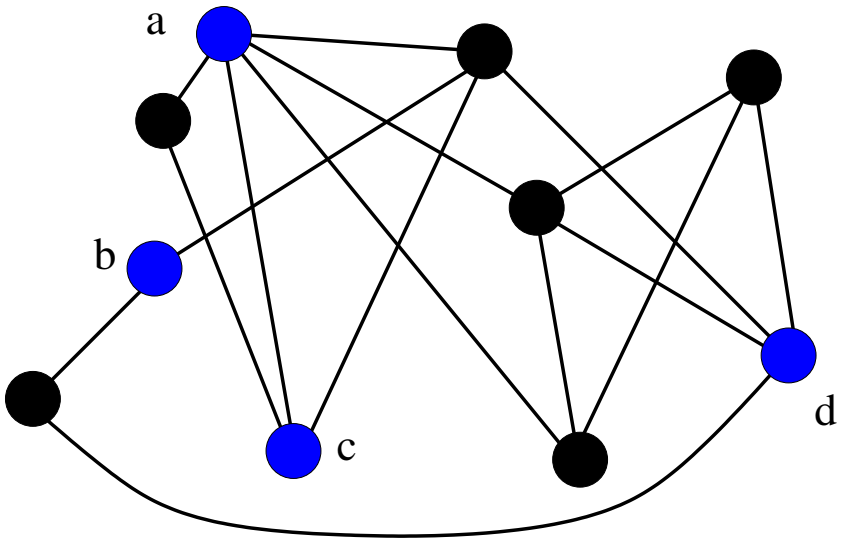
Theorem

For all weighted graphs $G = (V, E)$, and all $K \subset V$ there is a weighted graph $G' = (K, E')$ such that G' is a poly$(\log k)$-quality Vertex Sparsifier and such a graph can be computed in time polynomial in n and k.

An Application to Steiner Minimum Bisection

Graph G=(V,E)



$h_K(a) = 5$ $h_K(d) = 4$ $h_K(ad) = 5$
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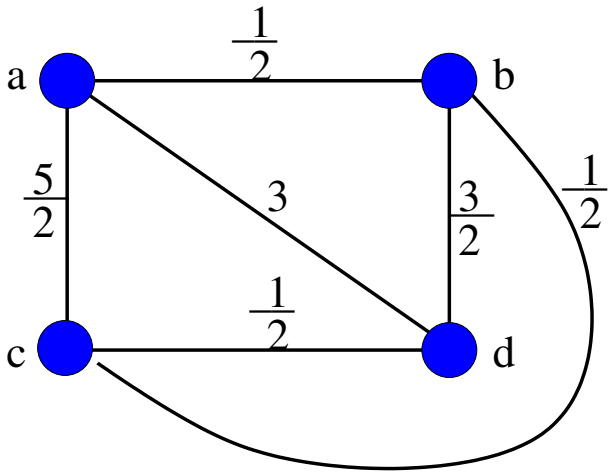
Sparsifier G'=(K,E')



$h'(a) = 6$ $h'(d) = 5$ $h'(ad) = 5$
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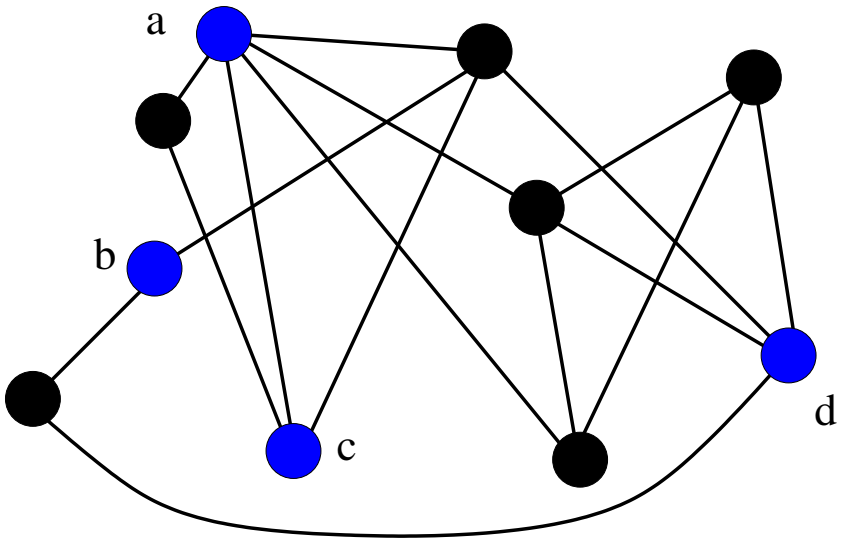

$h_K(a) = 5$ $h_K(d) = 4$ $h_K(ad) = 5$
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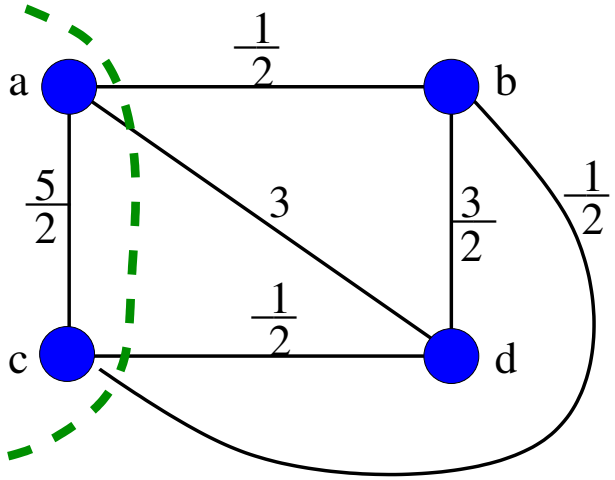
Sparsifier G'=(K,E')



h'(a) = 6 h'(d) = 5 h'(ad) = 5
h'(b) = 2.5 h'(ab) = 7.5
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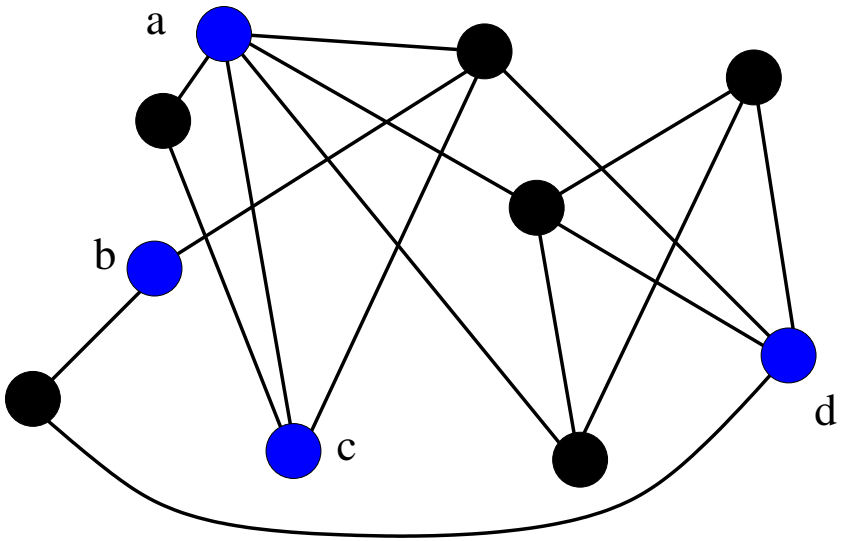

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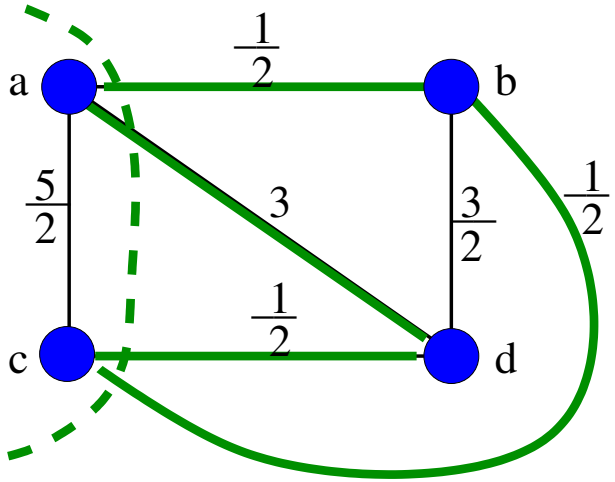

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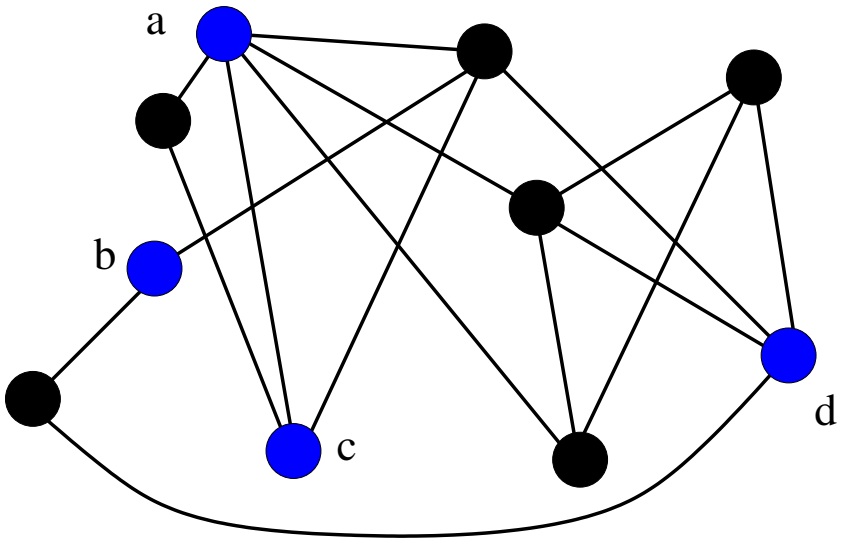

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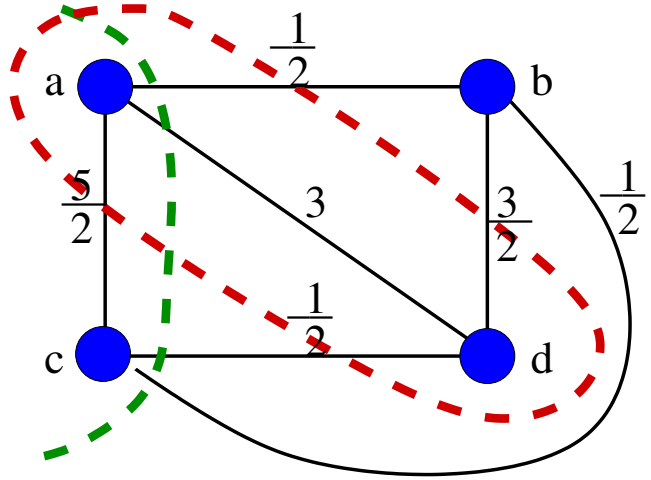

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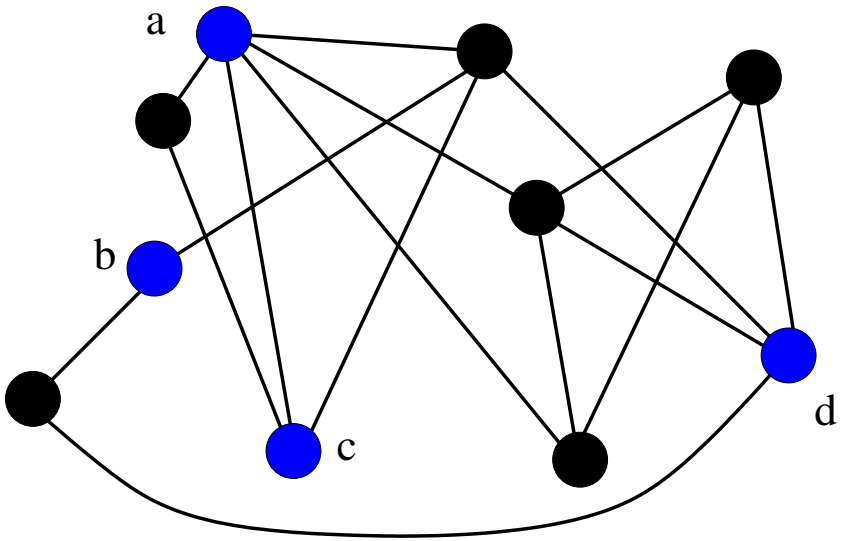

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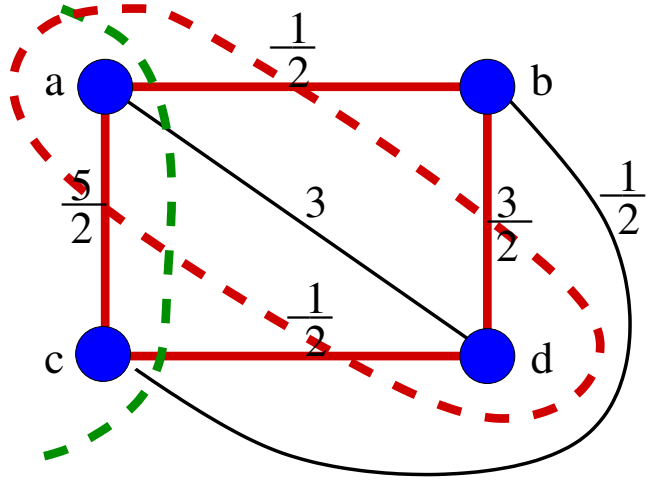

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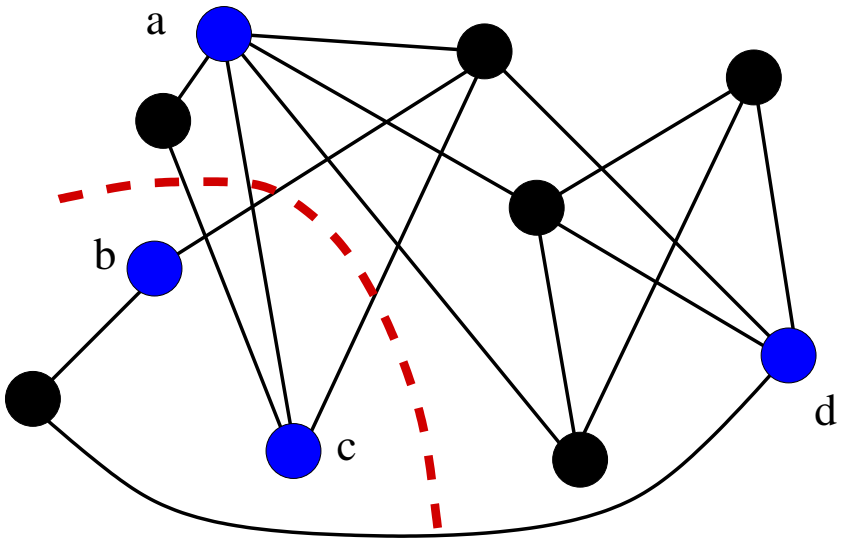

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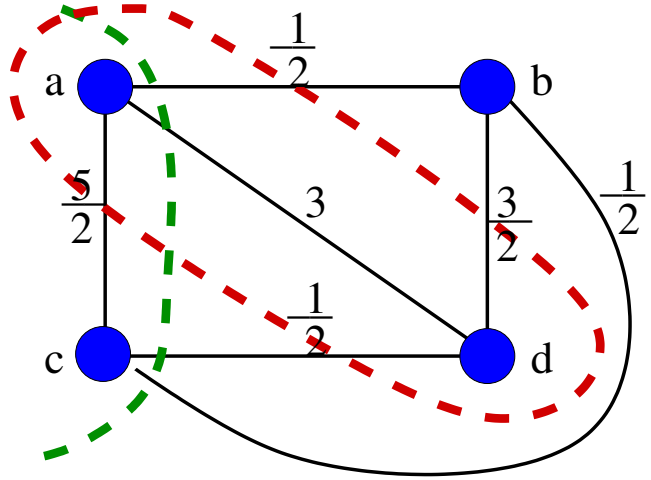

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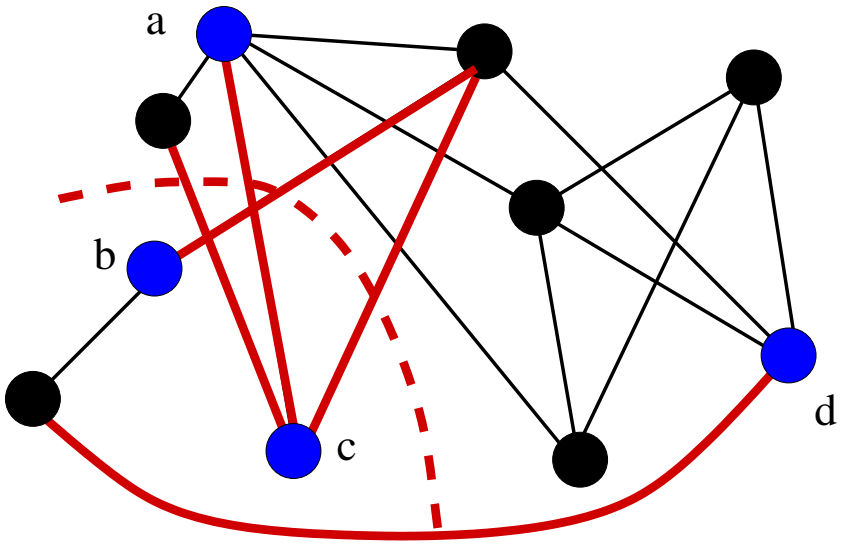

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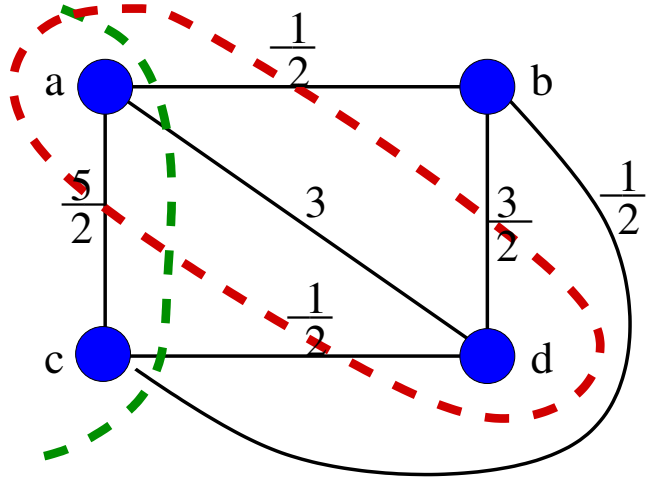
$h_K(c) = 3$ $h_K(ac) = 4$

Sparsifier G'=(K,E')



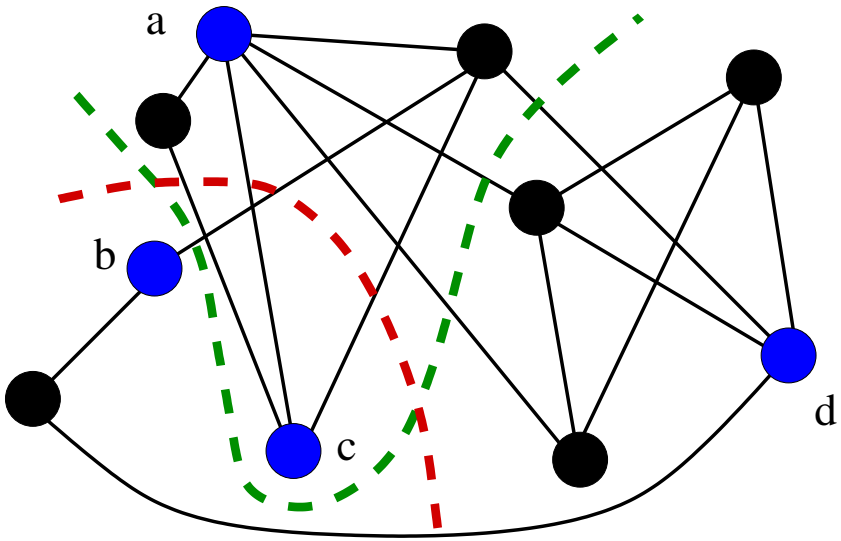
$h'(a) = 6$ $h'(d) = 5$ $h'(ad) = 5$

$h'(b) = 2.5$ $h'(ab) = 7.5$

$h'(c) = 3.5$ $h'(ac) = 4.5$

An Application to Steiner Minimum Bisection

Graph G=(V,E)



$h_K(a) = 5$ $h_K(d) = 4$ $h_K(ad) = 5$

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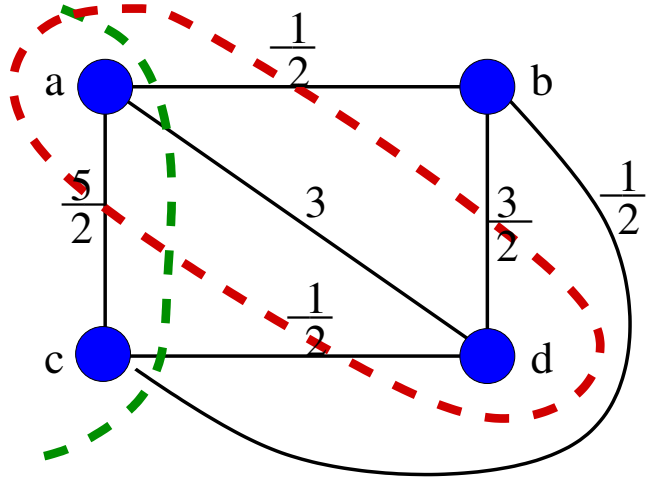

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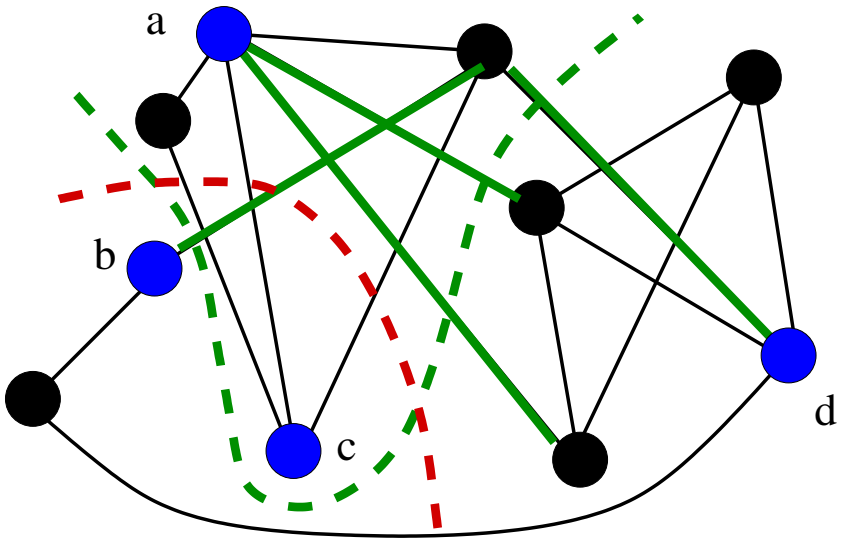

$h_K(a) = 5$ $h_K(d) = 4$ $h_K(ad) = 5$

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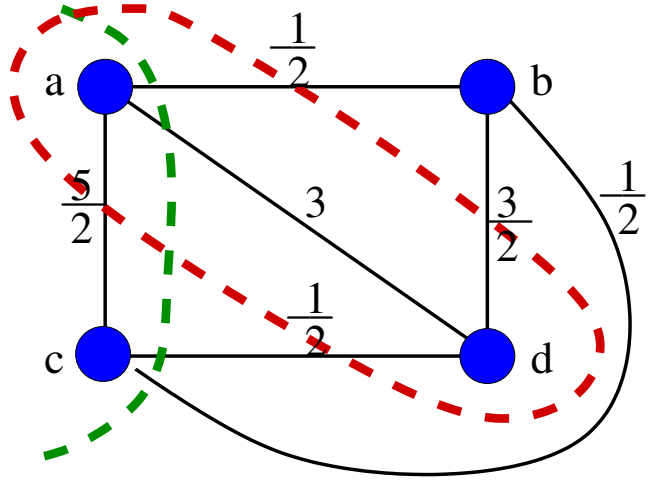

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Reducibility in Multicommodity-Type Problems

This is a general strategy!

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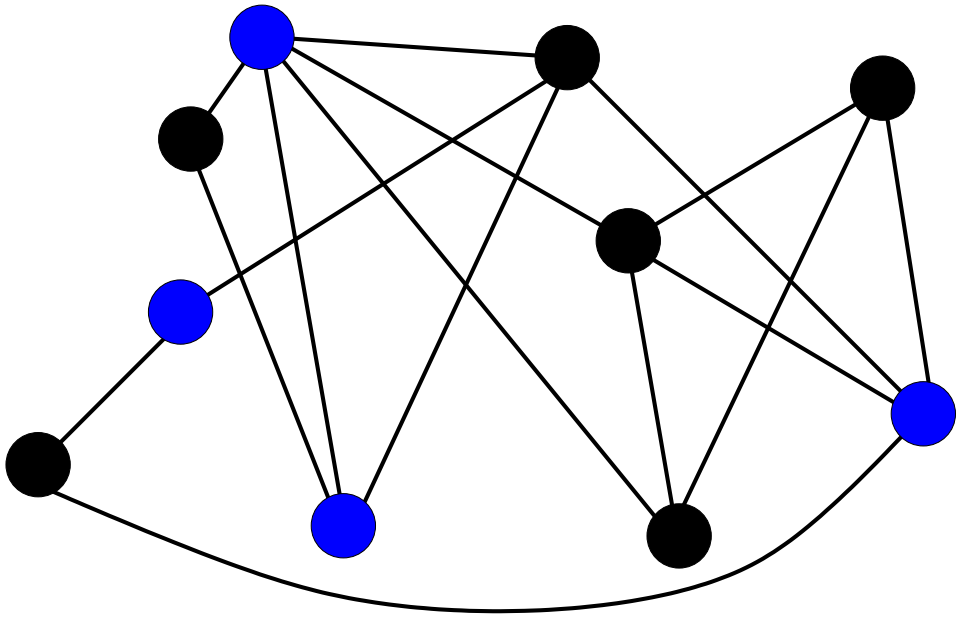
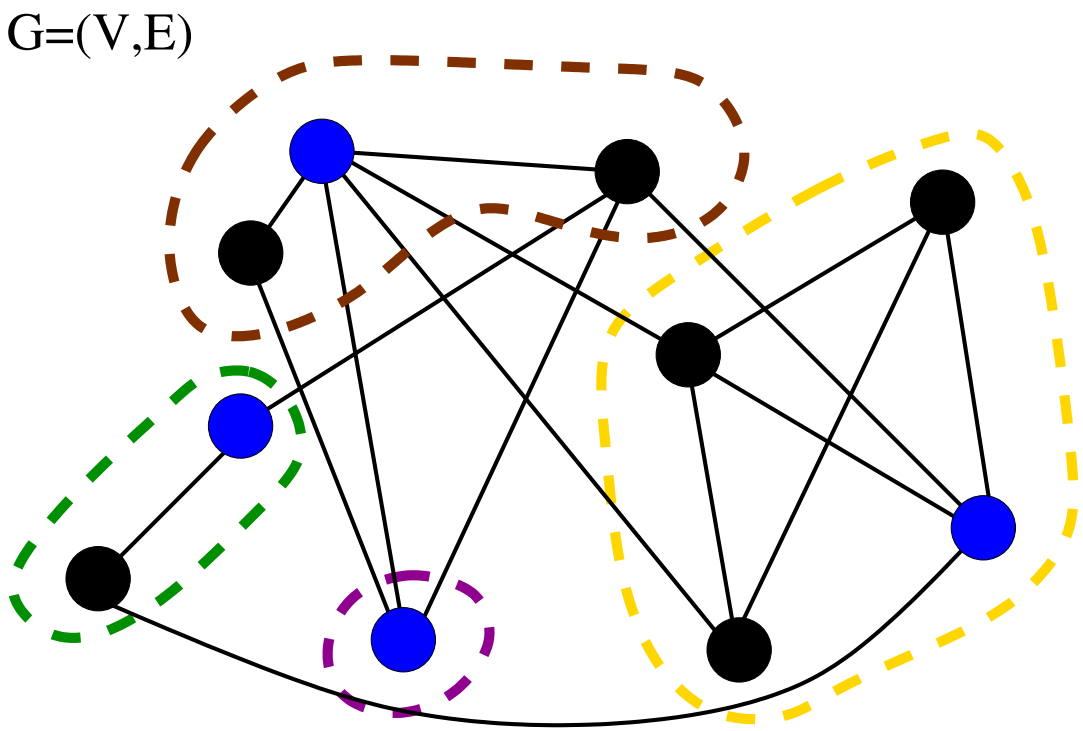
Informally: If the problem is Lipschitz w.r.t. h_K, then this is a general reduction!

Definition

G=(V,E)



Definition



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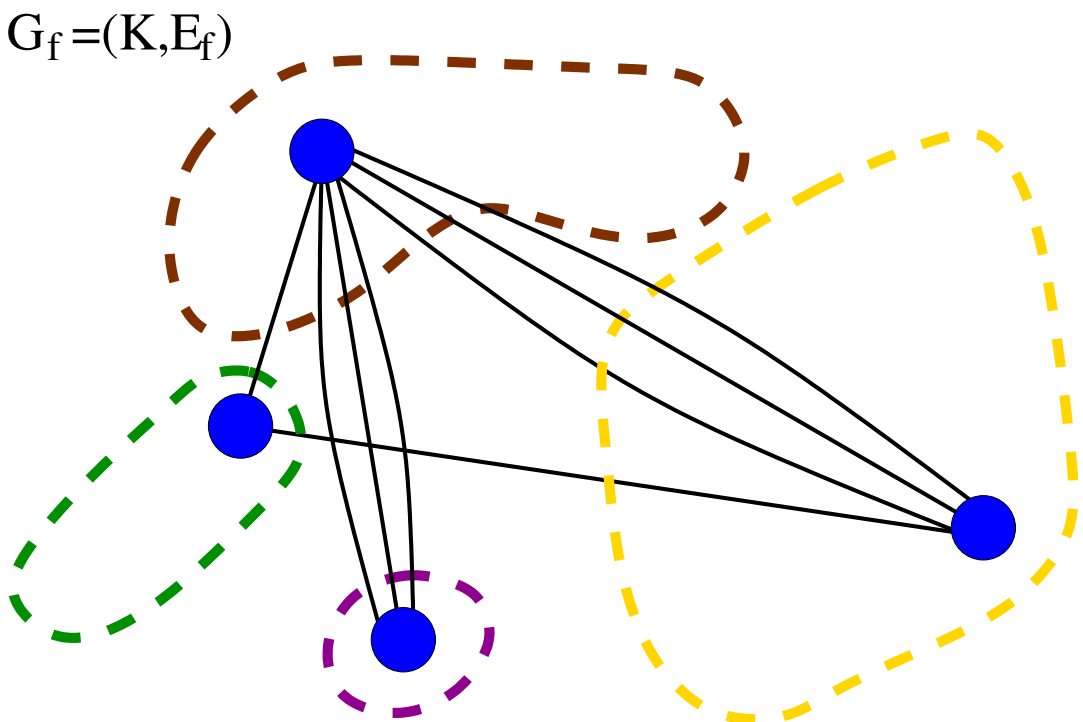
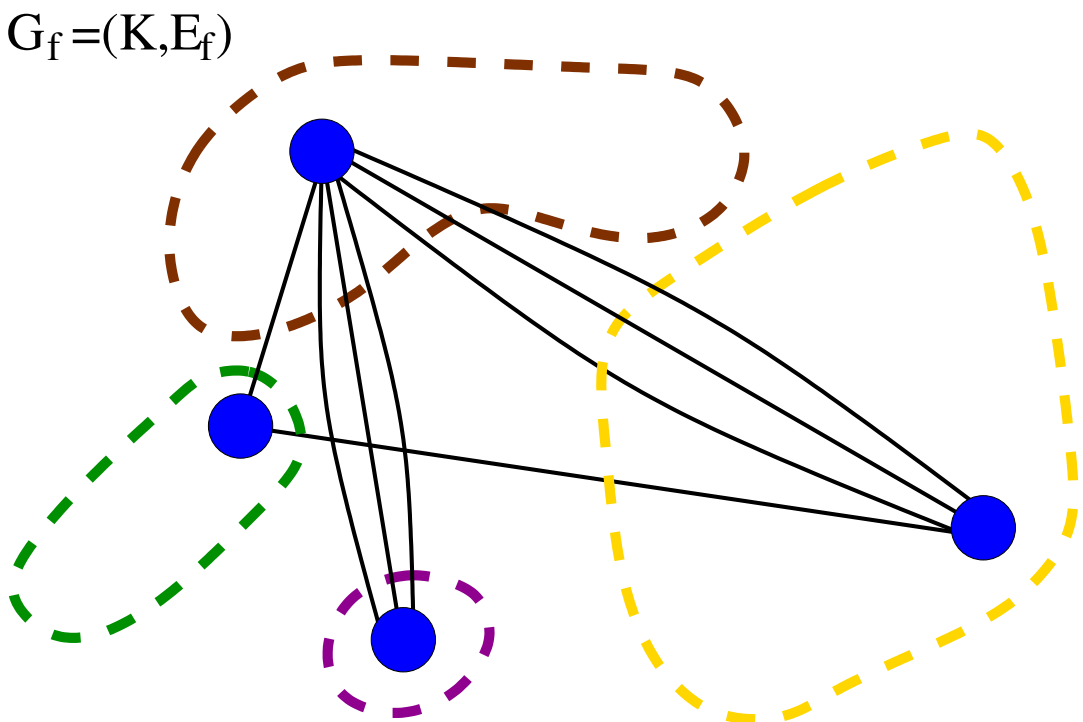


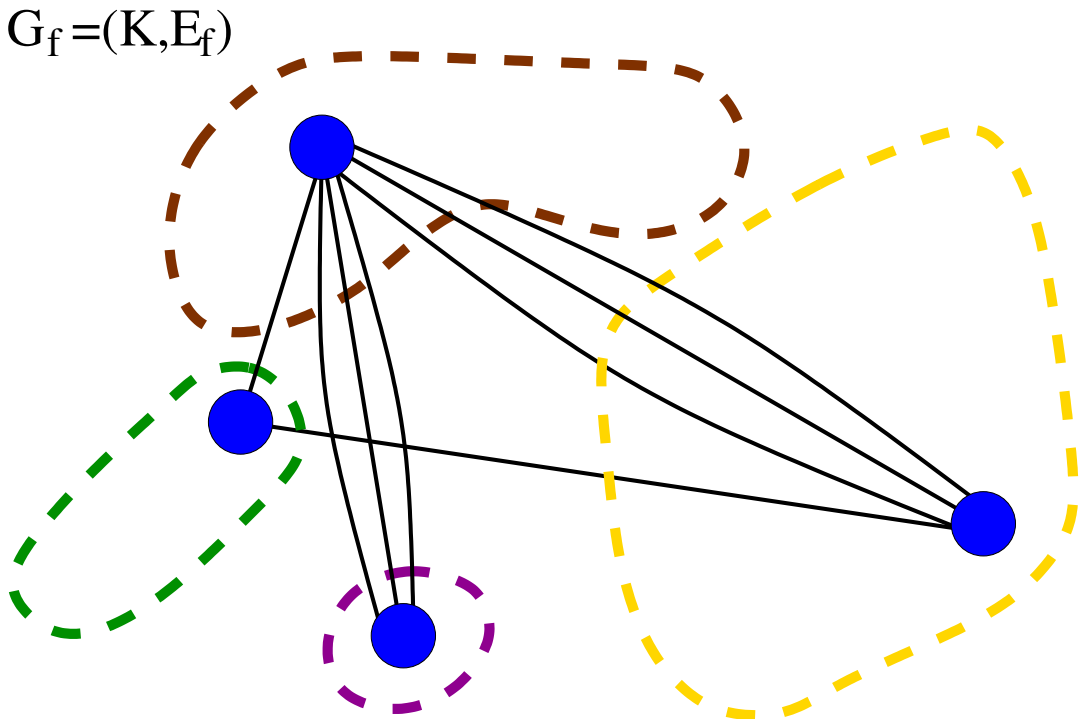
Definition

Let $f : V \to K$, denote a 0-extension if for all $a \in K, f(a) = a$.



Lemma

G_f is a vertex sparsifier



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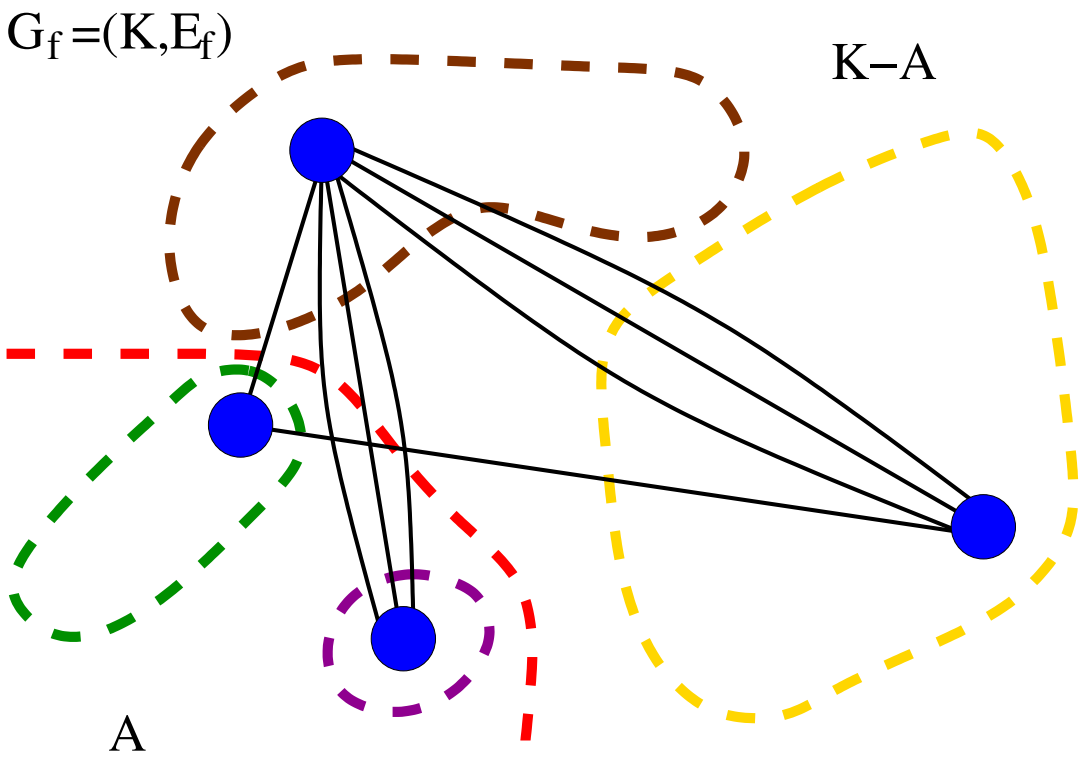

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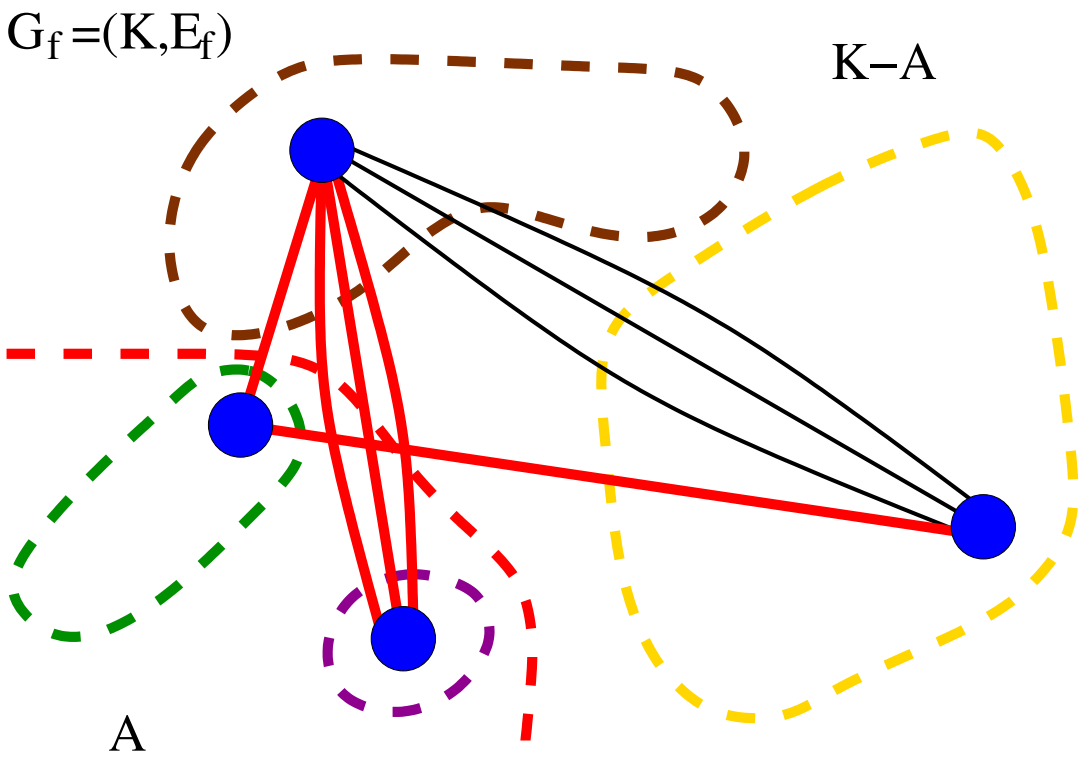

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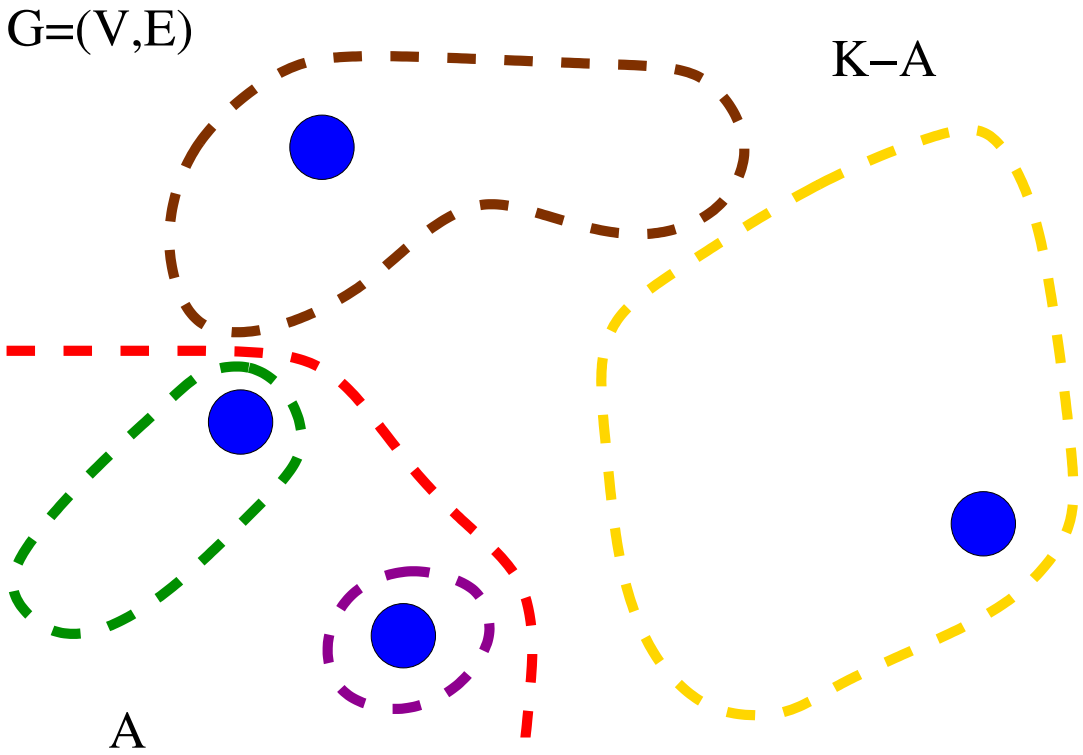

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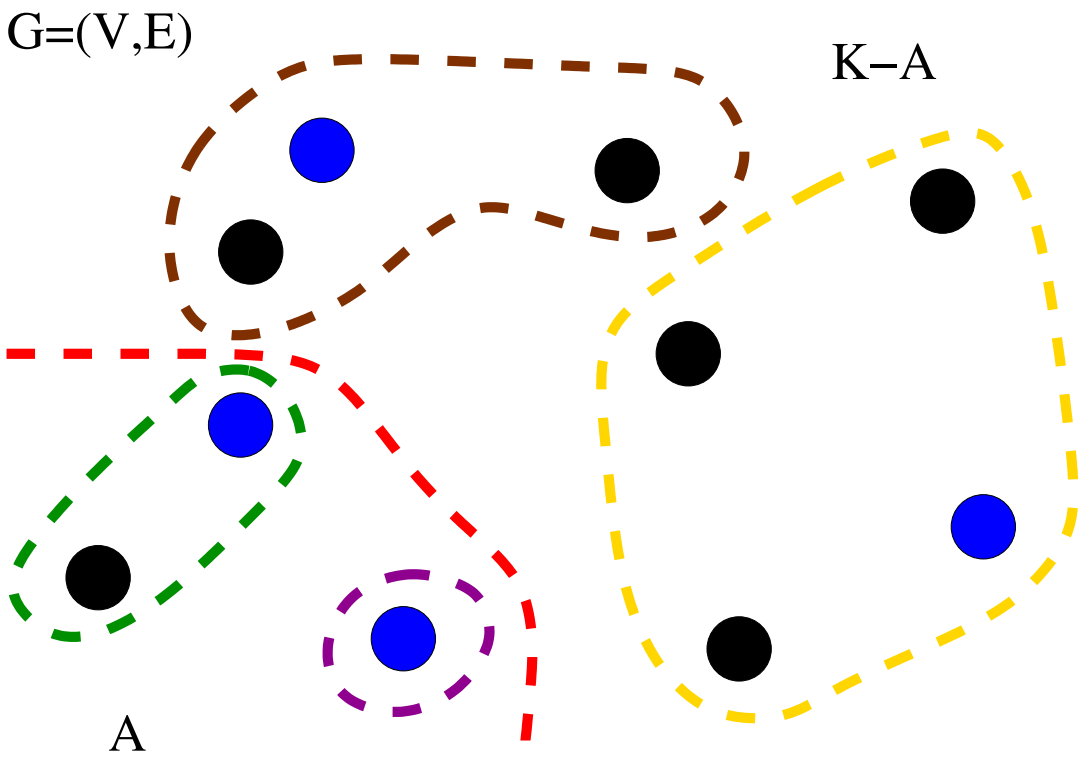

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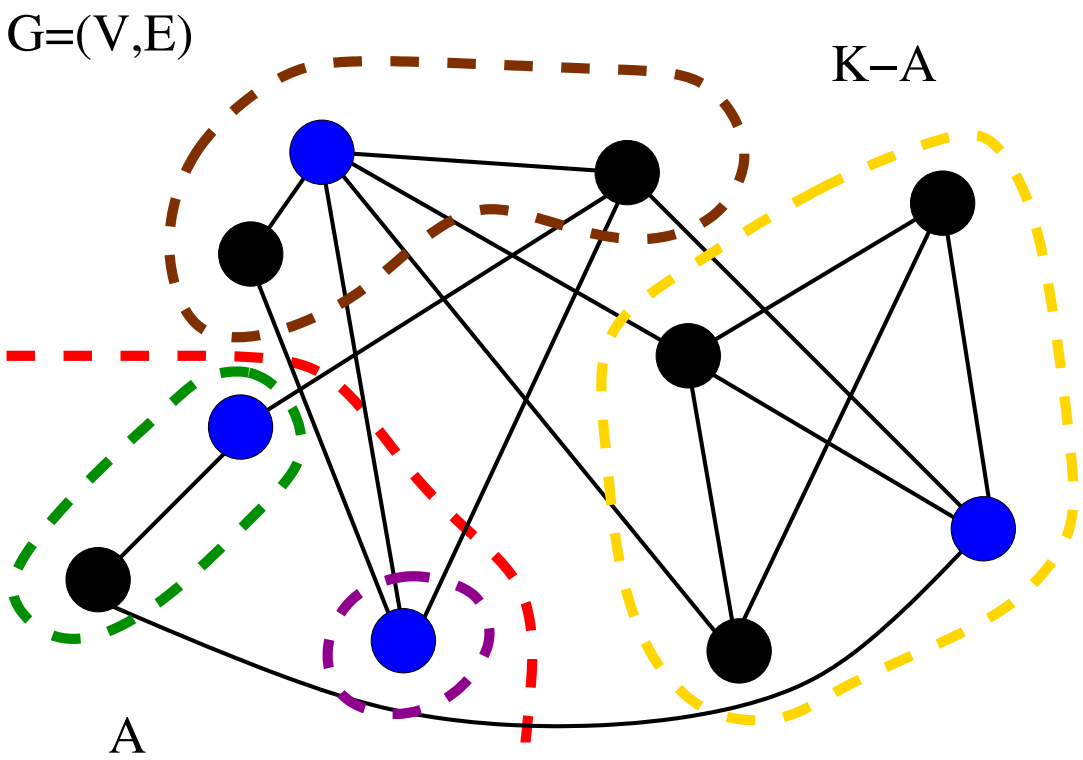

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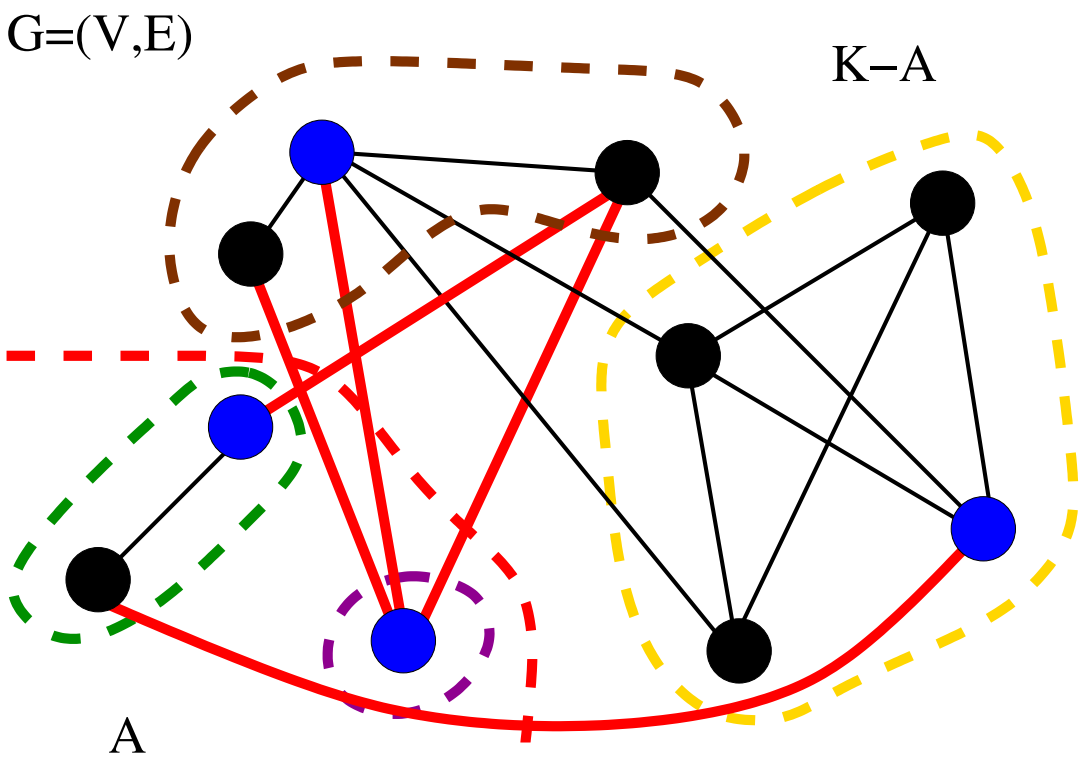
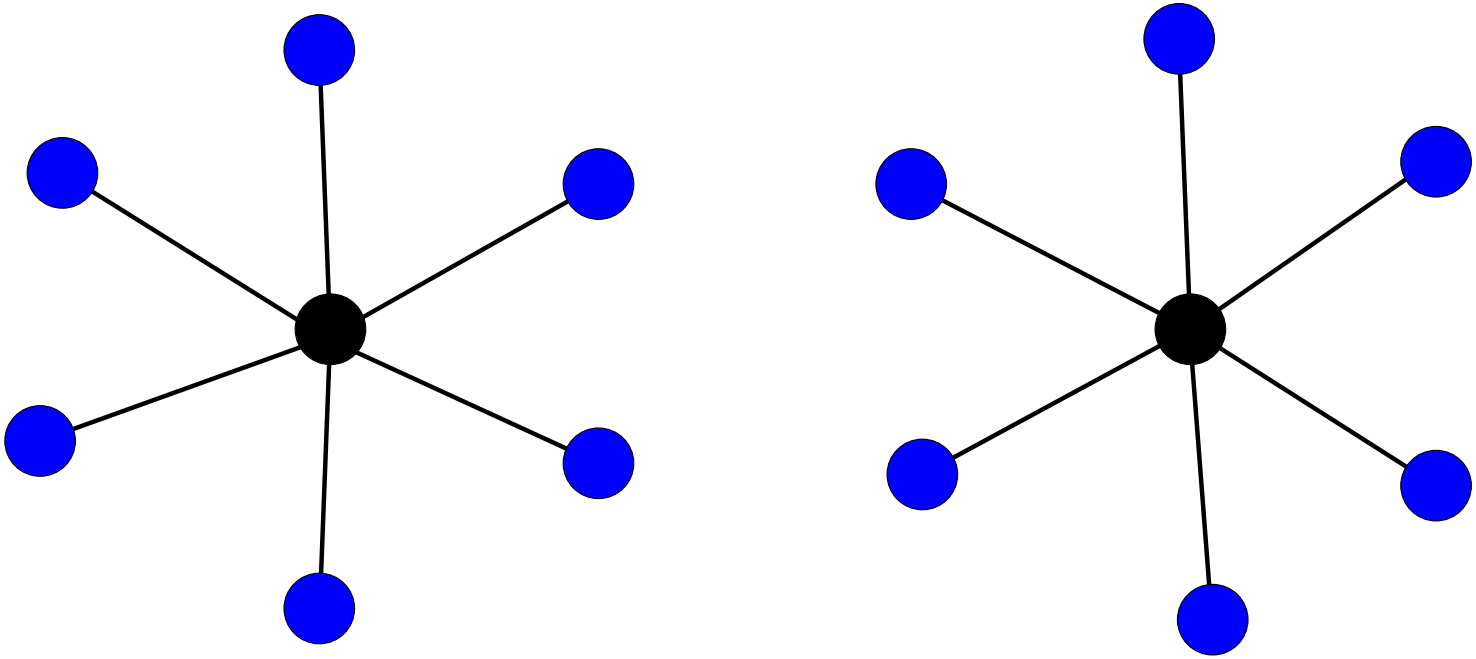


An Example



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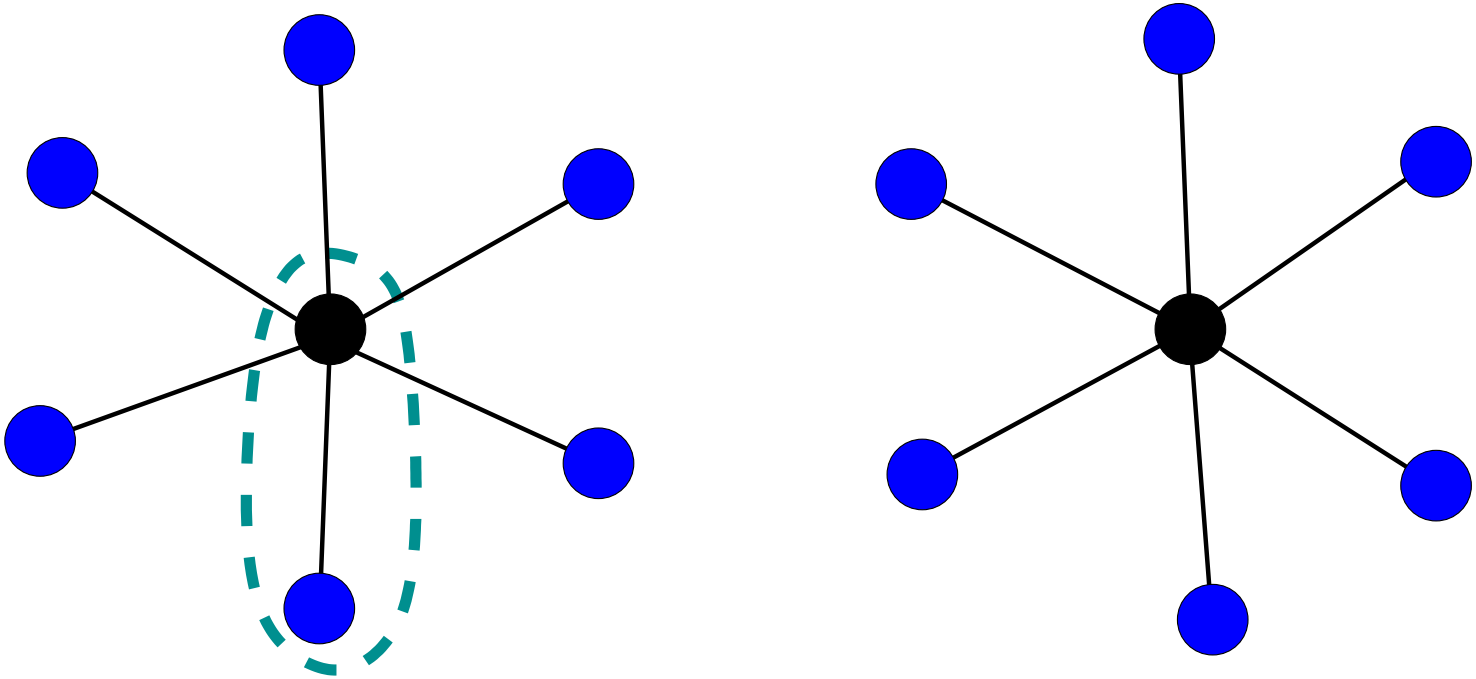

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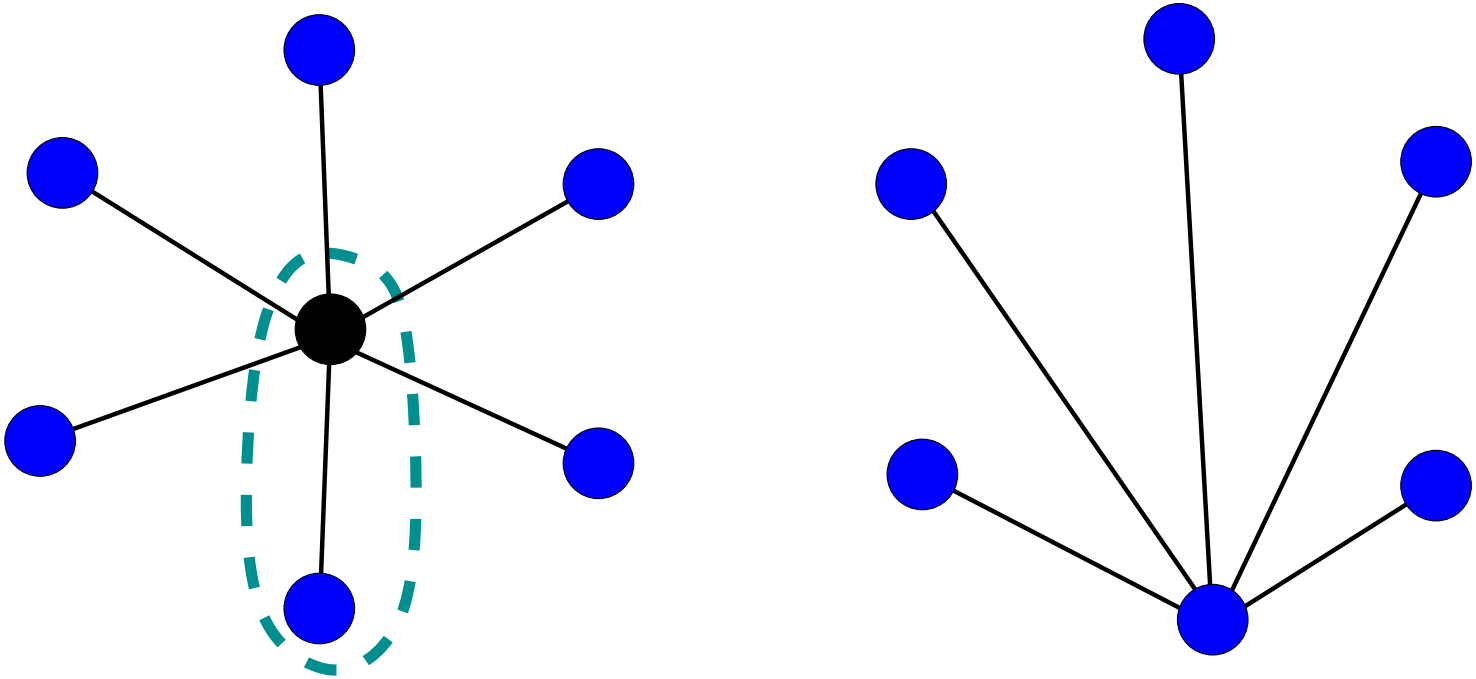
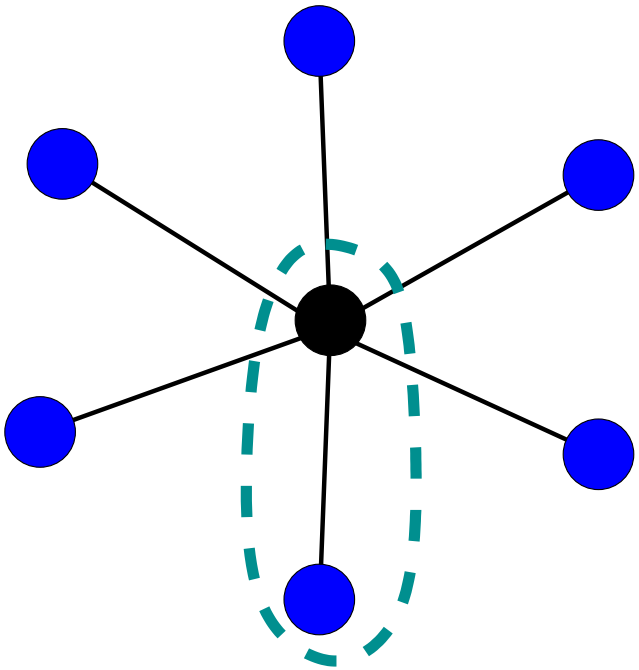
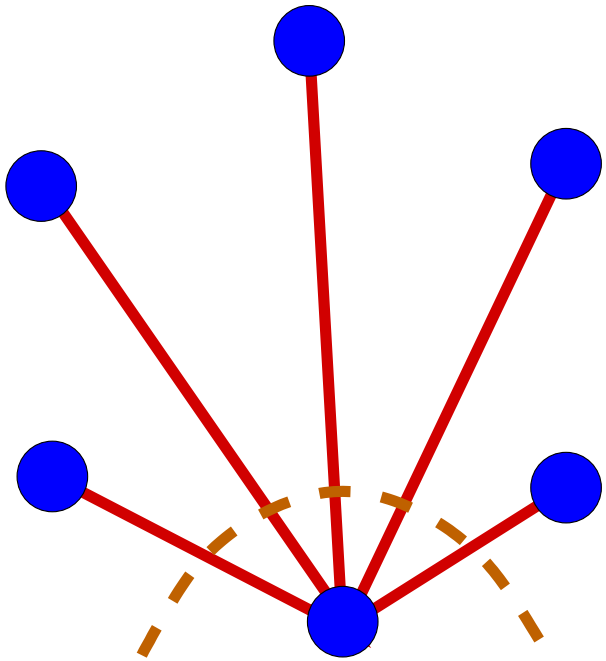


An Example



The cost is k–1



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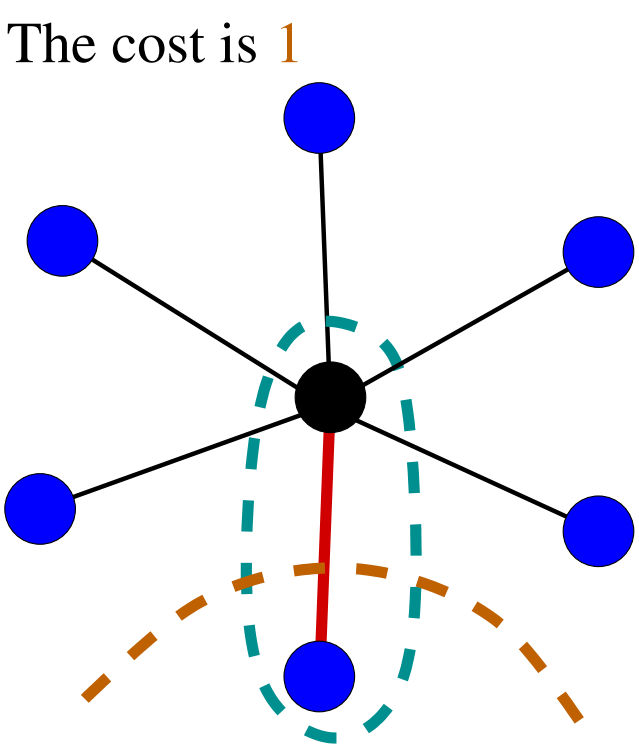

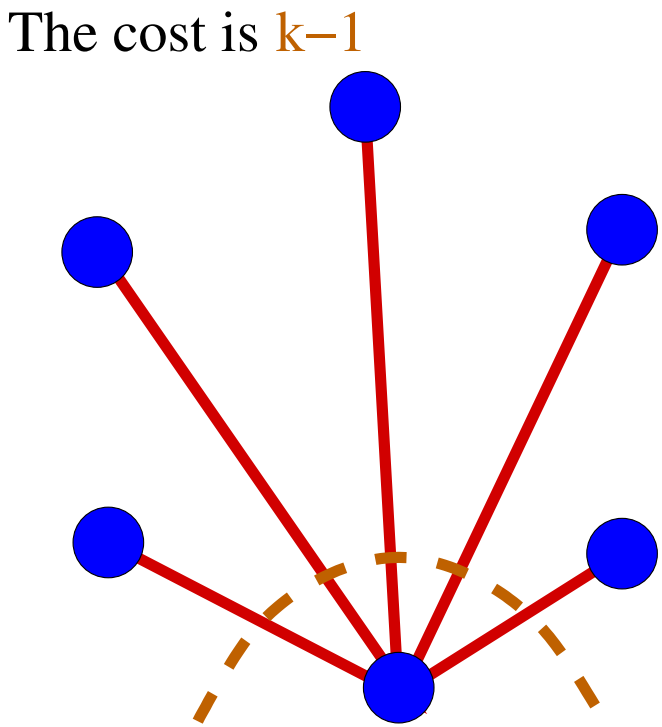

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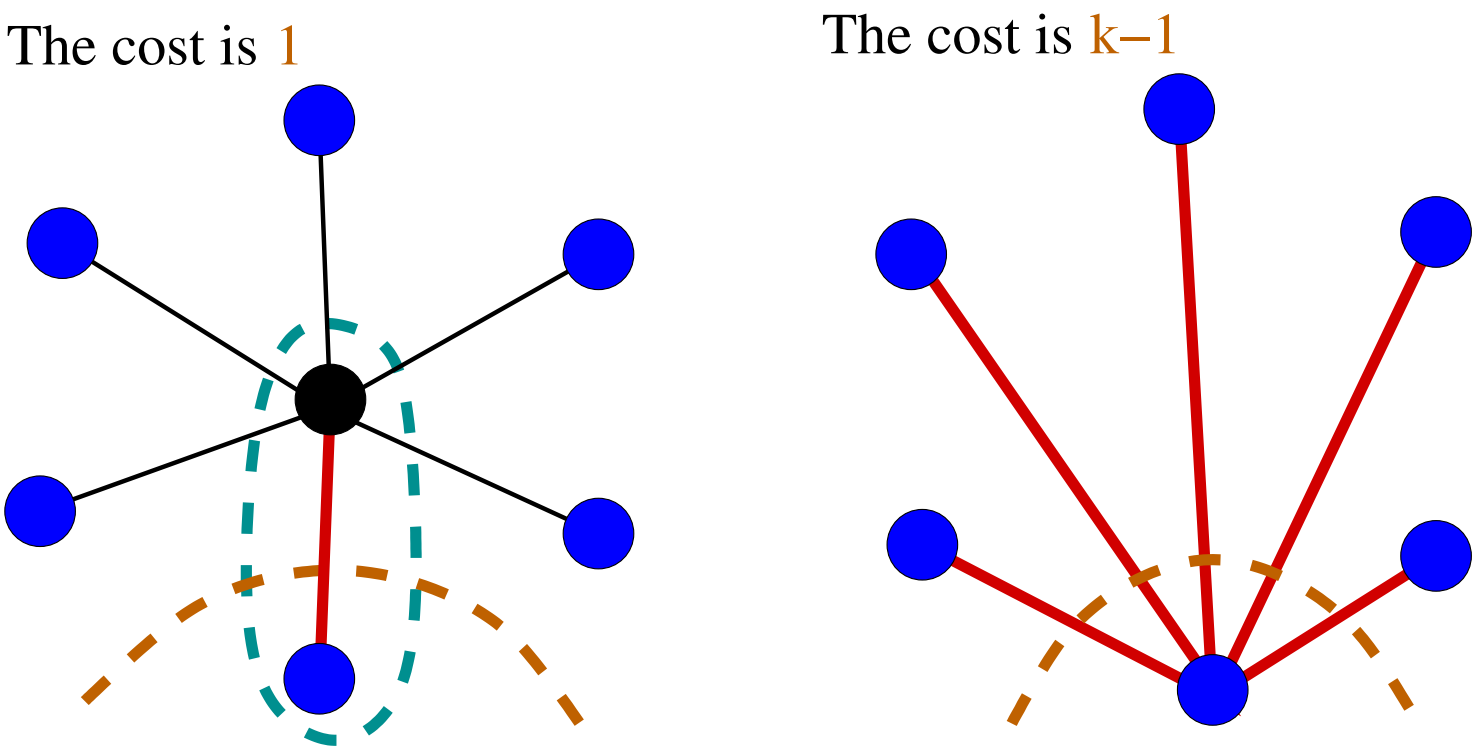
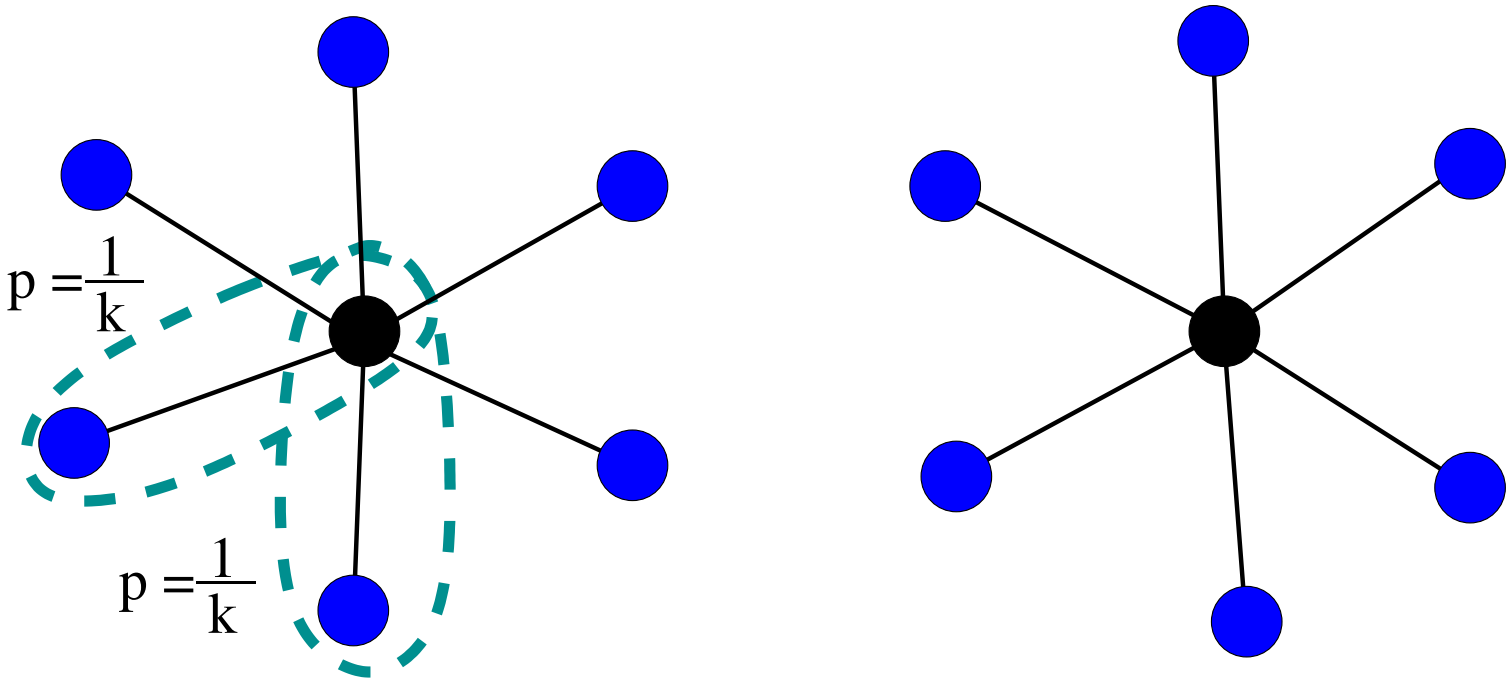

The cost is 1
The cost is k−1
The Quality of the Sparsifier is k−1

An Example: A Second Attempt



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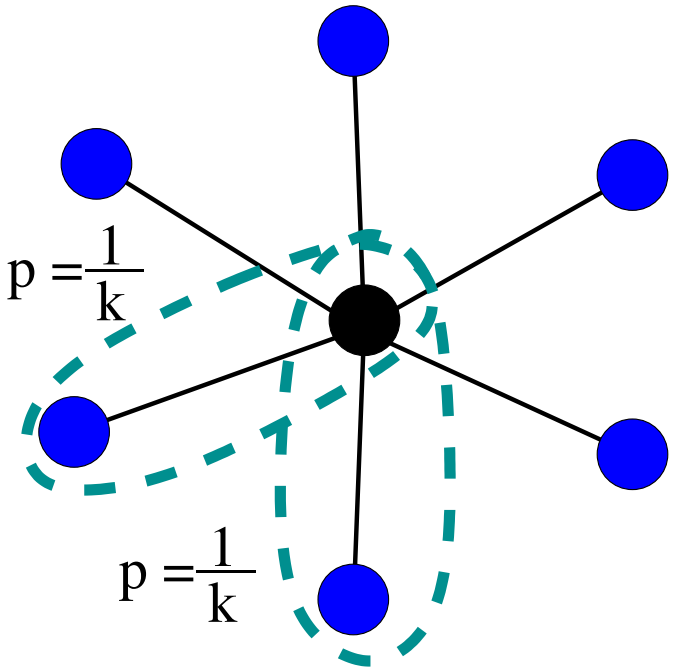

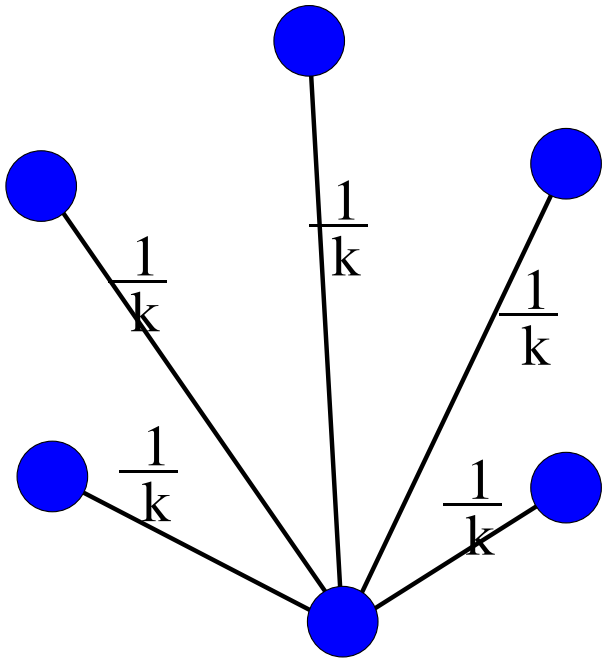

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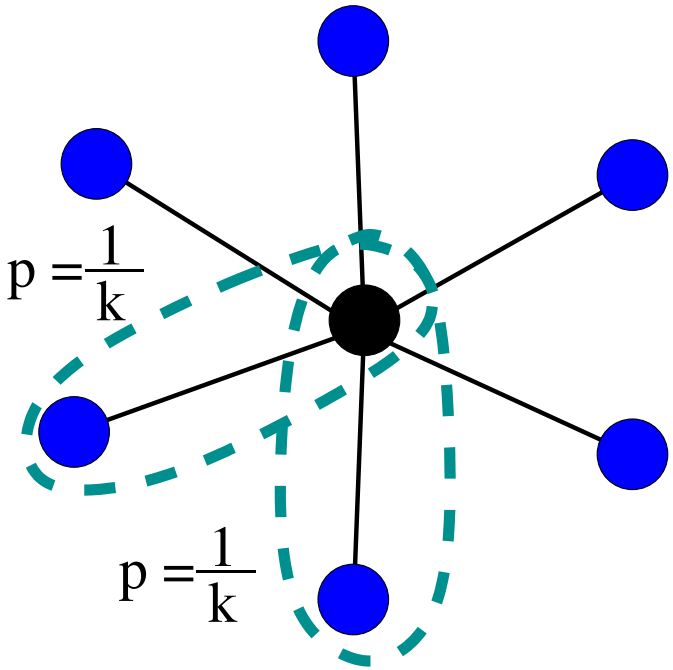

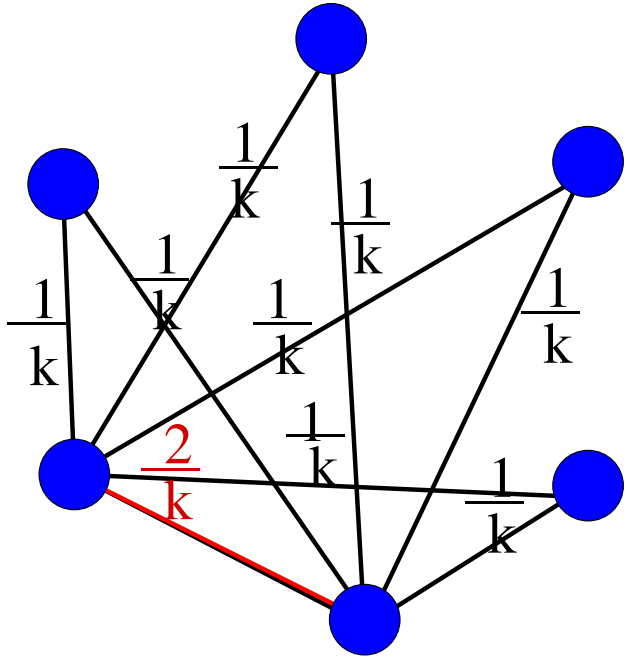

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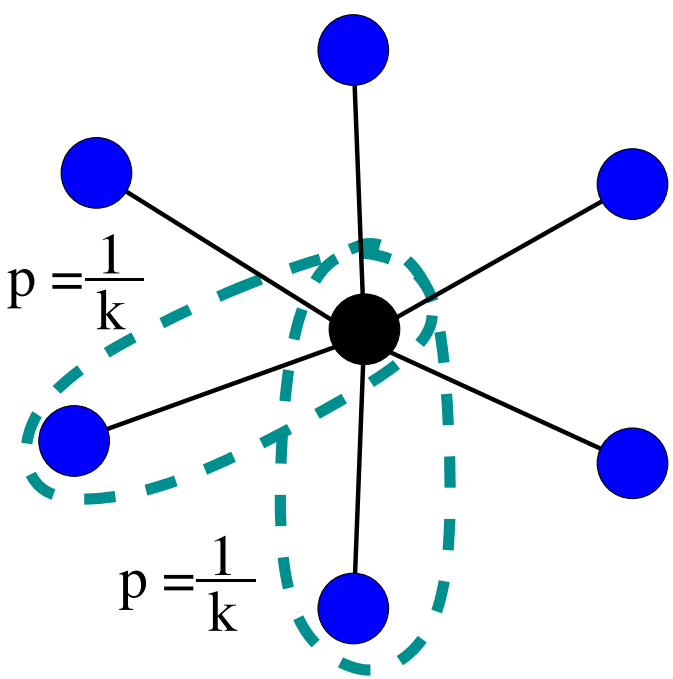

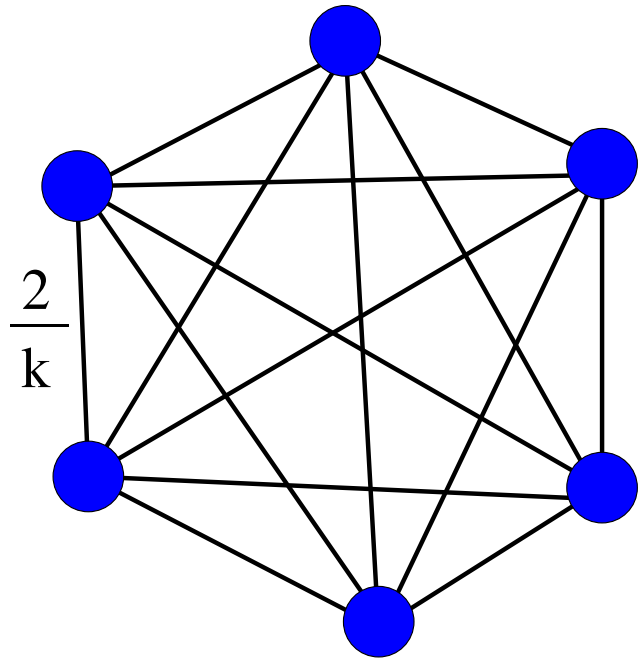

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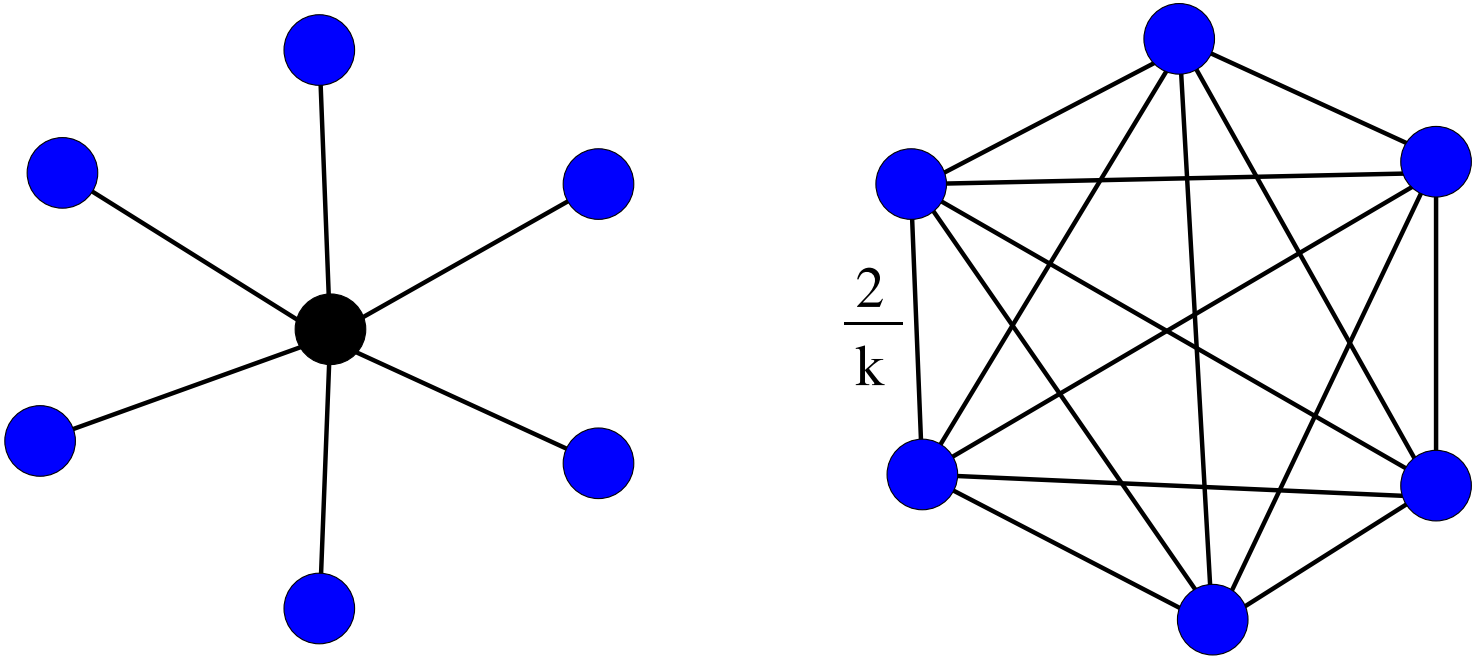
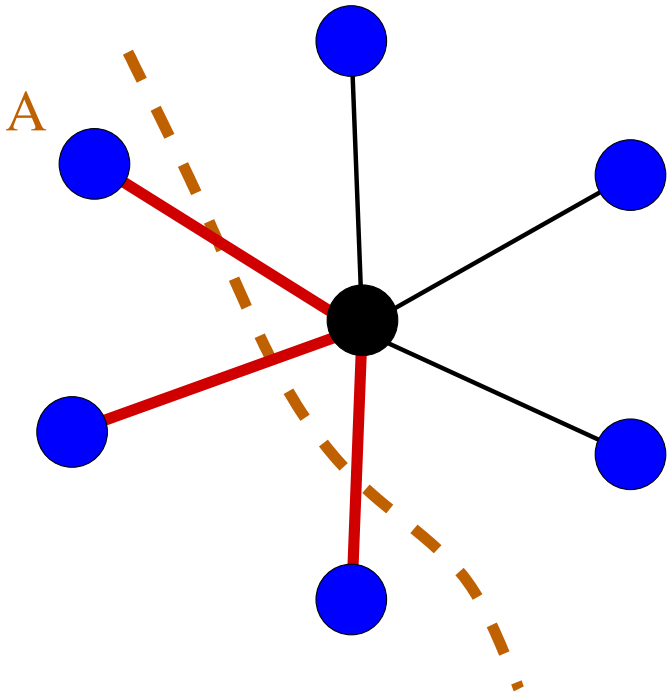
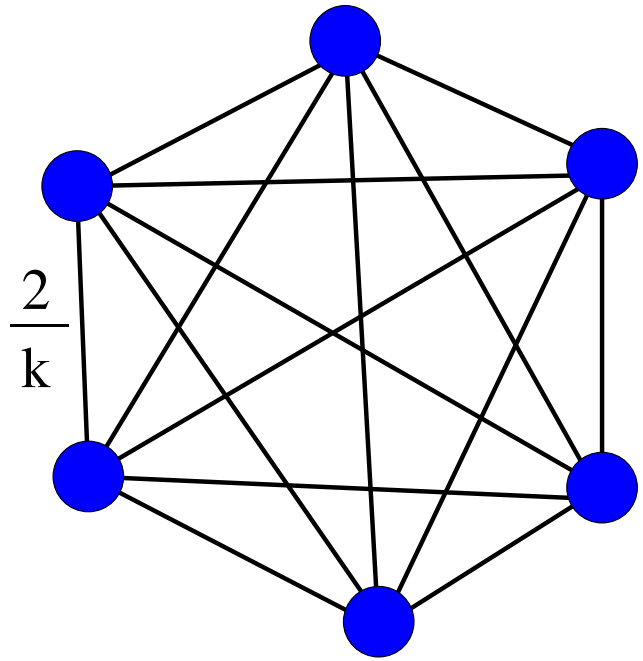


An Example: A Second Attempt

The cost is $\min(|A|, |K-A|)$





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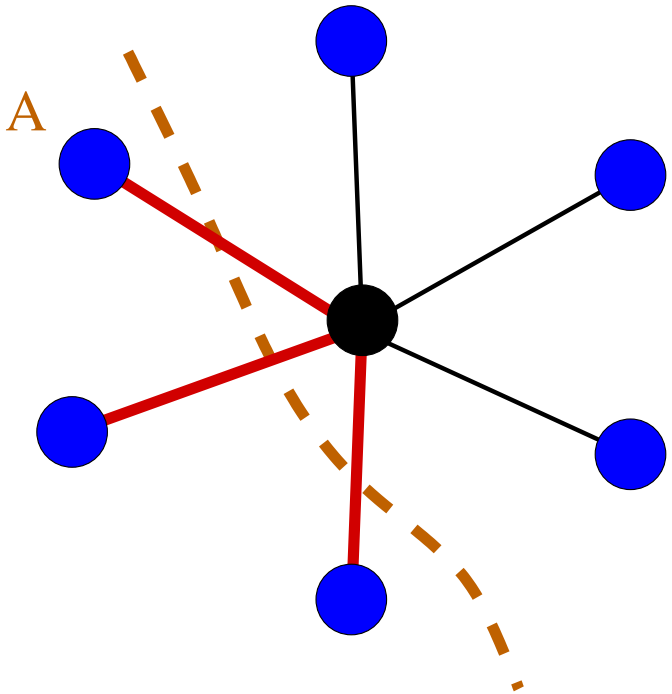
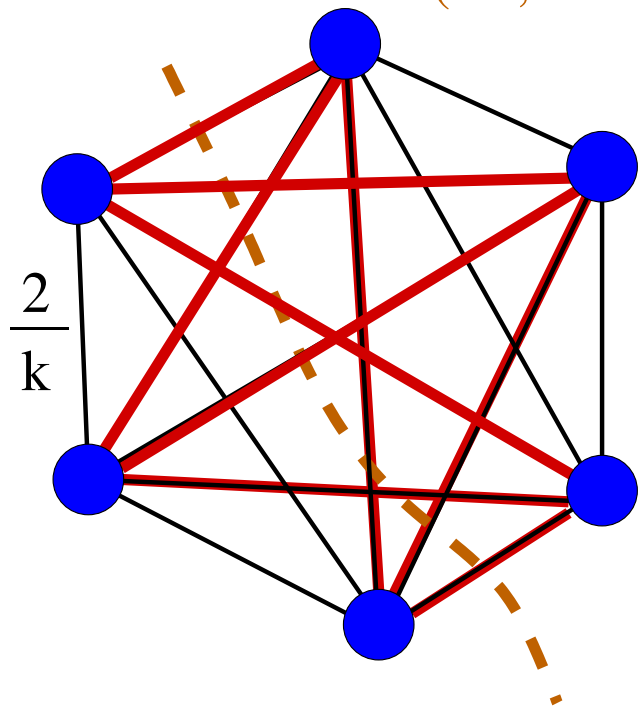


The cost is at most $2\min(|A|, |K-A|)$



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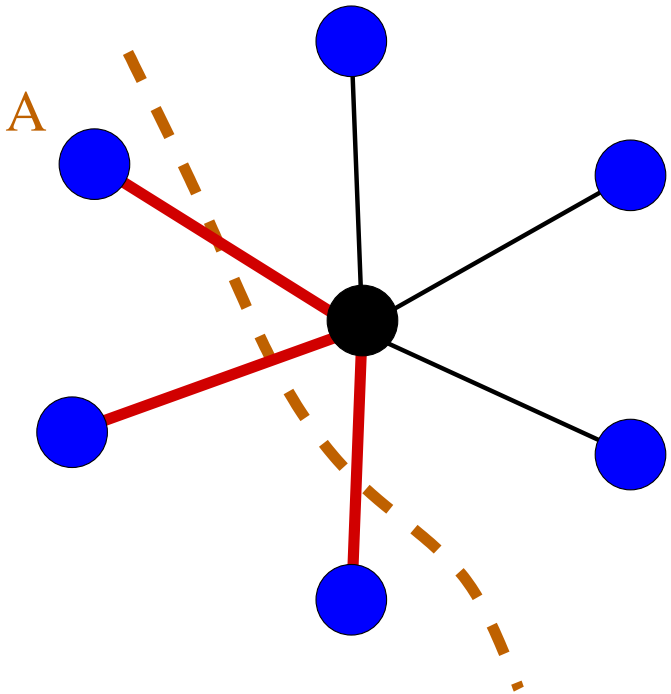

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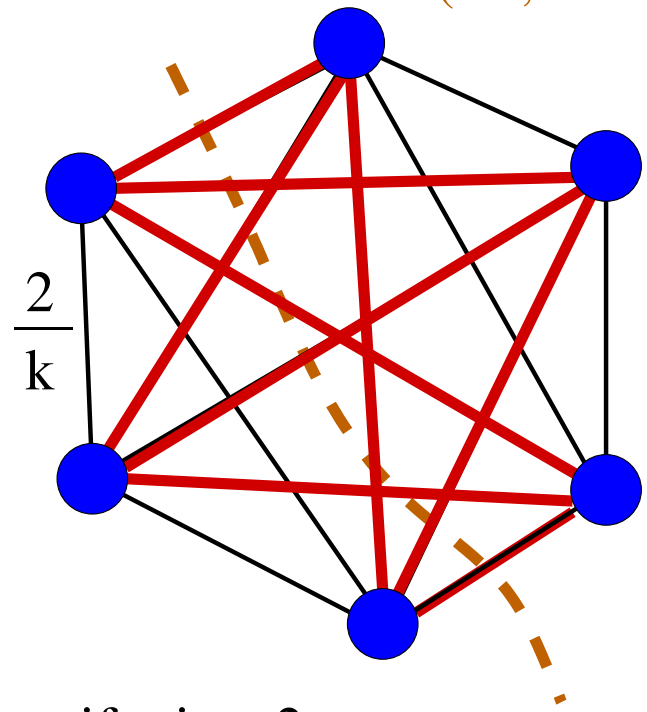

The Quality of the Sparsifer is < 2

Approximate Vertex Sparsifiers via 0-Extensions

Theorem

For all graphs $G = (V, E)$ and all sets $K \subset V$, there is a distribution γ on 0-extensions such that $G' = \sum_f \gamma(f) G_f$ is a $O(\frac{\log k}{\log \log k})$-quality Vertex Sparsifier for G, K.

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For all graphs $G = (V, E)$ and all sets $K \subset V$, there is a polynomial (in n and k) time algorithm to construct G' that is a poly$(\log k)$-quality Vertex Sparsifier for G, K.

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Proof Outline

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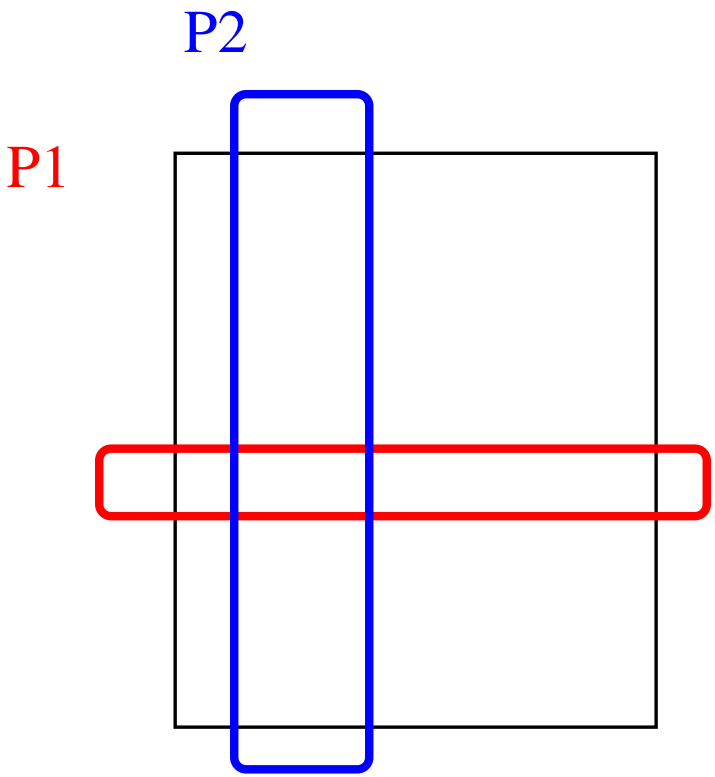
1. Define a **Zero-Sum Game**

The Extension-Cut Game



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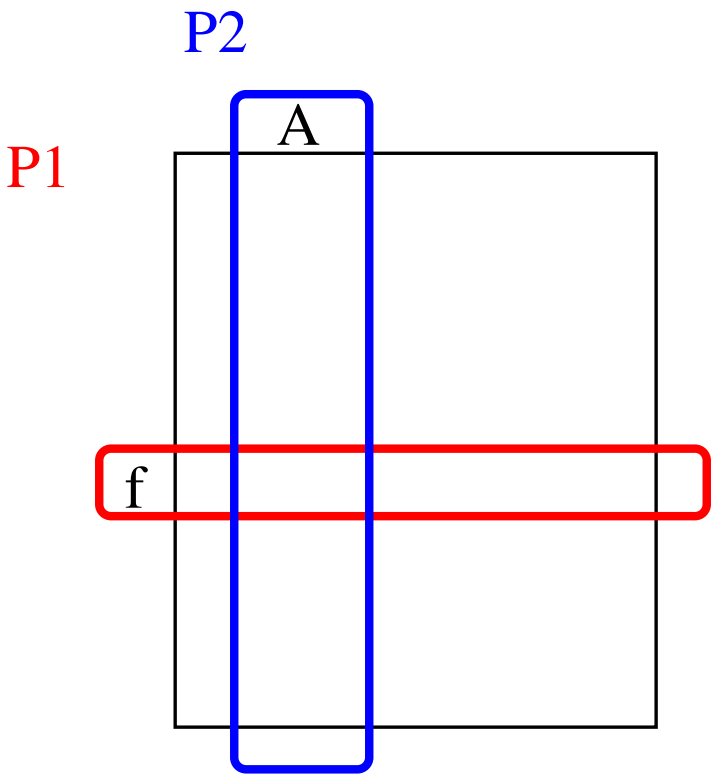

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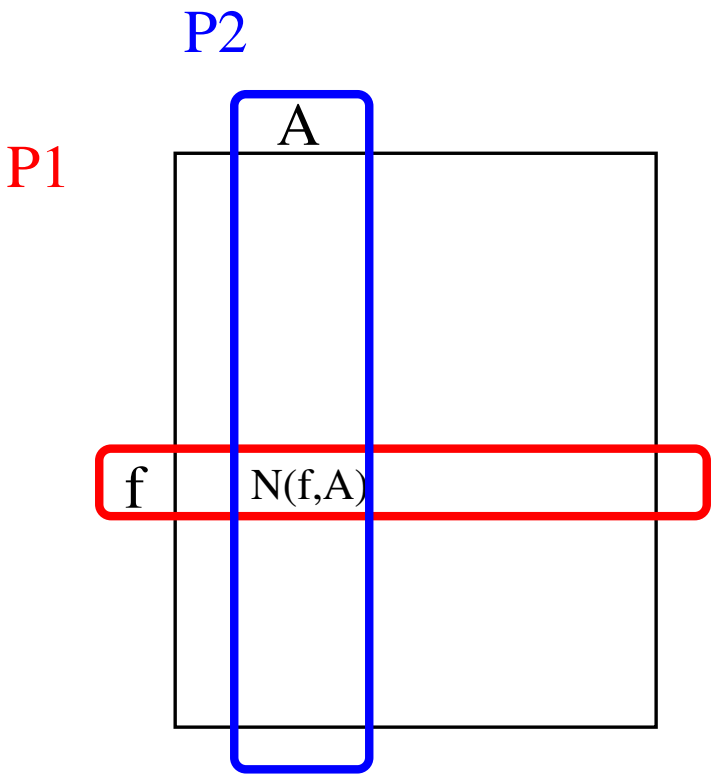

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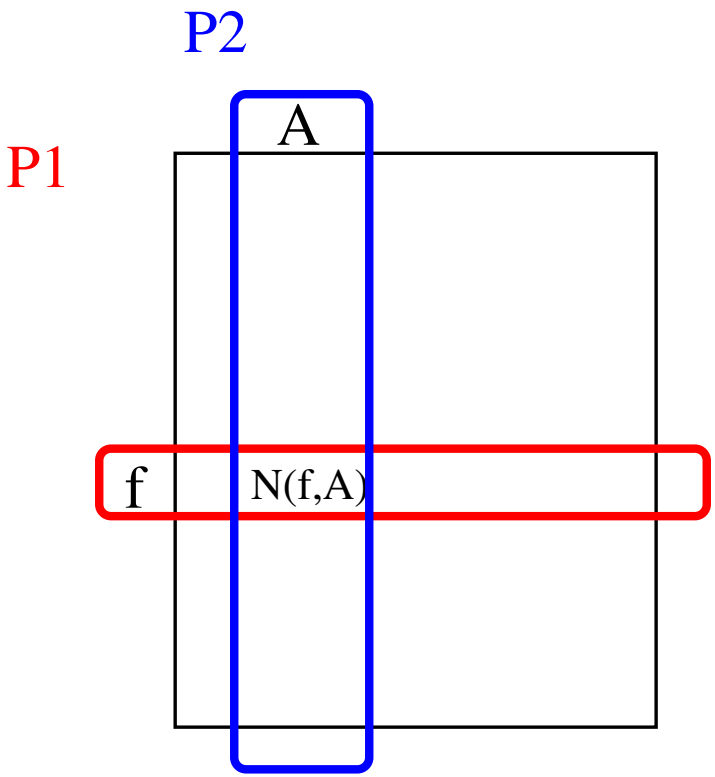

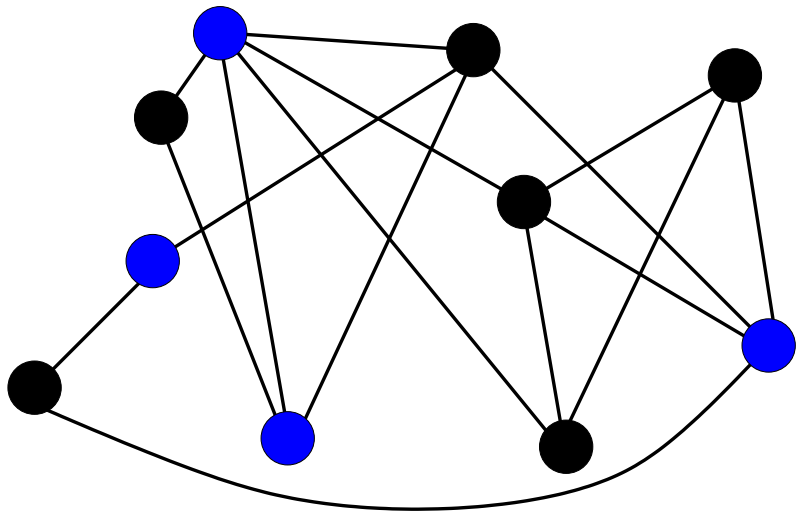

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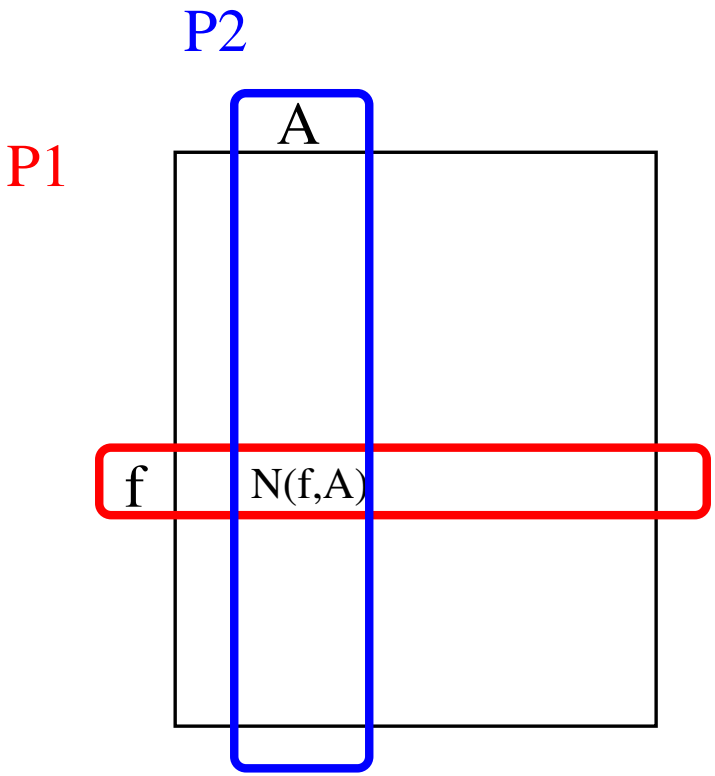

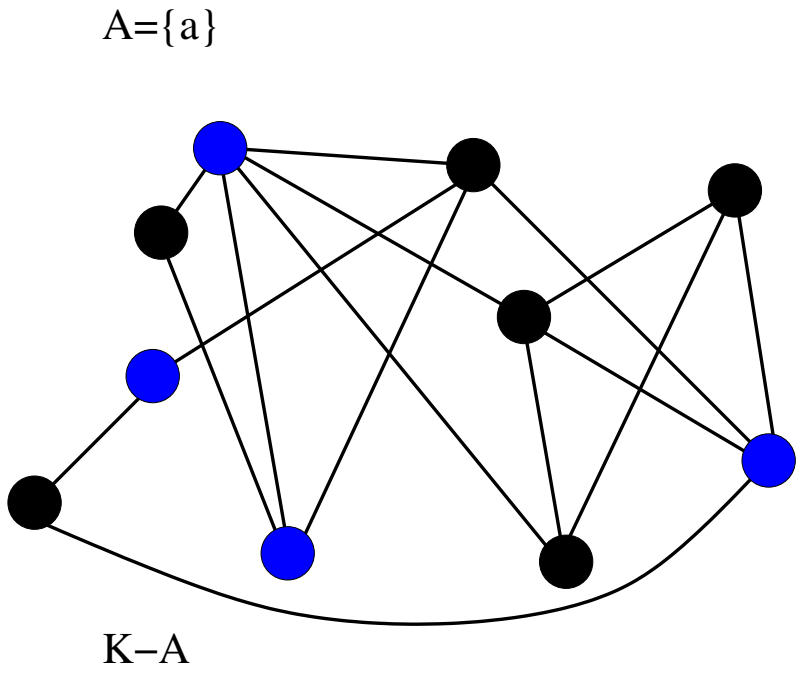

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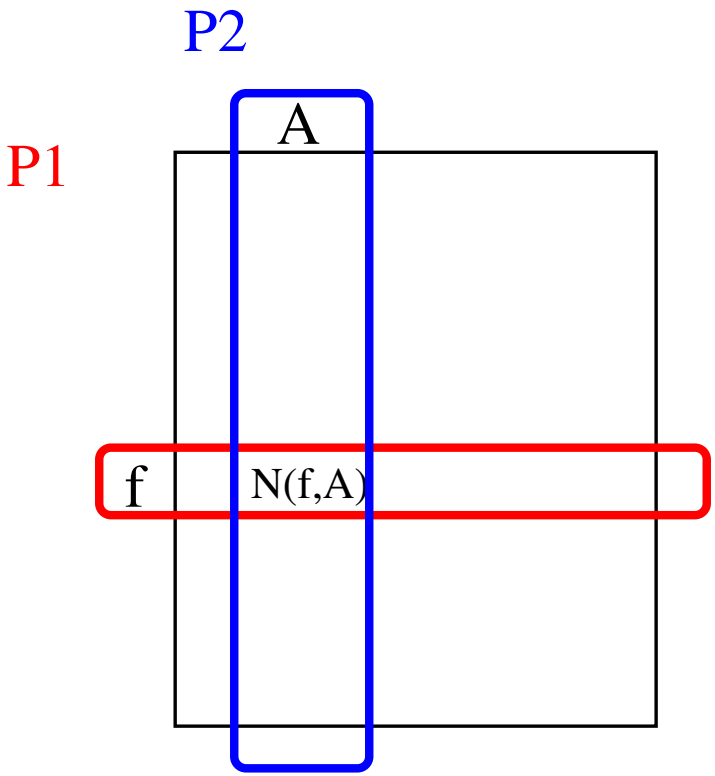

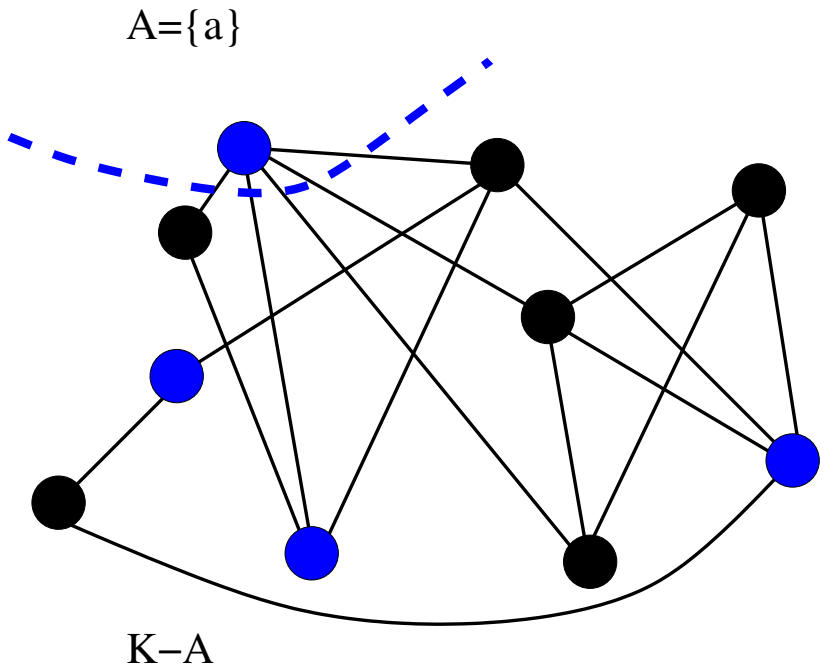

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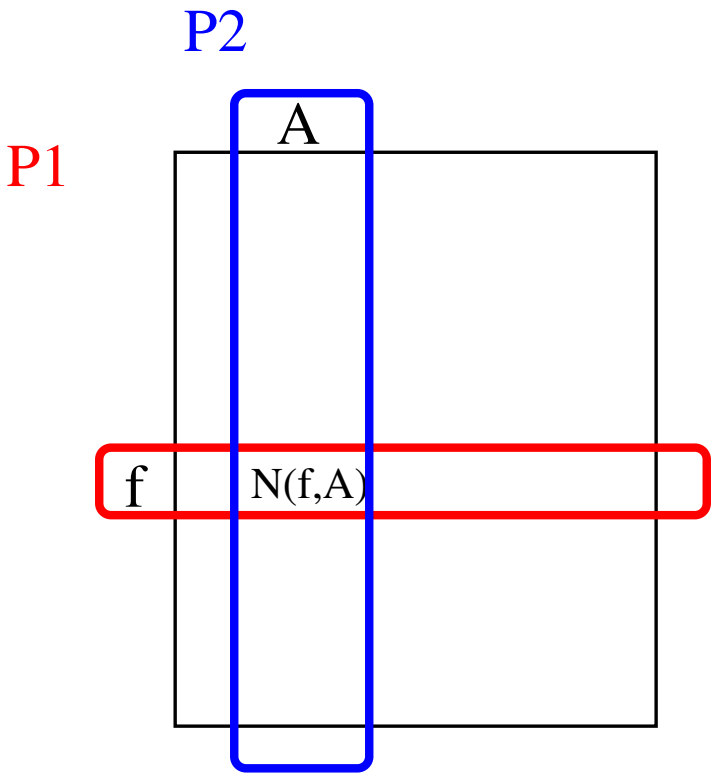
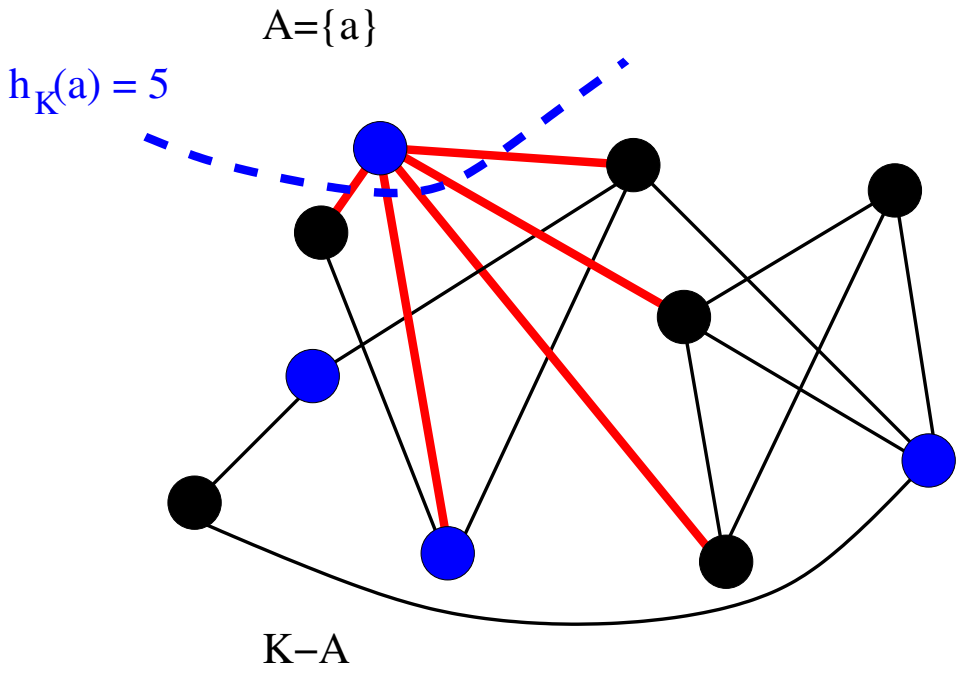
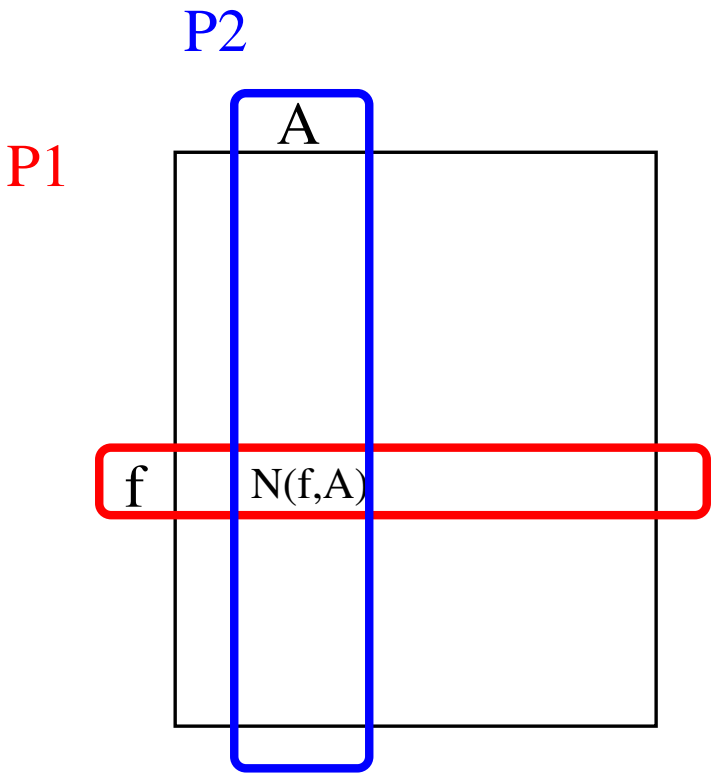
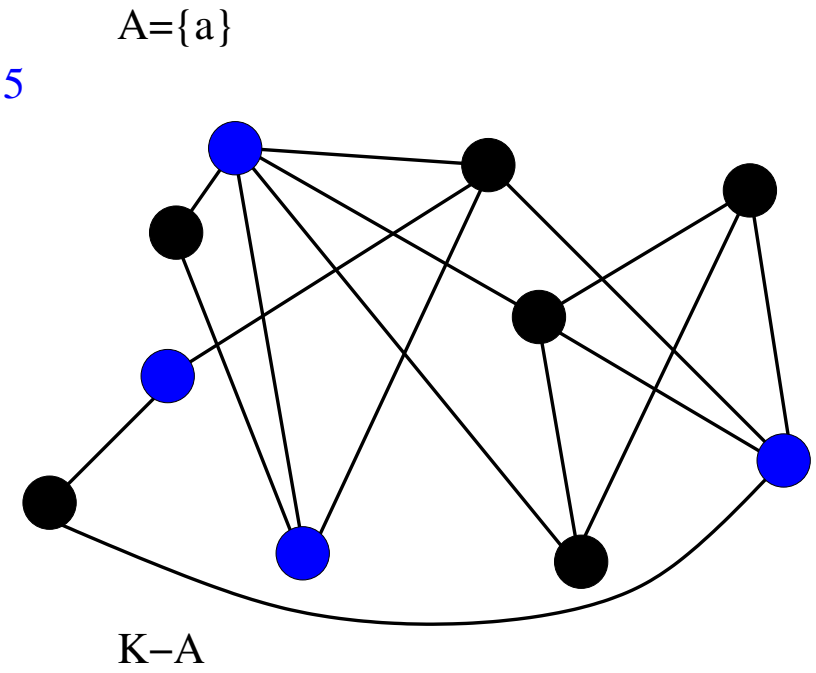




The Extension-Cut Game



$h_K(a) = 5$



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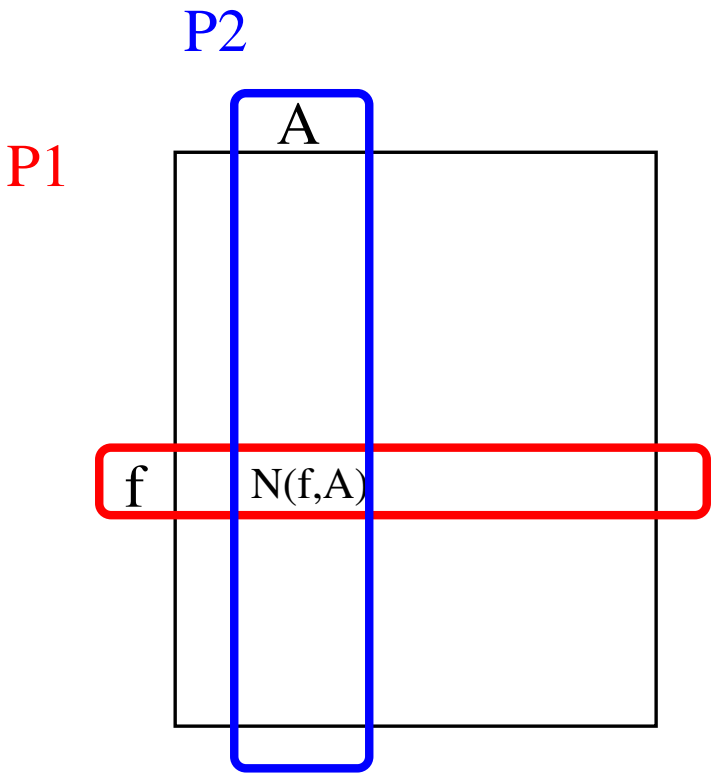

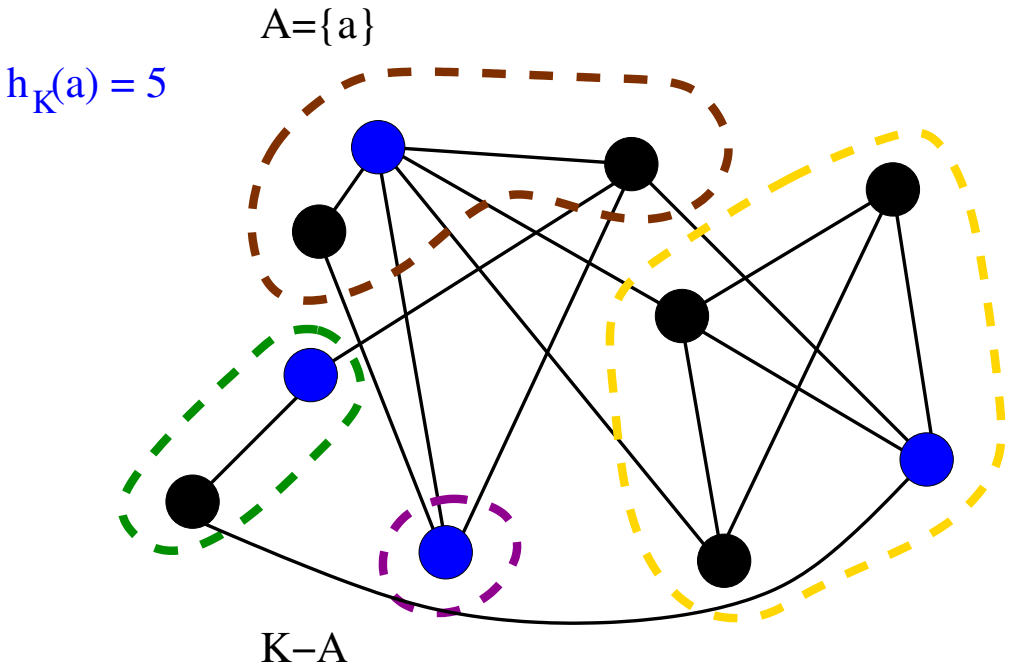

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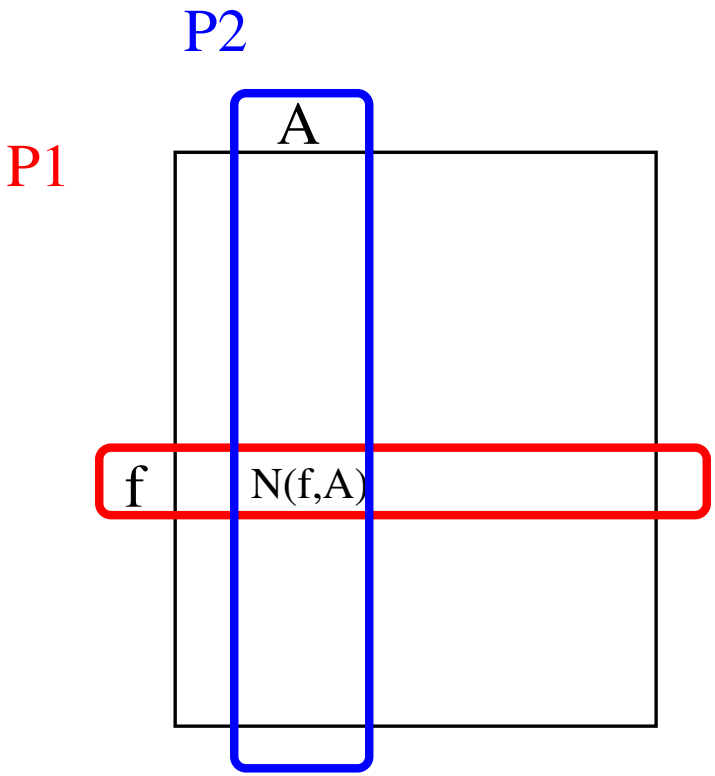

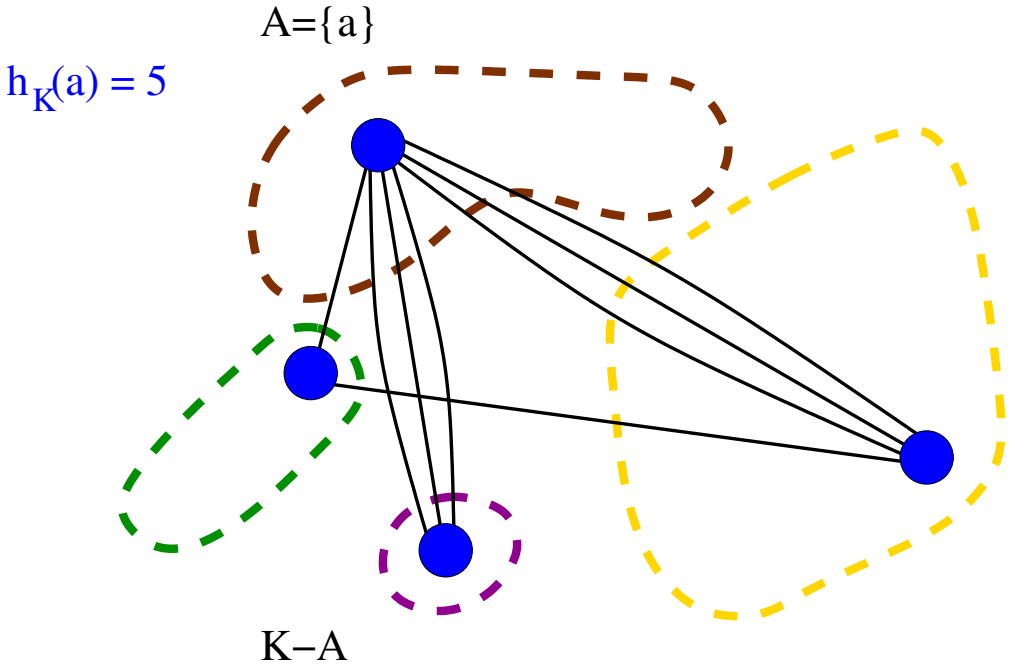

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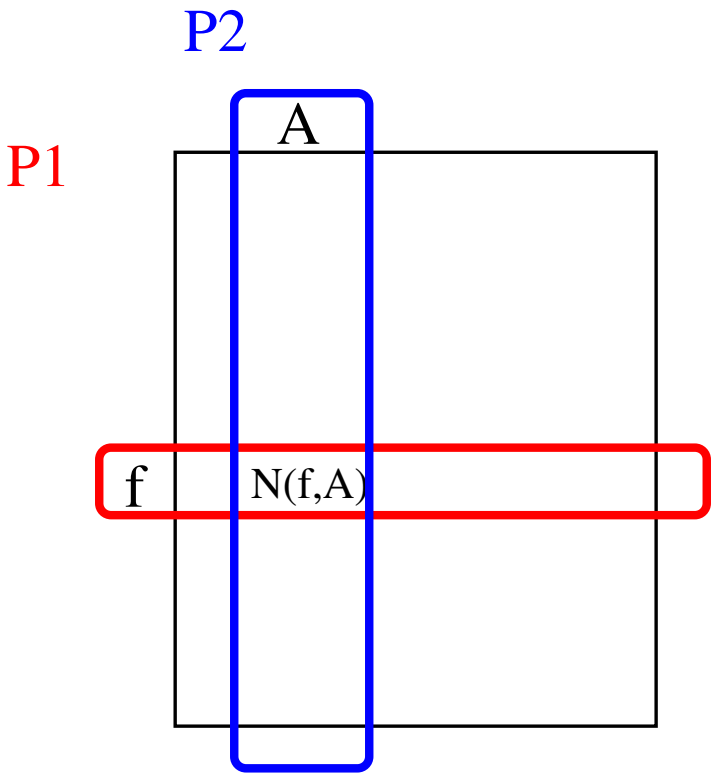

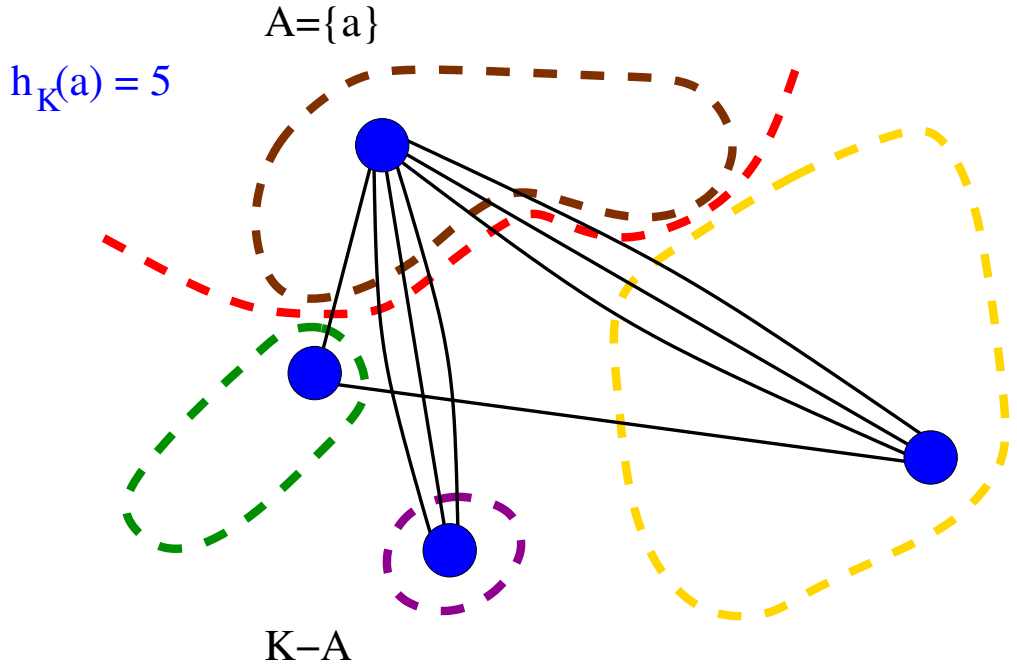

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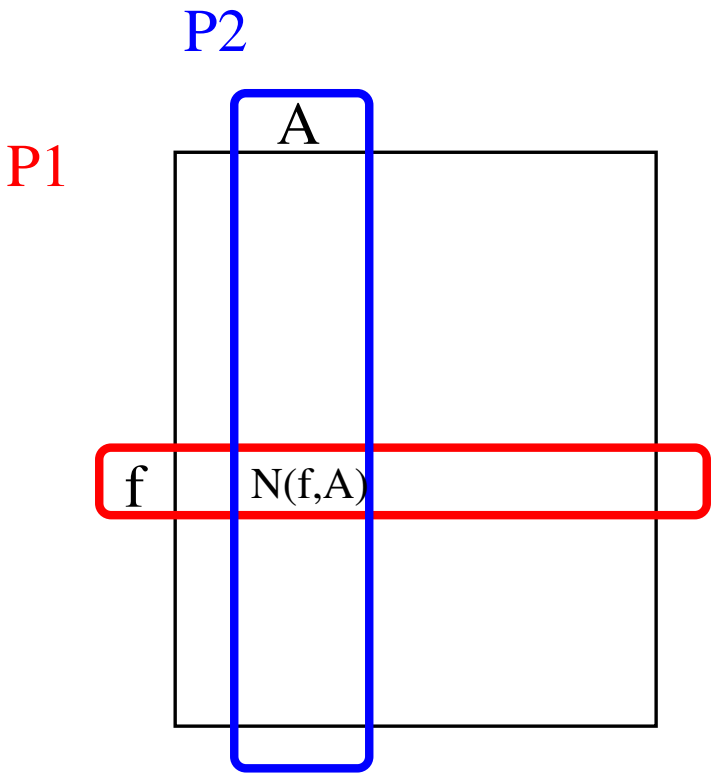

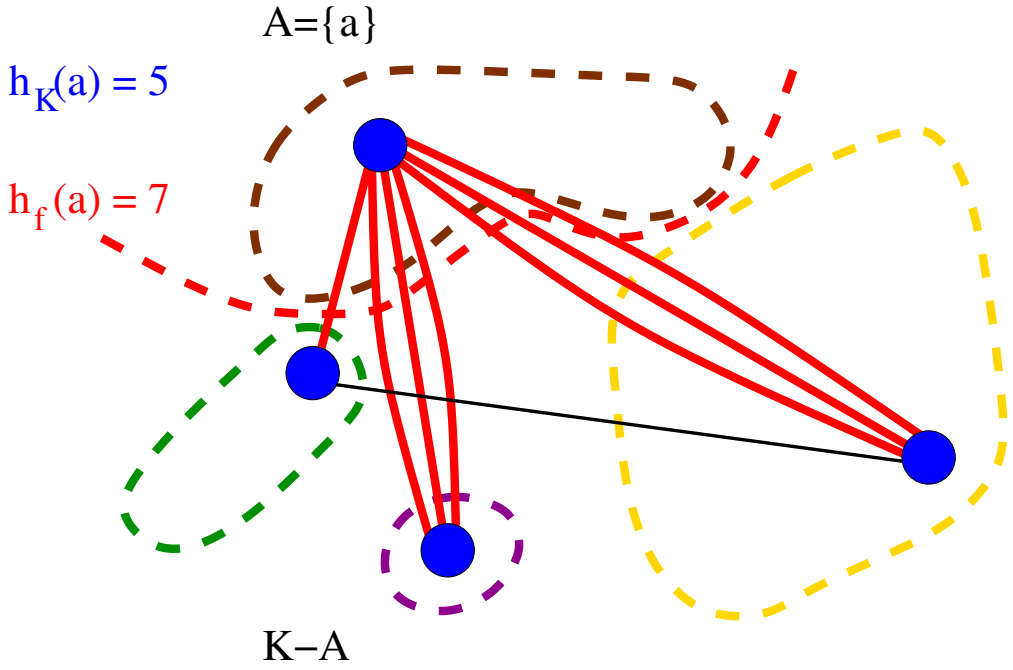

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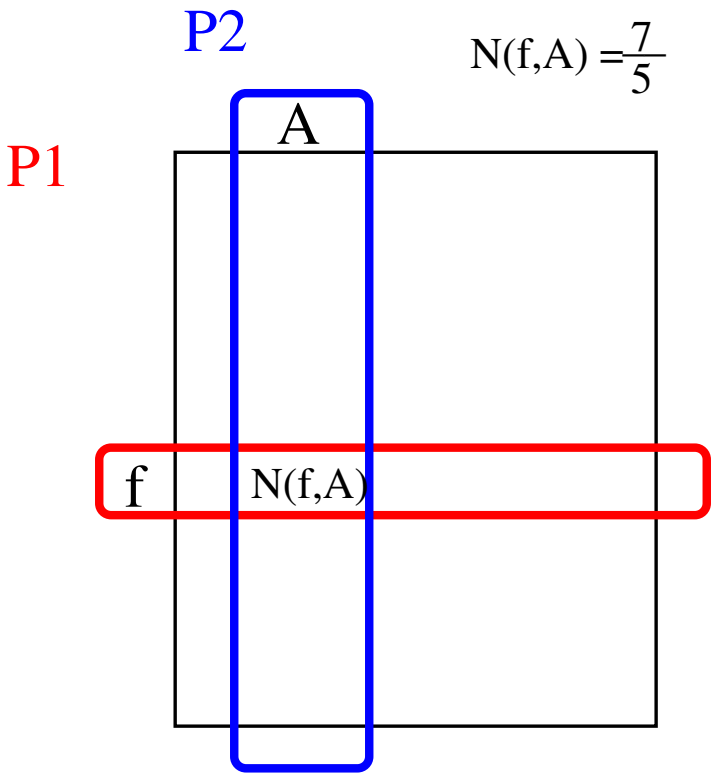

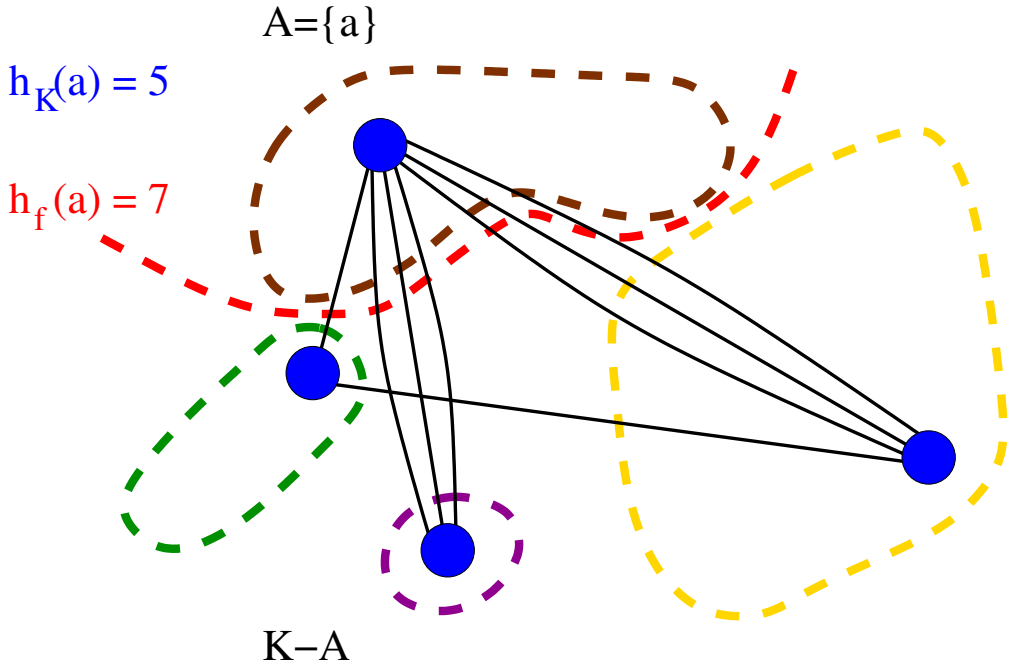

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Let ν denote the game value of the extension-cut game

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Let $G' = \sum_f \gamma(f) G_f$; for all $A \subset K$:

$$h'(A) = E_{f \leftarrow \gamma}[h_f(A)] \leq \nu h_K(A)$$

Proof Outline

1. Define a **Zero-Sum Game**

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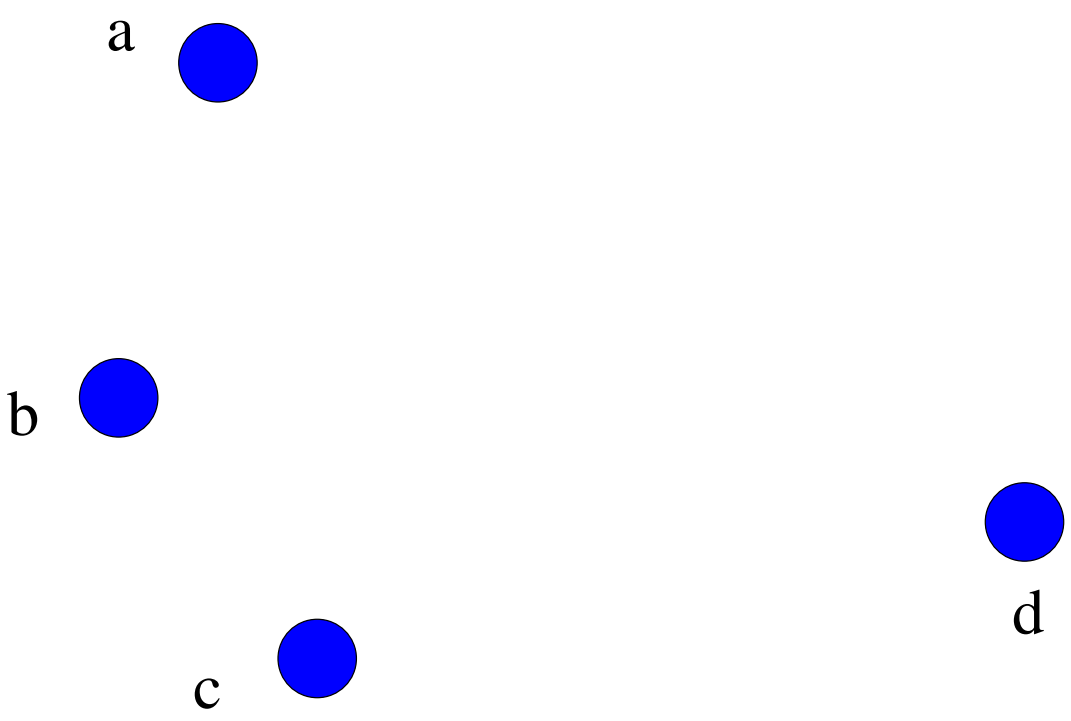
2. The **Best Response** is a 0-Extension Problem

Best Response?

Let $\mu = \frac{1}{2}\{a, b\} + \frac{1}{2}\{a, d\}$



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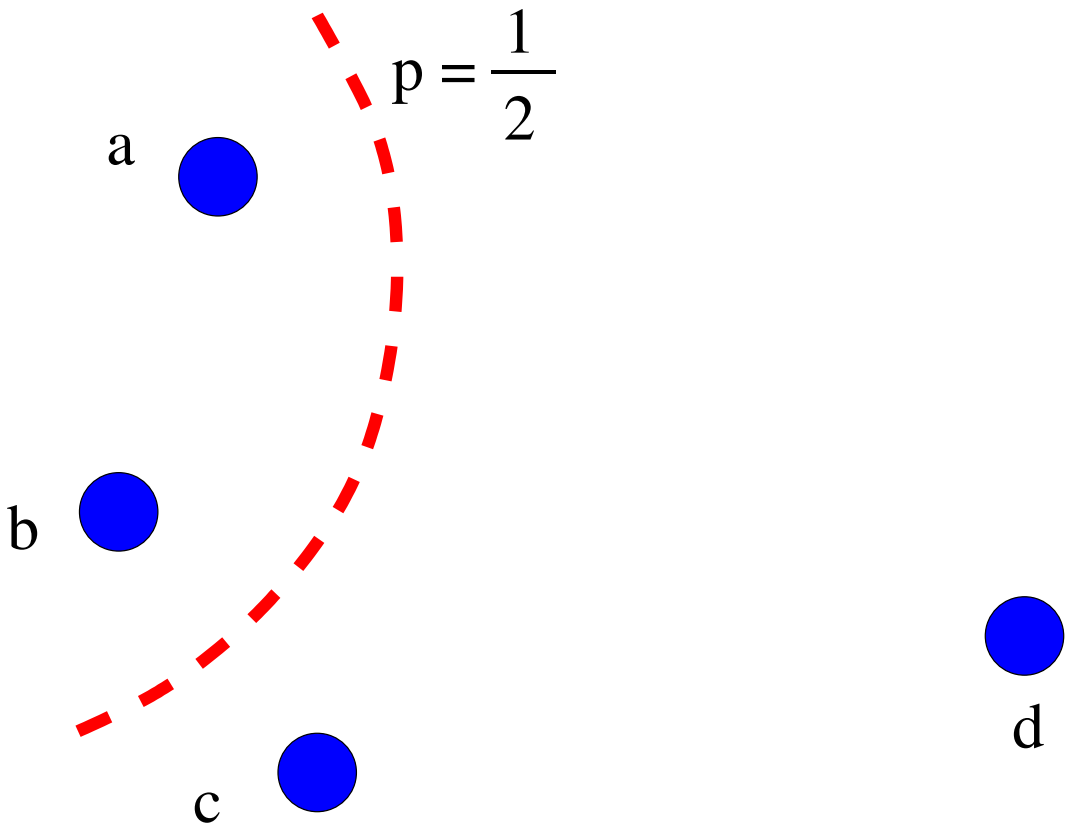

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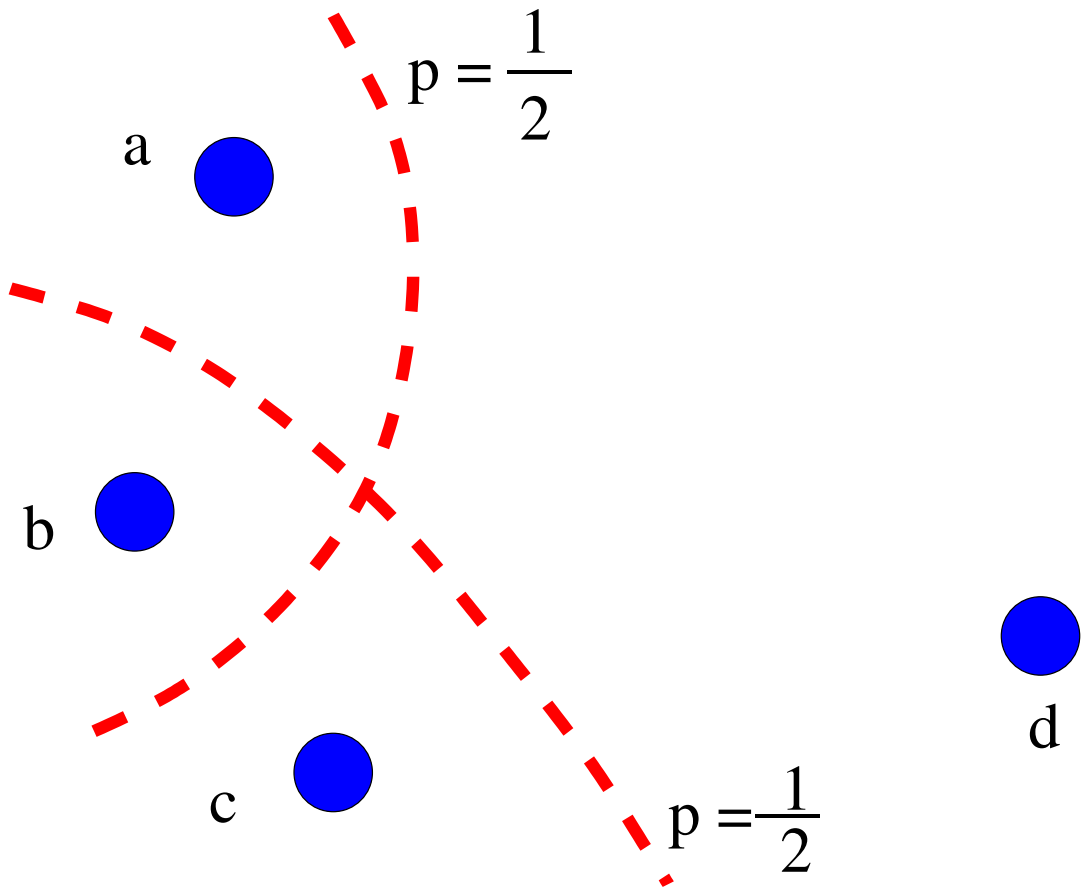

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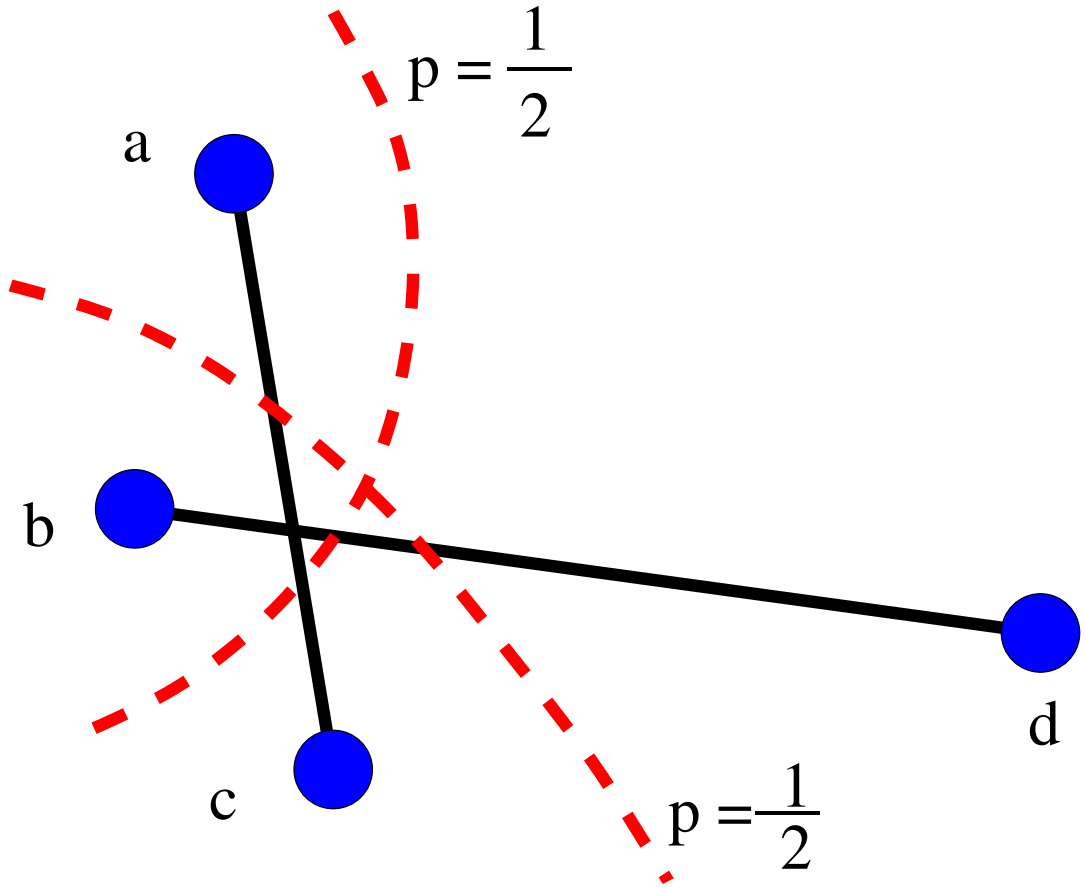

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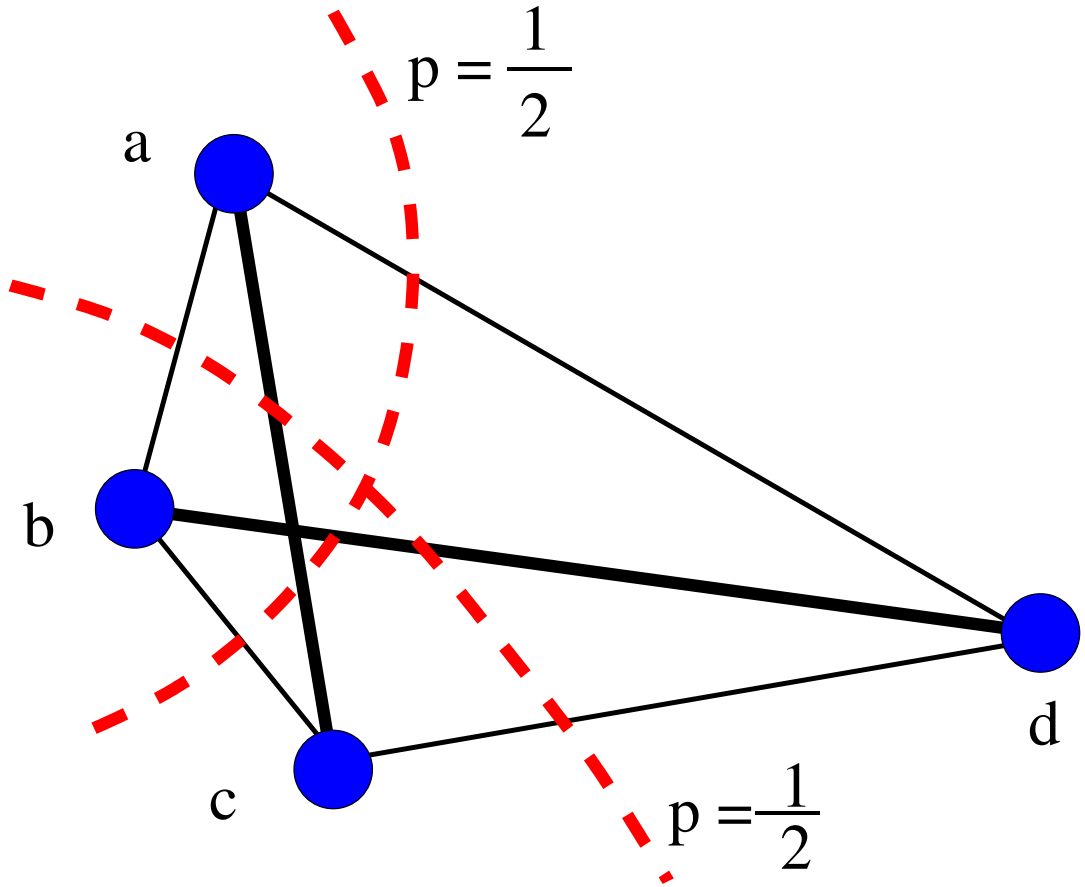

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   [Fakcharoenphol, Harrelson, Rao, Talwar 2003]
   [Calinescu, Karloff, Rabani 2001]

Approximate Vertex Sparsifiers via 0-Extensions

Theorem

For all graphs $G = (V, E)$ and all sets $K \subset V$, there is a distribution γ on 0-extensions such that $G' = \sum_f \gamma(f) G_f$ is a $O(\frac{\log k}{\log \log k})$-quality Vertex Sparsifier for G, K.

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For all graphs $G = (V, E)$ and all sets $K \subset V$, there is a polynomial (in n and k) time algorithm to construct G' that is a poly$(\log k)$-quality Vertex Sparsifier for G, K.

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Constructively Finding G'

Unfortunately, this is a whole other talk!

Improvements and Open Questions

Let $P_G \subset \Re^{\binom{k}{2}}$ be the set of all demands that can be routed in G (where demands are restricted to K).

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There is a graph $G' = (K, E')$ *such that* $P_G \subset P_{G'} \subset O(\frac{\log k}{\log \log k}) P_G$

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What is the integrality gap of the linear program for 0-extensions?

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Is it $\sqrt{\log k}$, $\frac{\log k}{\log \log k}$, ...?

Improvements and Open Questions

Theorem (Leighton, M)

There is a graph G and a set K so that for any graph $G' = (K, E')$ which is a better flow network - i.e. $P_G \subset P_{G'}$, $P_{G'} \notin \Omega(\log \log k) P_G$

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Equivalently, is there an $O(1)$ upper bound for the linear program for 0-extensions when Δ is an ℓ_1 semi-metric?

Thanks!