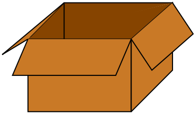
Pareto Optimal Solutions for Smoothed Analysts

Ankur Moitra, IAS

joint work with Ryan O'Donnell, CMU

September 26, 2011

values	v_1	$\cdots$	v_i	v_{i+1}	$\cdots$	v_n
weights	w_1	$\cdots$	w_i	w_{i+1}	$\cdots$	w_n



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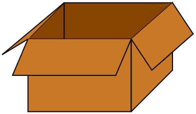

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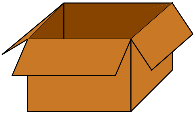

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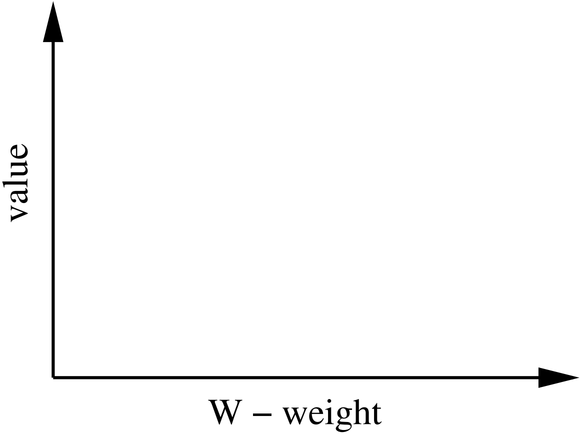

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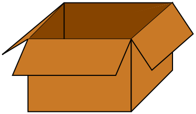

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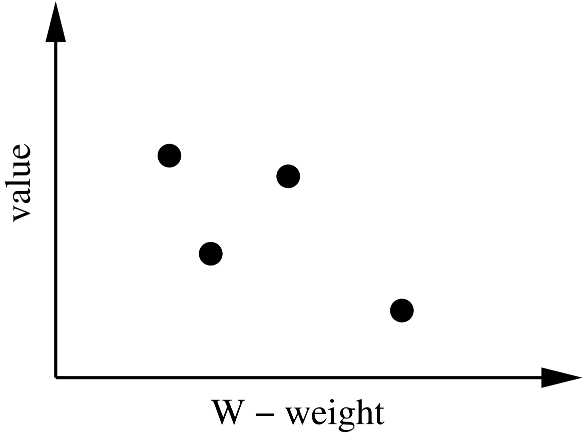

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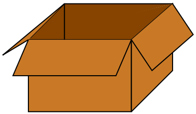

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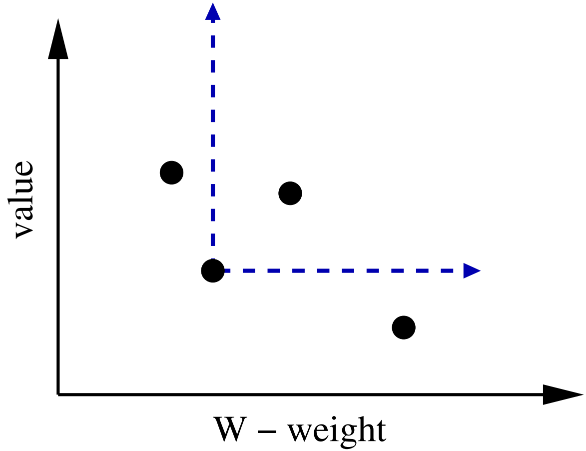

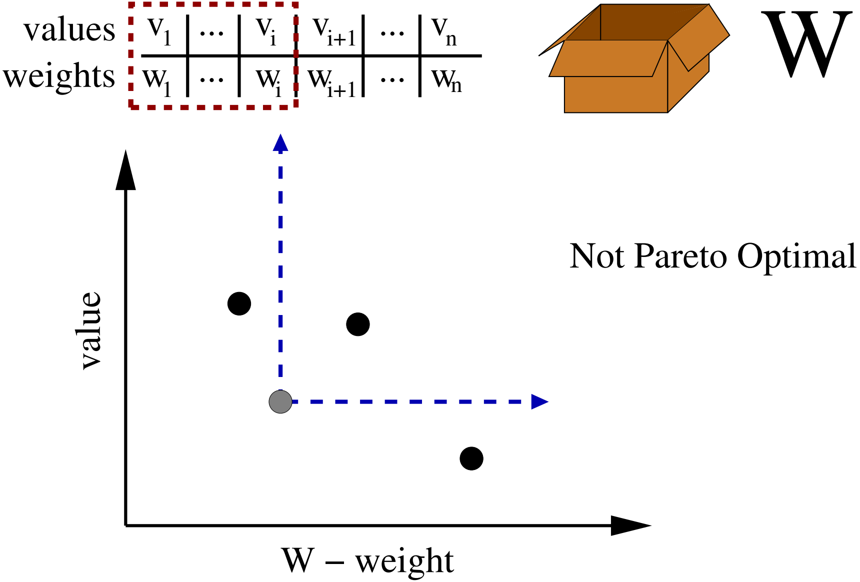
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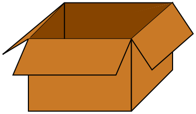

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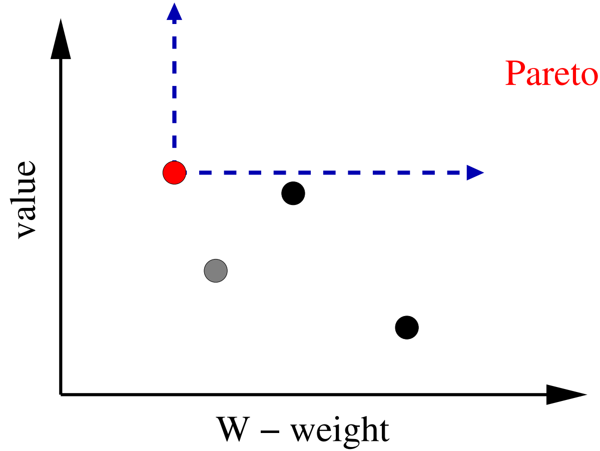
Pareto Optimal!



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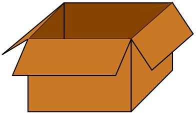

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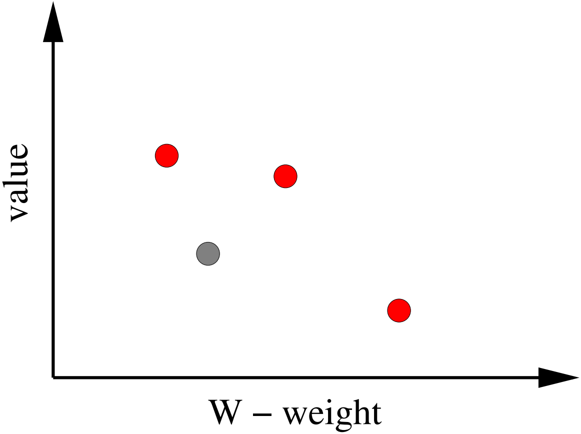

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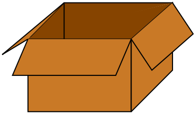

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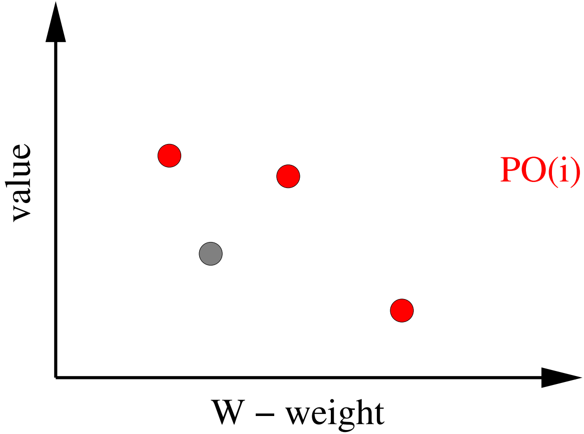

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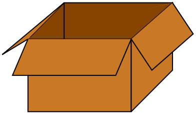

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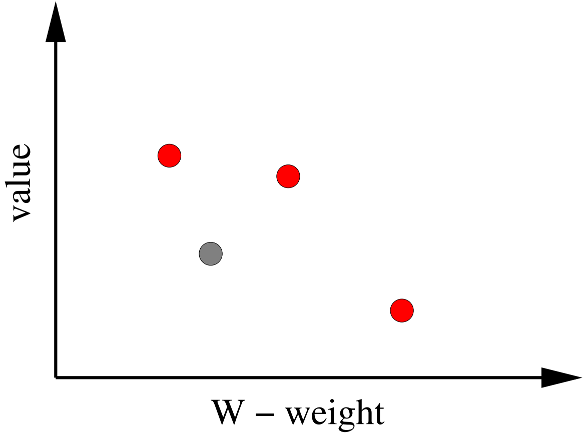

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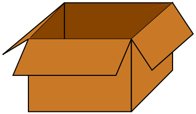

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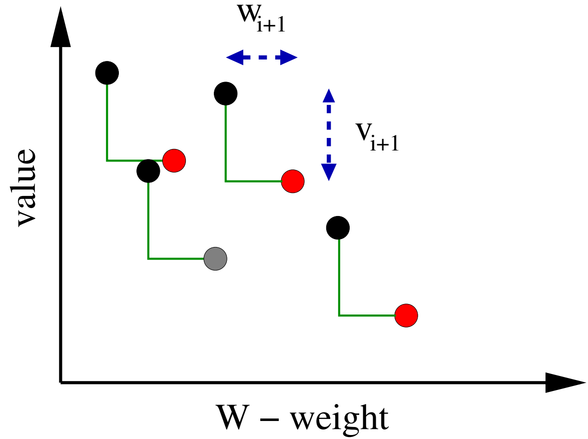

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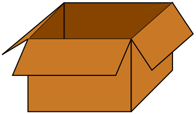

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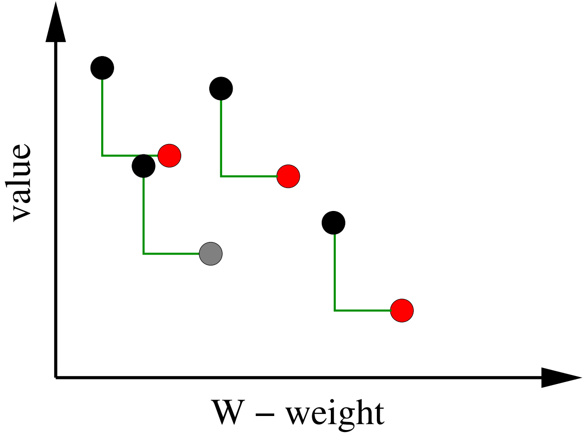

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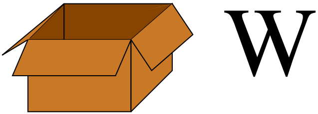

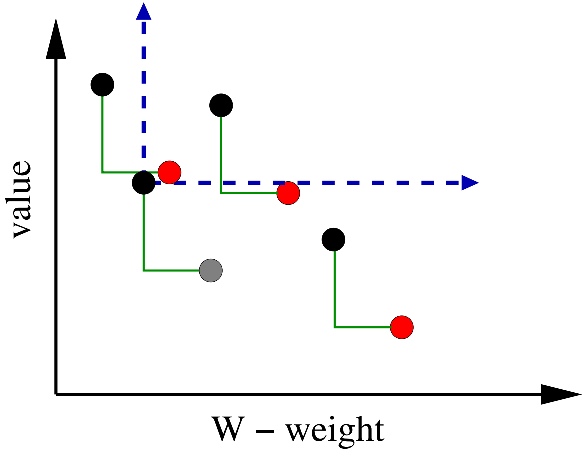

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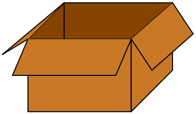

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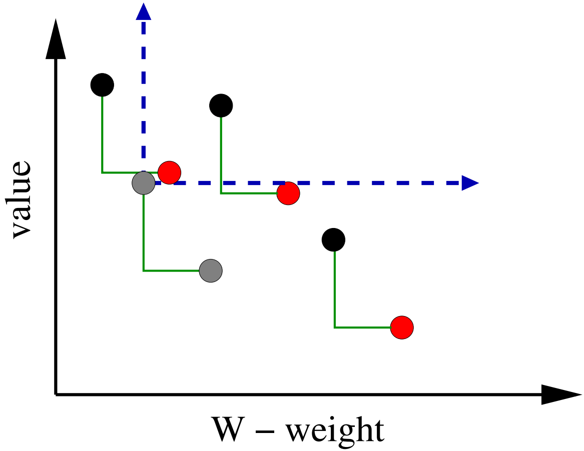

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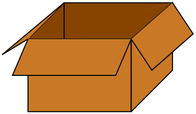

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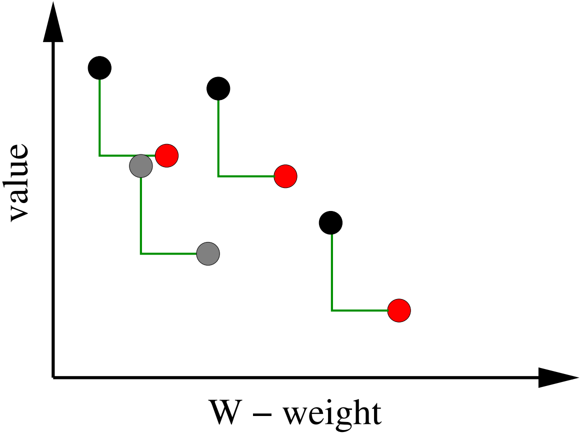

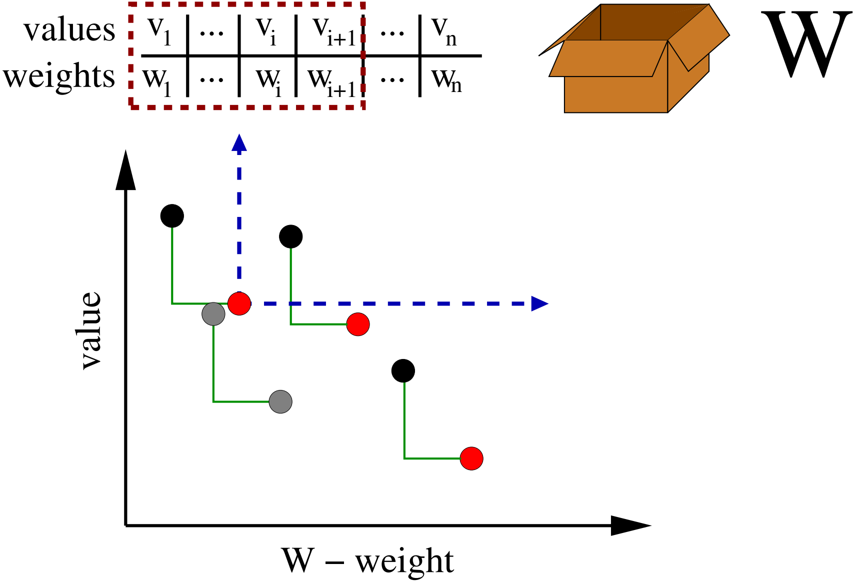
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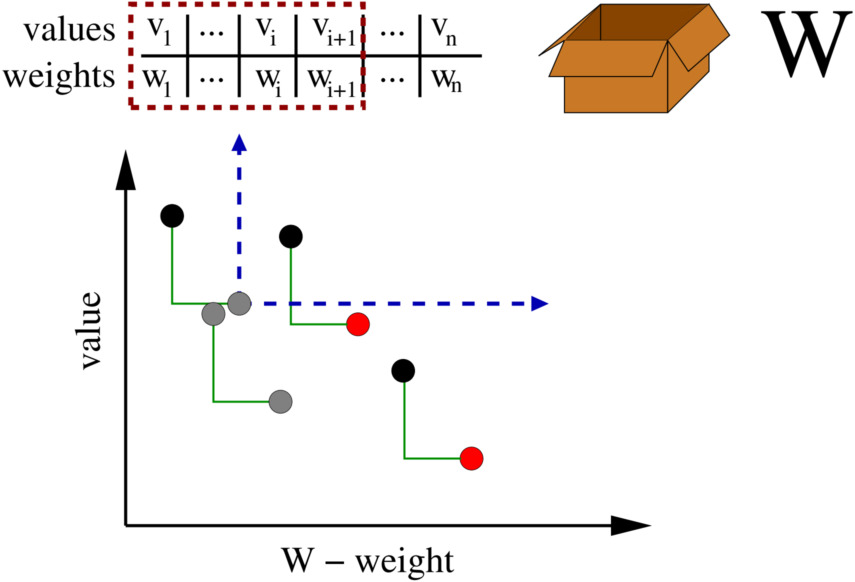

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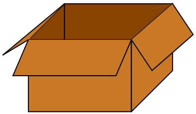

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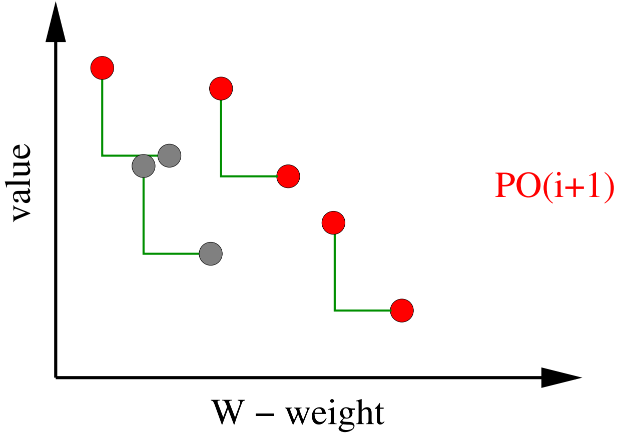

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Bottleneck in many algorithms: enumerate the set of Pareto optimal solutions

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  - generalizes long line of results on random instances

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Pareto curves capture tradeoffs among competing objectives

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Only useful approach if Pareto curves are small

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Caveat: Smoothed Analysis is not a complete explanation

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... each w_j^i is a random variable on $[-1,+1]$ – density is bounded by ϕ

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  ... where $f(d) = 2^{d-1}(d+1)$!
- [Dughmi, Roughgarden, FOCS 2010] any FPTAS can be transformed to a truthful in expectation FPTAS

Our Results

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[Bently et al, JACM 1978]: 2^n points sampled from a d-dimensional Gaussian,

$$E[|PO|] = \Theta(n^{d-1})$$

square factor difference necessary for $d = 2$

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Randomness is your friend!

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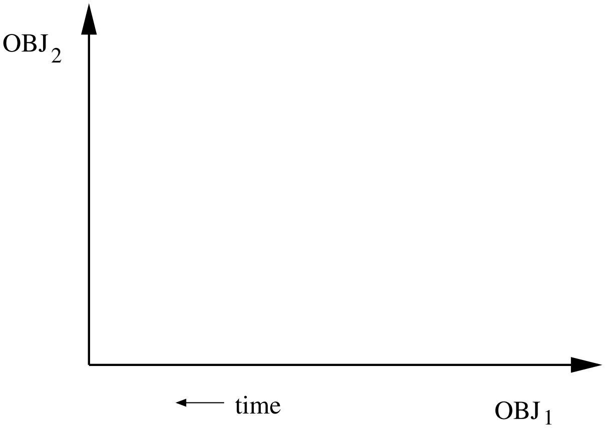
We give an algorithm to define these events

An Alternative Condition



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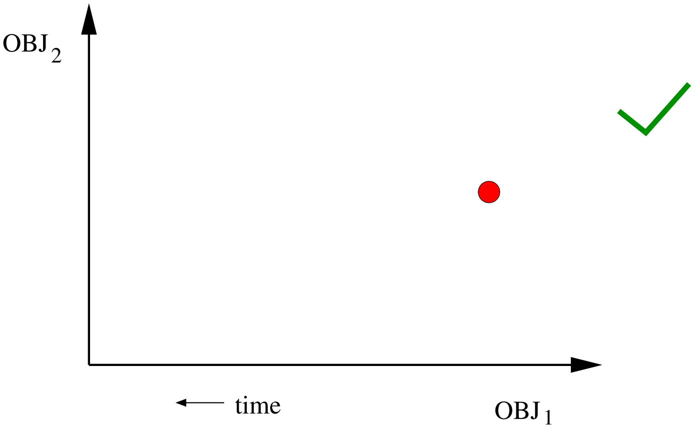

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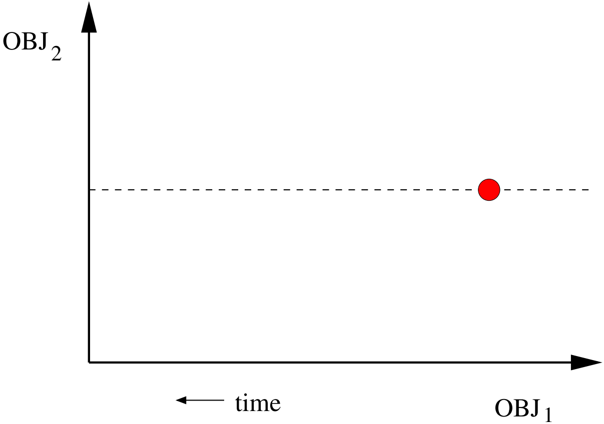

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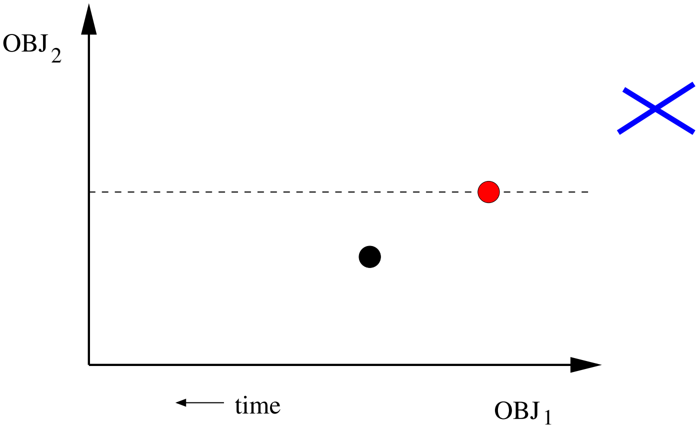

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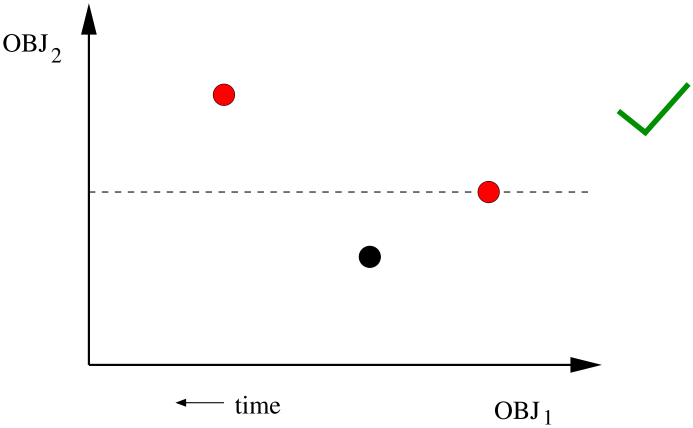

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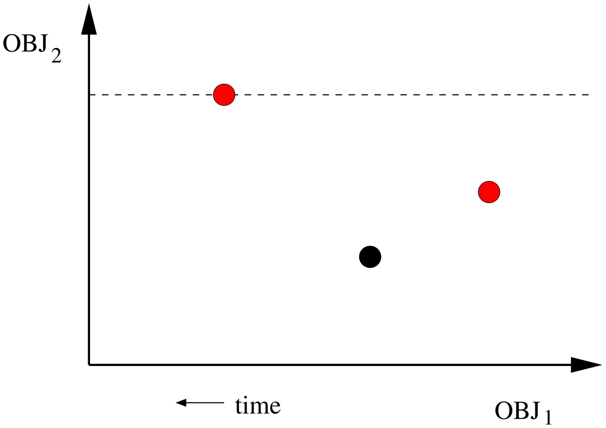

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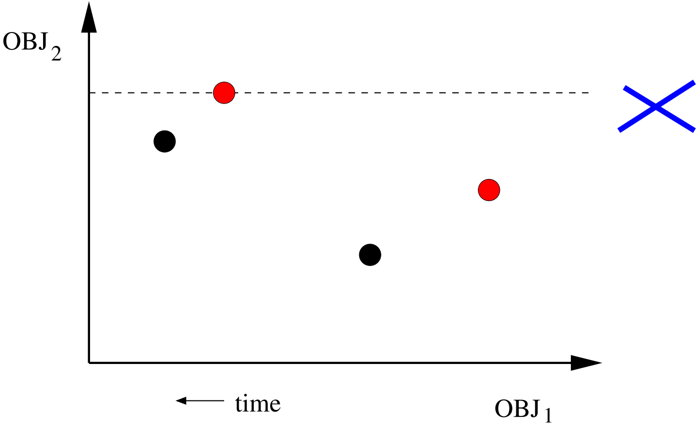

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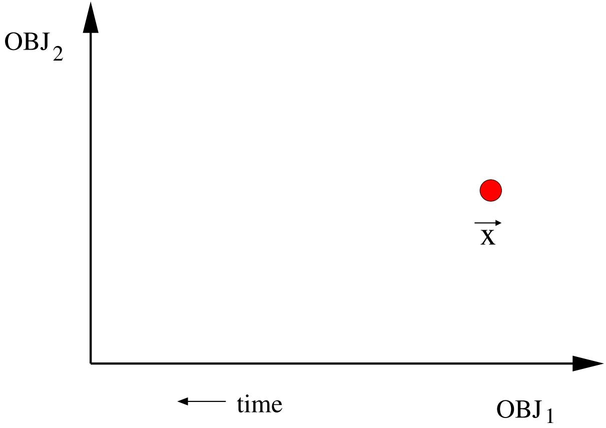
The key is in the definition

The Common Case (Beier, Röglin, Vöcking)



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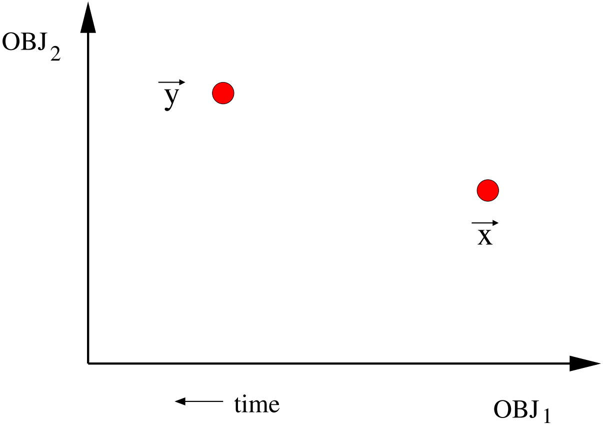

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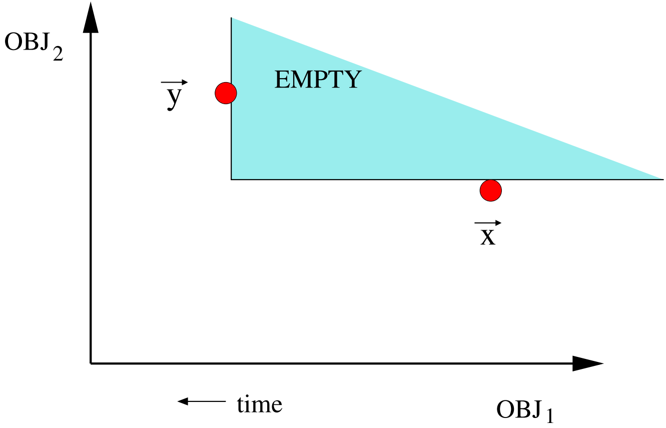

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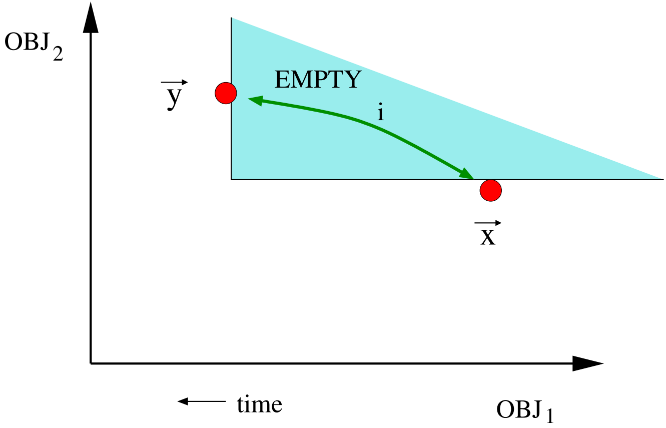

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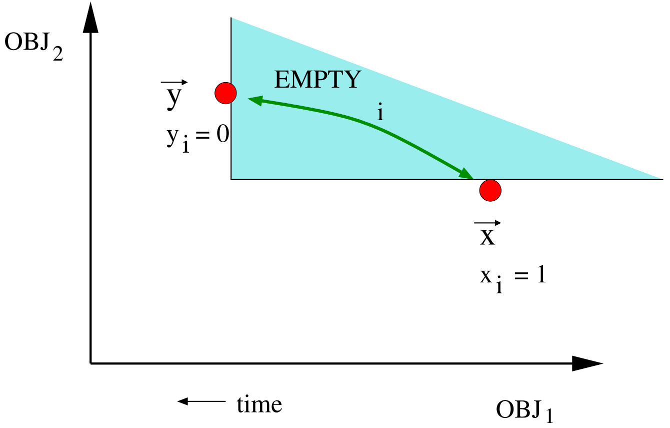

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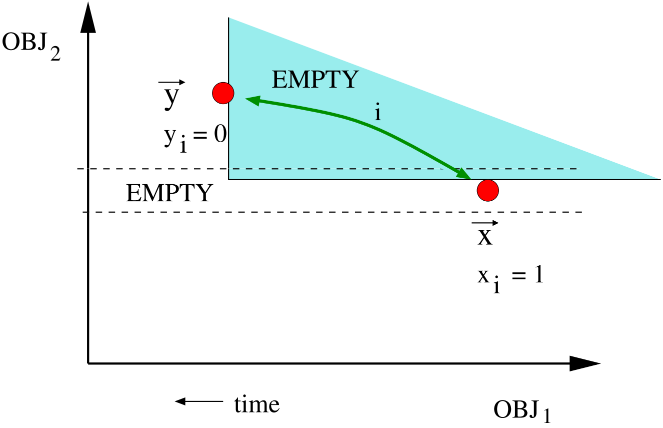

Complete Family of Events

Let $I = (a, b)$ be an interval of $[-n, +n]$ of width ϵ

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Claim

$Pr[E] \leq \epsilon\phi$

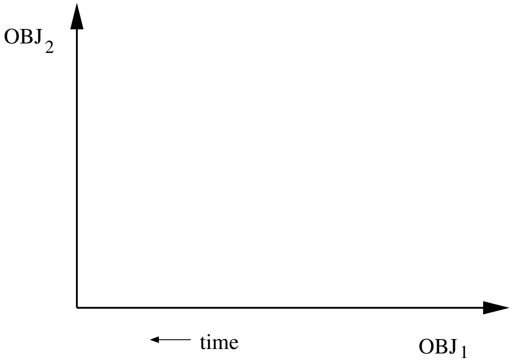
Analysis

OBJ_1 : arbitrary

OBJ_2 : [?? , ?? , ...?? ....??] $\vec{x}$

E

Z



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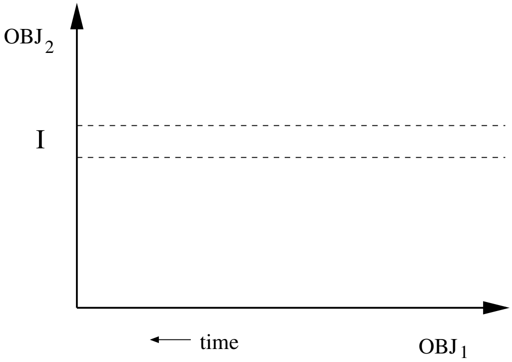


Analysis

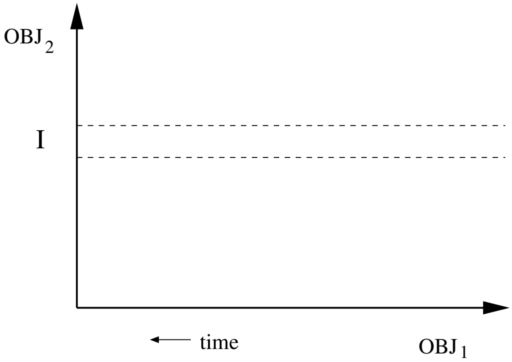
OBJ_1 : arbitrary

OBJ_2 : $[w_1, w_2, \ldots ?? \ldots w_n]\ \vec{x}$

index i

E

Z



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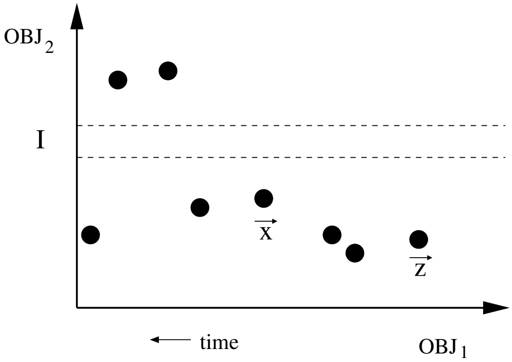

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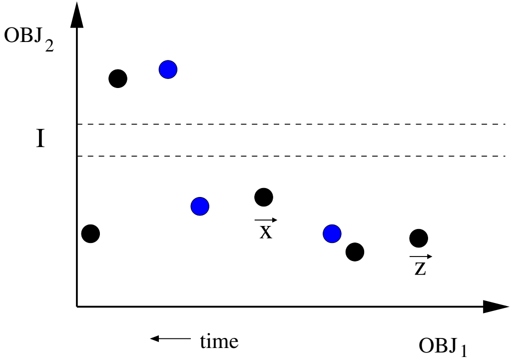
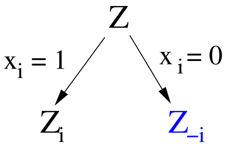


E



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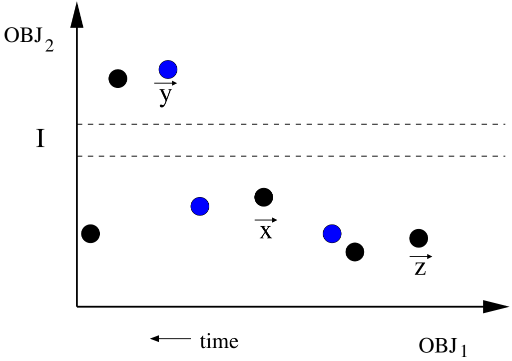

E

$$Z \xrightarrow{x_i = 1} Z_i \qquad Z \xrightarrow{x_i = 0} Z_{-i}$$

Analysis

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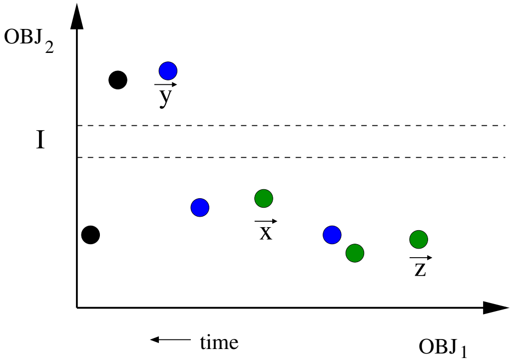

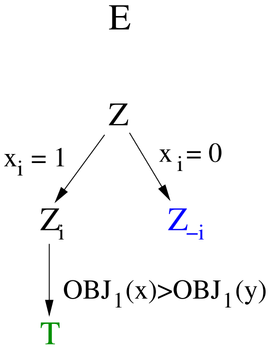

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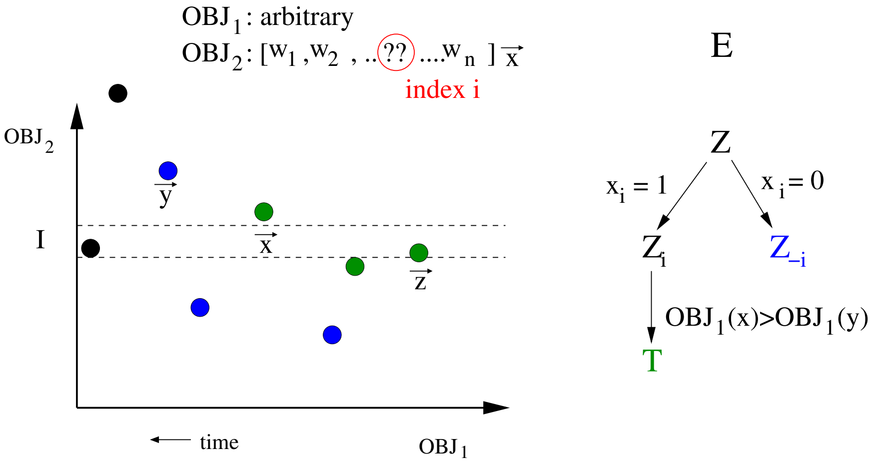
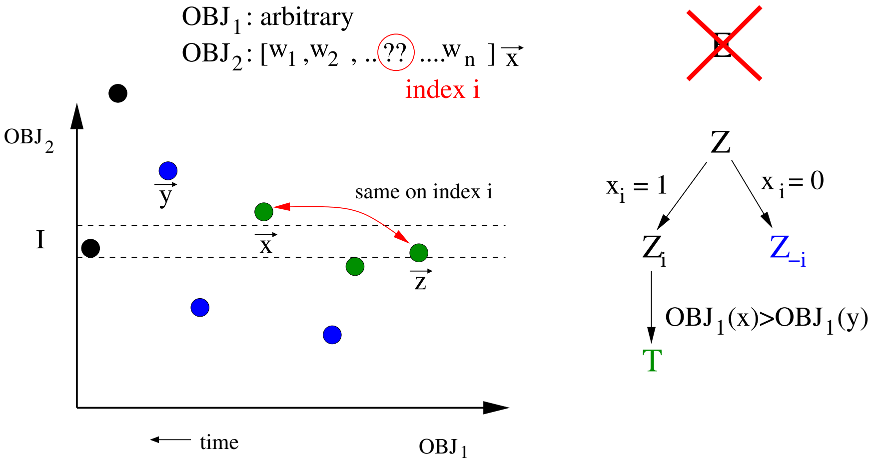


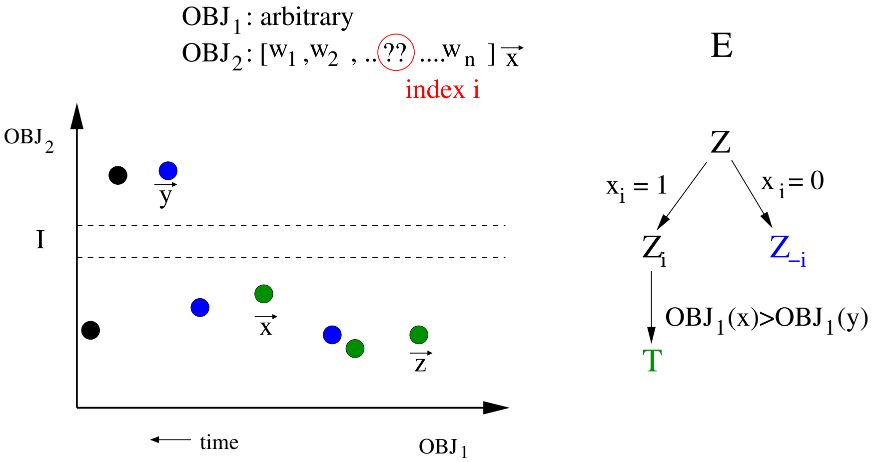
OBJ 1 : arbitrary
OBJ 2 : [w1 ,w2 , ..?? ....w n] x
index i
OBJ 2
I
y
x
z
time
OBJ 1
E
Z
x i = 1
x i = 0
Z i
Z −i
OBJ 1 (x)>OBJ 1 (y)
T

Analysis



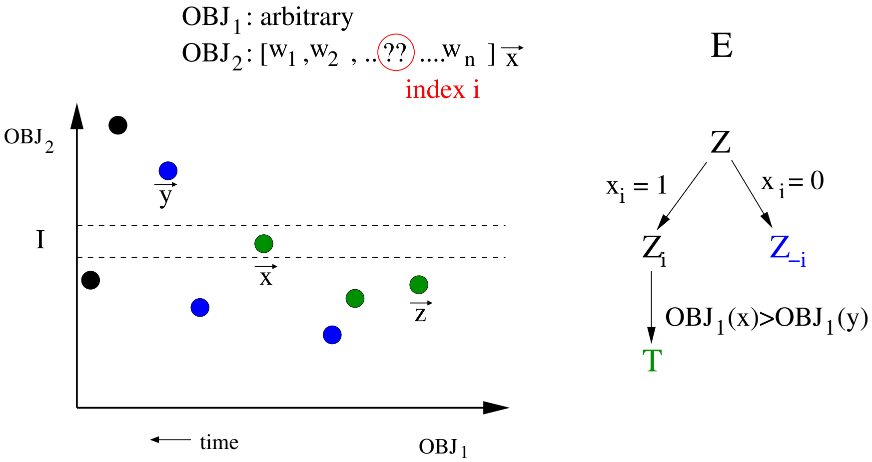
OBJ 1 : arbitrary
OBJ 2 : [w 1 ,w 2 , .. ?? ....w n] x
index i
OBJ 2
y
same on index i
I
x
z
time
OBJ 1
Z
x i = 1
x i = 0
Z i
Z −i
OBJ 1 (x)>OBJ 1 (y)
T

Analysis



OBJ 1 : arbitrary
OBJ 2 : [w1 ,w2 , ..?? ....w n] x
index i
E
OBJ 2
y
I
x
z
time
OBJ 1
Z
xi = 1
x i = 0
Zi
Z−i
OBJ 1 (x)>OBJ 1 (y)
T

Analysis



OBJ 1 : arbitrary
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index i
OBJ 2
y
I
x
z
time
OBJ 1
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Z
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Zi
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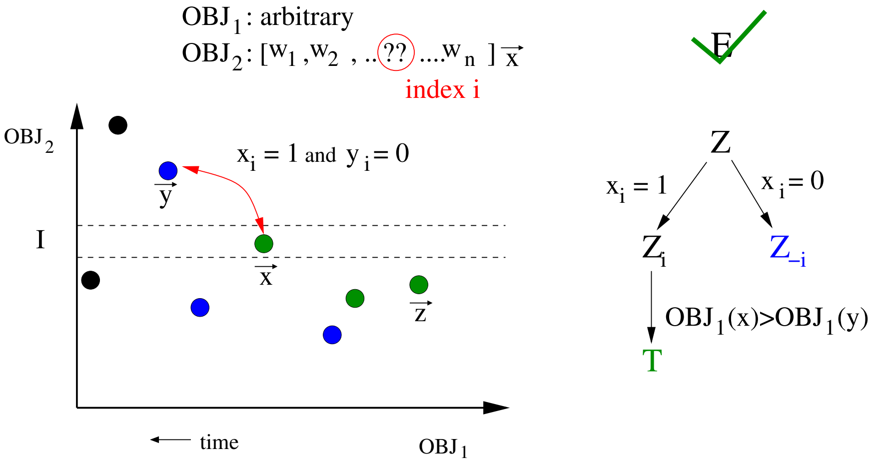
Analysis



OBJ 1 : arbitrary
OBJ 2 : [w 1 ,w 2 , ..?? ....w n] x
index i
OBJ 2
x i = 1 and y i = 0
y
I
x
z
time
OBJ 1
E
Z
x i = 1
x i = 0
Z i
Z −i
OBJ 1 (x)>OBJ 1 (y)
T

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Implicitly defined a Transcription Algorithm:

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Given a Pareto optimal solution x, which event should we blame?

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Blame event E: interval I, index i, x_i and y_i

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Blame event E: interval I, index i, x_i and y_i

Transcription Algorithm

- *Input: values of r.v.s (and Pareto optimal x)*
- *Output: interval I, index i, x_i and y_i*

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Suppose output is event E: interval I, index i, x_i and y_i

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- *Find y*

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Reverse Algorithm

- *Find y*
- *Then find x*

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Does x fall into the interval I?

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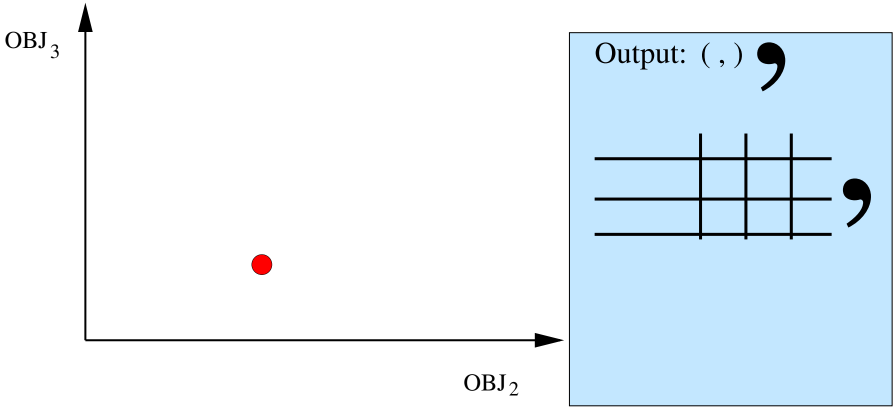
Hence each output is unlikely

Transcription for $d = 3$

OBJ_1 : adversarial

OBJ_2, OBJ_3 : linear

Z



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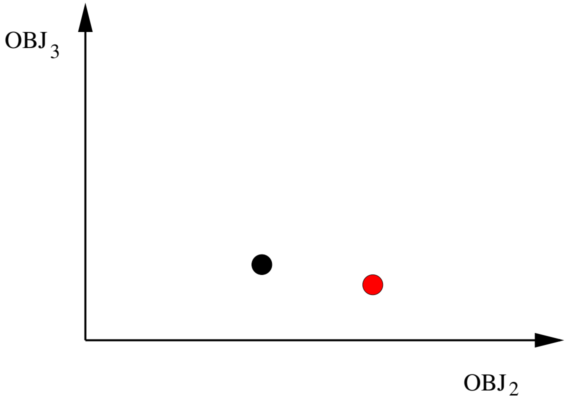

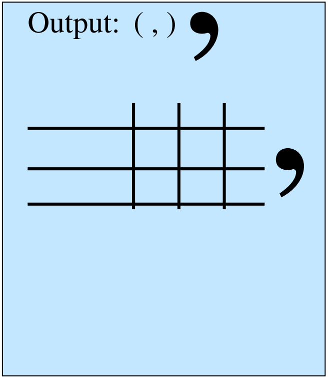

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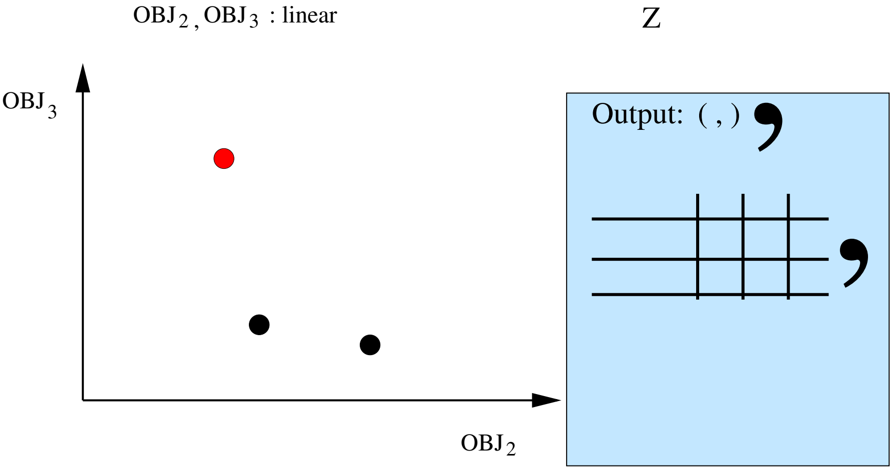

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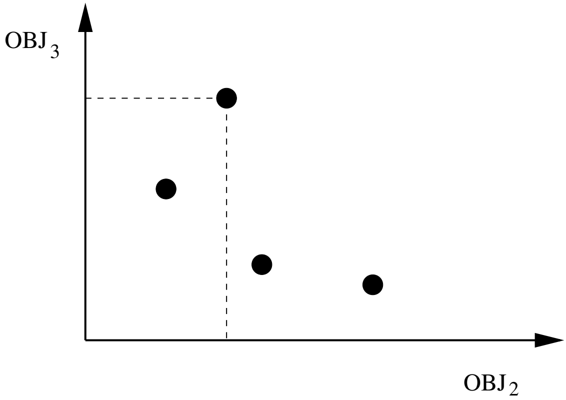

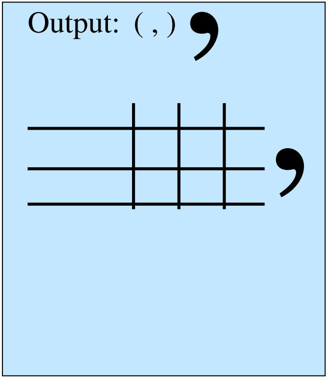

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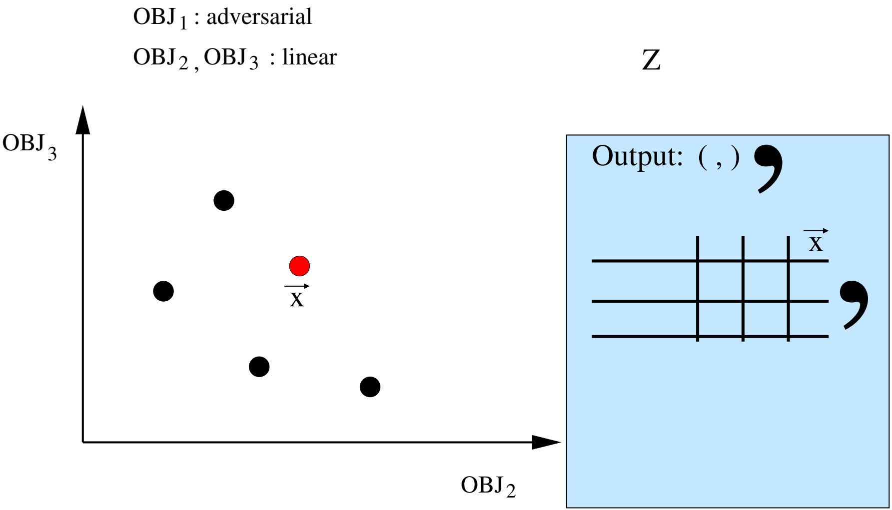

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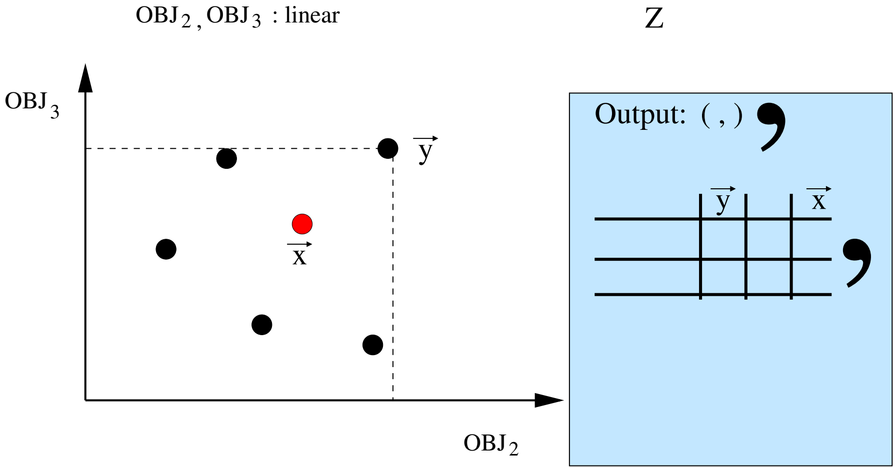

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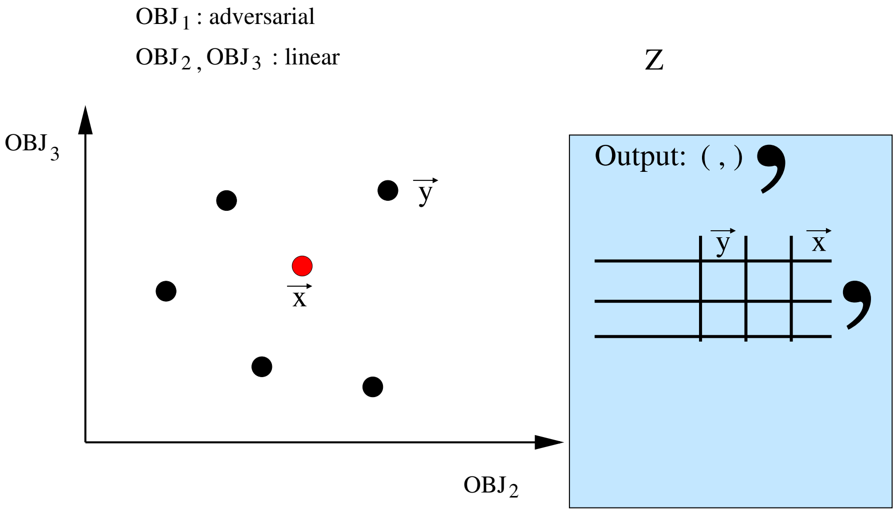

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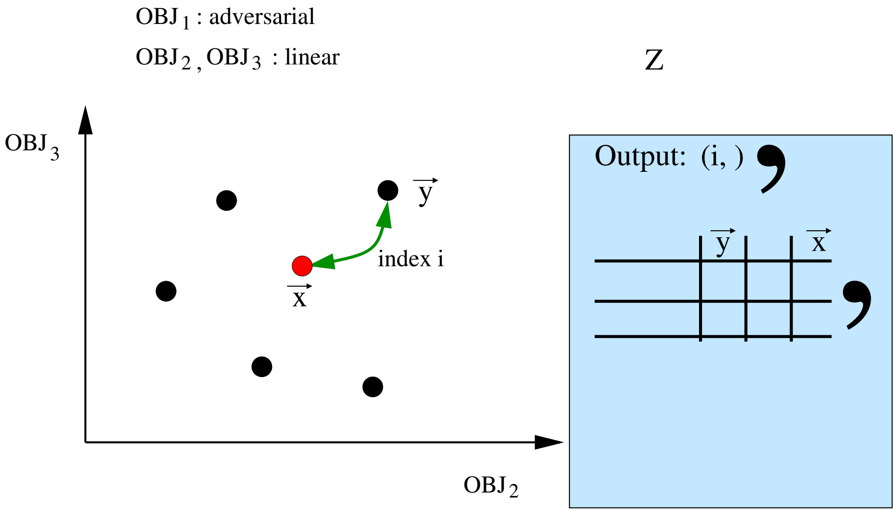

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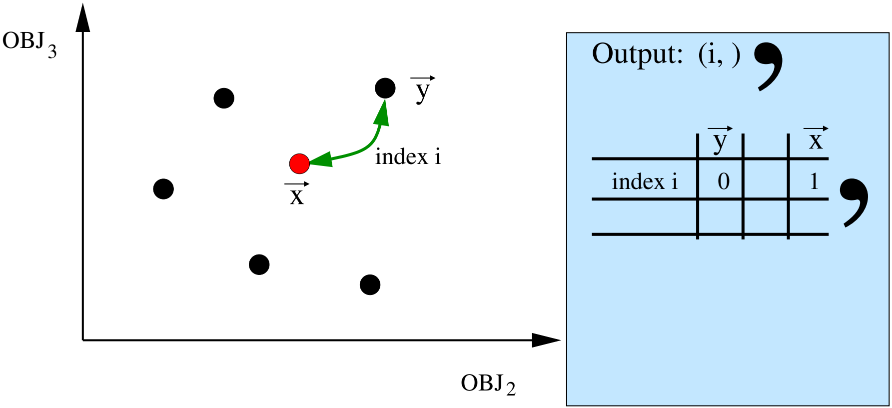

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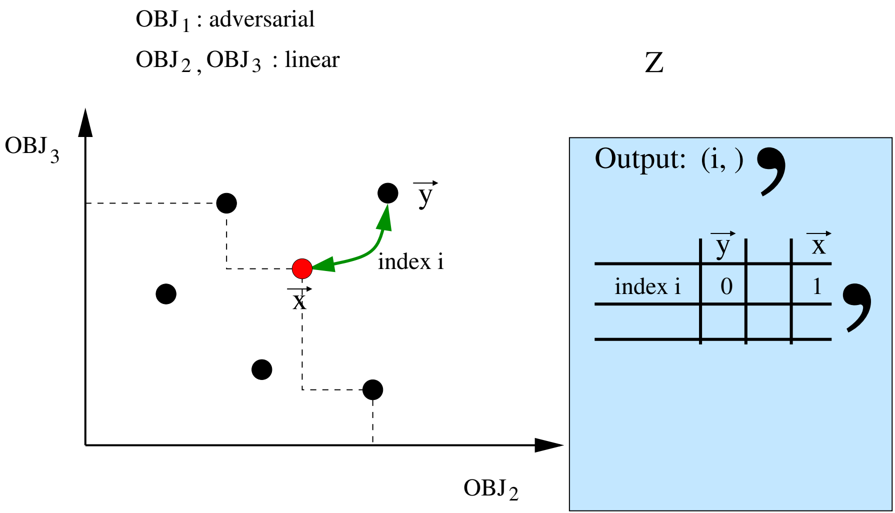

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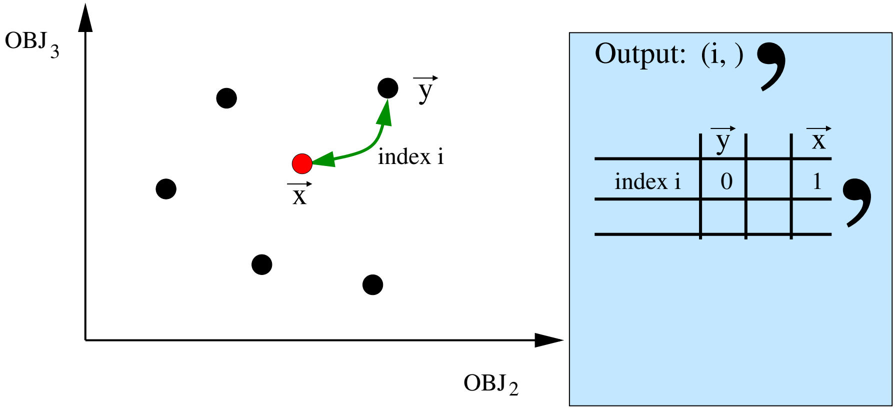

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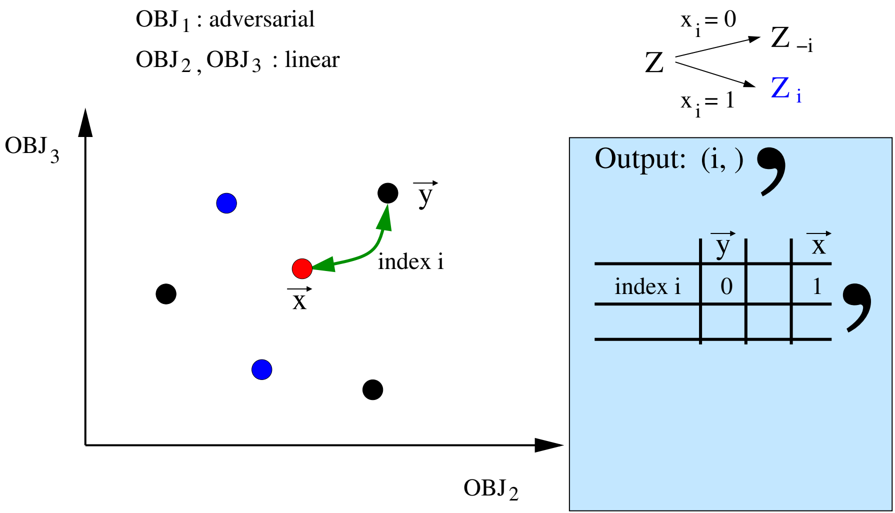

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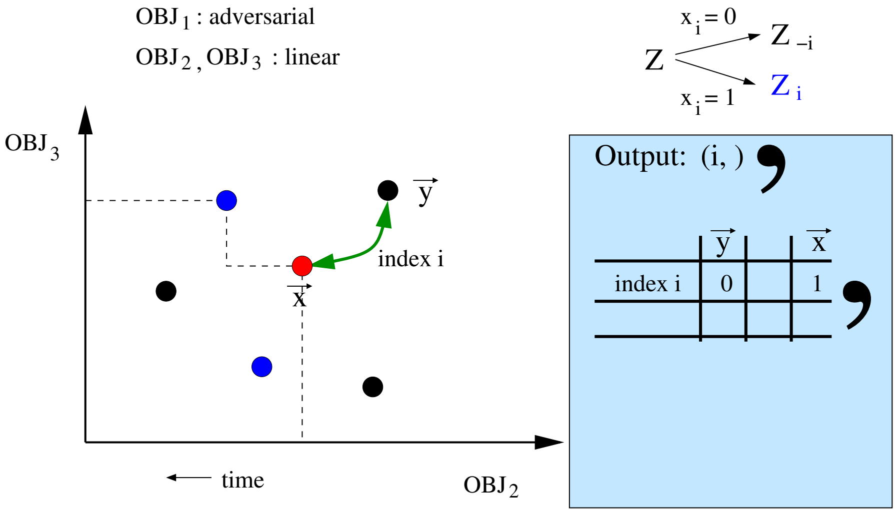

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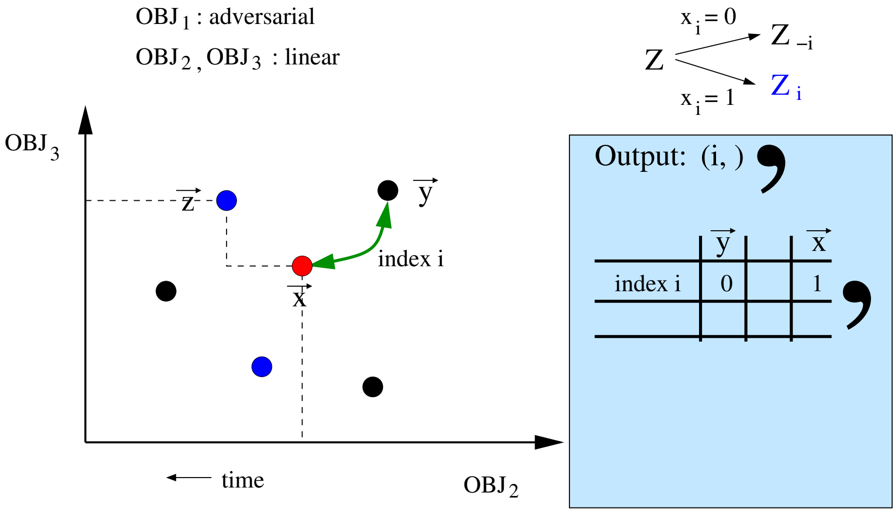

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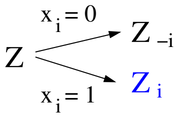

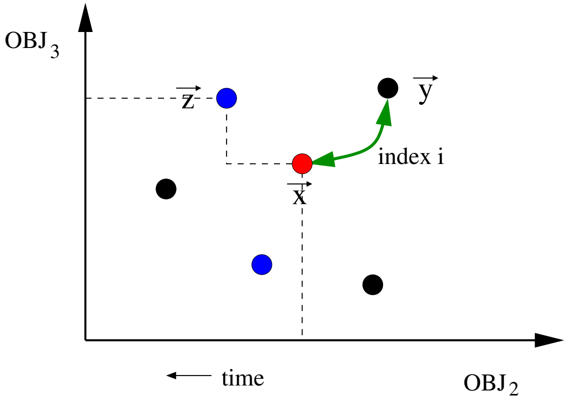

Output: (i,)

	$\vec{y}$	$\vec{z}$	$\vec{x}$
index i	0	1	1

Transcription for $d = 3$

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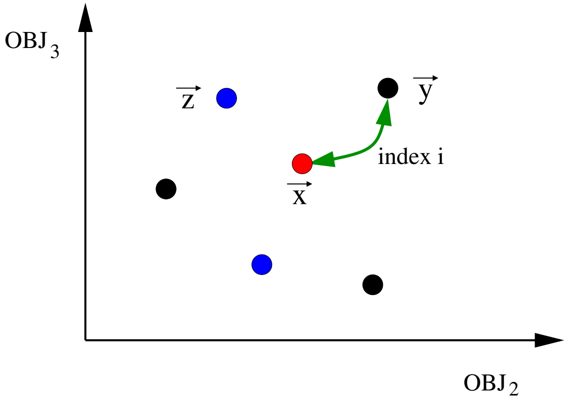
$Z \xrightarrow{x_i = 0} Z_{-i}$

$Z \xrightarrow{x_i = 1} Z_i$



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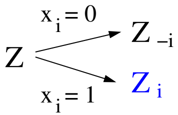

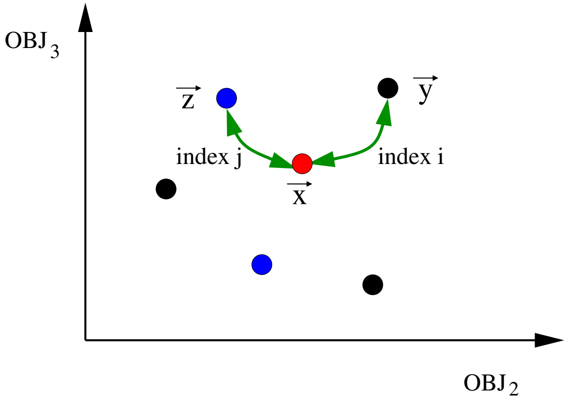

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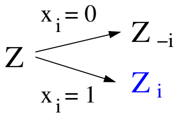

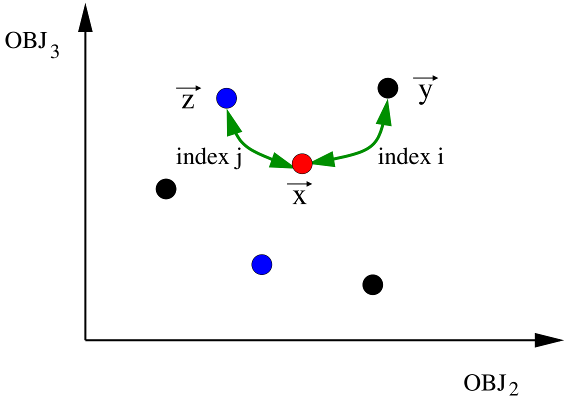

Output: (i,j)

	$\vec{y}$	$\vec{z}$	$\vec{x}$
index i	0	1	1
index j	1	0	1

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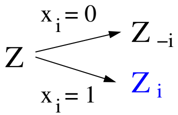

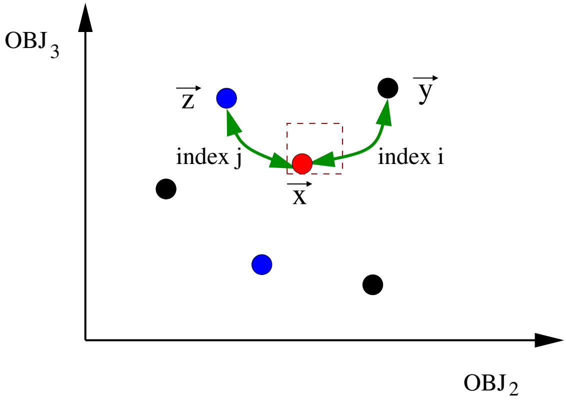

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$$Z \xrightarrow{x_i = 0} Z_{-i}$$
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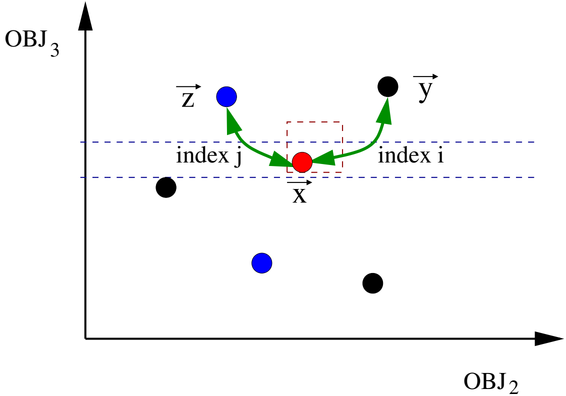

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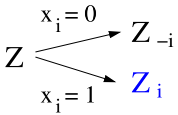

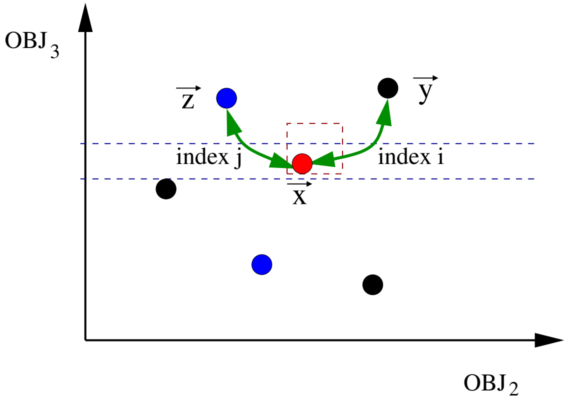

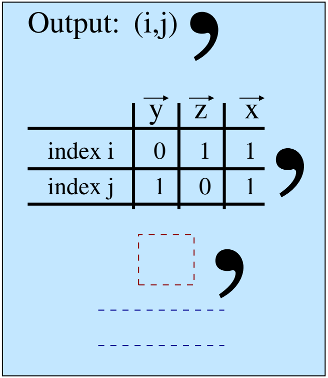

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Future Directions?

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Open Question

Are there additional applications of randomness "conservation" in Smoothed Analysis?

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e.g. simplex algorithm

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Are there additional applications of randomness "conservation" in Smoothed Analysis?

e.g. simplex algorithm

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Are there perturbation models that make sense for non-linear objectives?

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Thanks!