Planted Clique, Sum-of-Squares and Pseudo-Calibration

Ankur Moitra (MIT)

joint work with Boaz Barak, Sam Hopkins, Jon Kelner, Pravesh Kothari and Aaron Potechin

PLANTED CLIQUE

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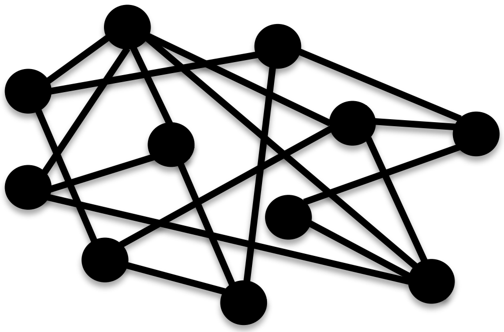

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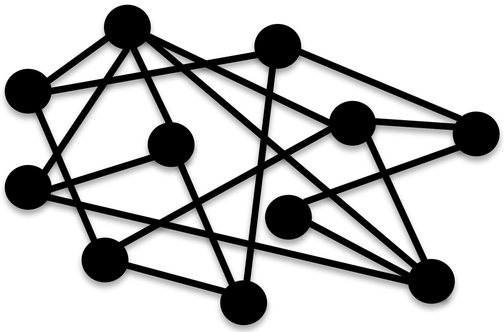

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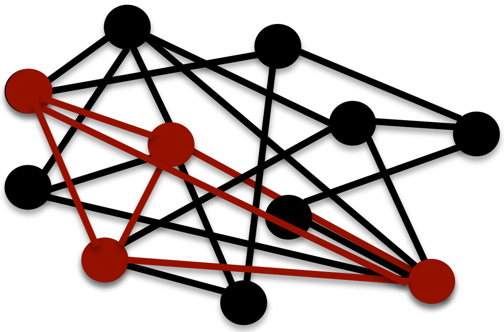

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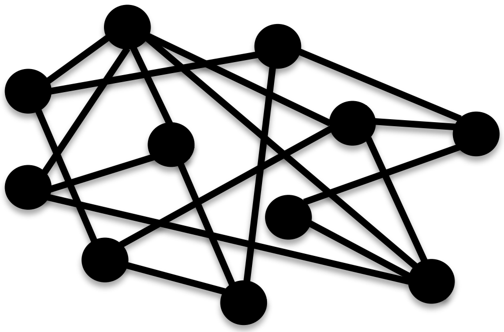

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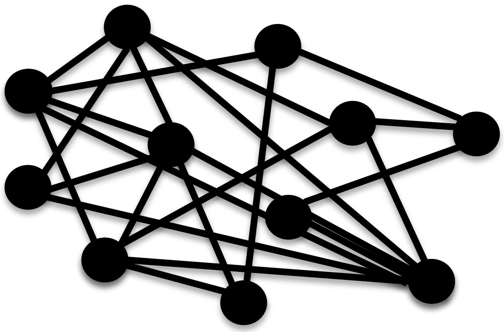

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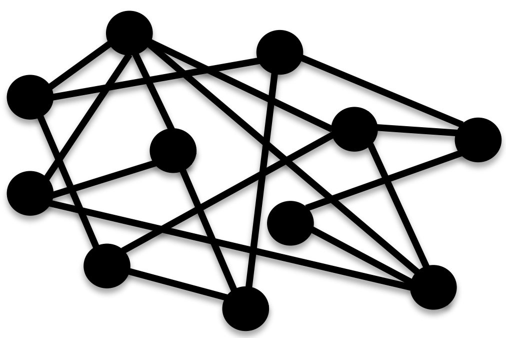

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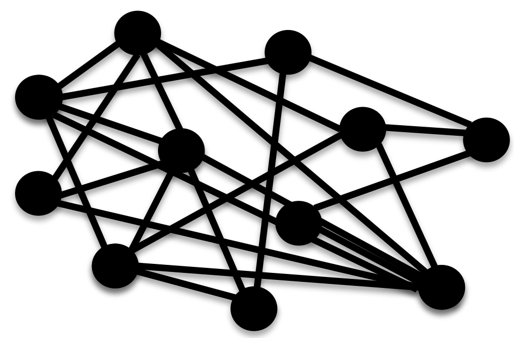

Can we find the planted clique?

And how large does ω need to be?

Quasi-polynomial time:

Fact: There is an $n^{O(\log n)}$-time algorithm (brute-force) that can find planted cliques of size $\omega \geq C \log n$, for any $C > 2$

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Theorem [Deshpande, Montanari '13]: There is a nearly linear time algorithm that succeeds (whp) for $\omega \geq \sqrt{n/e}$

APPLICATIONS OF PLANTED CLIQUE

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- Discovering motifs in biological networks **[Milo et al '02]**
- Computing the best Nash Equilibrium **[HK '11]**, **[ABC '13]**
- Property testing **[Alon et al '07]**
- Sparse PCA **[Berthet, Rigollet '13]**
- Compressed sensing **[Koiran, Zouzias '14]**
- Cryptography **[Juels, Peinado '00]**, **[Applebaum et al '10]**
- Mathematical finance **[Arora et al '10]**

LOWER BOUNDS?

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Our best evidence seems to come from **hierarchies**…

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SUM-OF-SQUARES HIERARCHY

Powerful hierarchy of semidefinite programs, introduced by **[Shor ‘87]**, **[Nesterov ‘00]**, **[Parrilo ‘00]**, **[Lasserre ‘01]**

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Powerful hierarchy of semidefinite programs, introduced by **[Shor '87]**, **[Nesterov '00]**, **[Parrilo '00]**, **[Lasserre '01]**

Goal: Find operator that behaves like the expectation over a distribution on solutions

$$\widetilde{\mathbb{E}} : \underbrace{\mathcal{P}_n^{\leq d}}_{\text{degree } \leq d \text{ polynomials in } n \text{ variables}} \to \mathbb{R}$$

Called a **Pseudo-expectation**

Constraints on the pseudo-expectation:

(1) $\widetilde{\mathbb{E}}$ is linear

(2) $\widetilde{\mathbb{E}}[1] = 1$

(3) $\widetilde{\mathbb{E}}[p^2] \geq 0$

for all deg(p) ≤ d/2

$\underbrace{\hphantom{\text{for all deg(p) ≤ d/2 xxxxxx}}}_{\textbf{general}}$

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(**clique size**)

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specific to planted clique

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E.g. if a_1, a_2, … a_n is the indicator vector of an ω-sized clique

$$\widetilde{\mathbb{E}}[p(x_1, x_2, ...x_n)] = p(a_1, a_2, ...a_n)$$

meets (1) – (6)

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There is an $n^{O(d)}$-time algorithm for finding such an operator, if it exists

Called the level d **Sum-of-Squares Algorithm**

- strengthens **Sherali-Adams**, **Lovasz-Schrijver**, **LS+**
- breaks integrality gaps for other hierarchies **[Barak et al, '12]**
- highly successful convex relaxation

  sparsest cut **[ARV '04]**
  unique games **[ABS '10]**, **[BRS '12]**, **[GS '12]**
- optimal among all poly. sized SDPs for random CSPs **[LRS '15]**
- best known algorithm for several **average-case** problems

  planted sparse vector, dictionary learning **[BKS '14, '15]**
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Can it find n^{ε}-sized planted cliques in polynomial time?

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OUR RESULTS

We show a nearly optimal lower bound against SoS, for the planted clique problem:

Theorem [Barak, Hopkins, Kelner, Kothari, Moitra, Potechin]:
The integrality gap of the level d Sum-of-Squares hierarchy is

$$n^{\frac{1}{2}-c\sqrt{d/\log n}}$$

for some constant c > 0

For any d = o(log n), the integrality gap is $n^{1/2-o(1)}$

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Improves upon [Meka, Potechin, Wigderson ‘14], [Deshpande Montanari ‘15], [Hopkins, Kothari, Potechin, Raghavendra, Scrhamm ‘16]

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When our *recipe* fails, does it immediately yield algorithms?

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Theorem [Feige, Krauthgamer '03]: The integrality gap of the level d LS+ hierarchy is

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$$\widetilde{\mathbb{E}}_{MPW}[\overbrace{\prod_{i \in A} x_i}^{x_A}] = 2^{\binom{|A|}{2}} \left(\frac{\omega}{n}\right)^{|A|}$$

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Approach: Spectral bounds on **locally random matrices**

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But these bounds are *tight* (for *these* moments)

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Set $G_{i,j} = \begin{cases} +1 & \text{if } (i,j) \text{ an edge} \\ -1 & \text{else} \end{cases}$

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If there is an ω-sized planted clique:

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Need: $\omega \leq n^{1/(\ell+1)} = n^{1/(d/2+1)}$ otherwise something is wrong

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Intuition: A good pseudo-expectation attempts to **hide** info about what vertices participate in the planted clique

But vertices with a **standard deviation higher degree**, should be a constant factor more likely to be in the p.c. (**soft constraint**)

FIXING THE MPW-MOMENTS

This family of polynomials is essentially the only thing that goes wrong at d = 4

Theorem [Hopkins et al ’16], [Raghavendra, Schramm ‘16]:
The integrality gap of the level 4 Sum-of-Squares hierarchy is

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36 pgs → 40 pgs → 26 pgs → 69 pgs → ??? pgs

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PSEUDO-CALIBRATION

Can we find pseudo-moments that satisfy the following:

$$\mathop{\mathbb{E}}_{G \leftarrow G(n,1/2)}[\widetilde{\mathbb{E}}[f(G,x)]] = \mathop{\mathbb{E}}_{(G,x) \leftarrow G(n,1/2,\omega)}[f(G,x)]$$

for all *simple* functions f?

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for all polynomials f that are low-degree in $G_{i,j}$'s and x_i's?

Consider the pseudo-expectation of some monomial:

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We can write any such function in terms of its **Fourier expansion**

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How should we set the Fourier coefficients?

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we can calculate:

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$$\mathbb{E}_{G \leftarrow G(n,1/2)}[\widetilde{\mathbb{E}}[x_A]\chi_T(G)]$$ (by linearity)

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It turns out , we need to **truncate** but at what degree?

TRUNCATION

Our pseudo-moments are:

$$\widetilde{\mathbb{E}}[x_A] = \sum_{\substack{T \subseteq \binom{[n]}{2} \\ |V(T) \cup A| \leq \tau}} \left(\frac{\omega}{n}\right)^{|V(T) \cup A|} \chi_T(G)$$

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(2) Is small enough $n^{-\Omega(\epsilon)}$ for any $\omega \leq n^{1/2-\epsilon}$ for $\tau \leq \frac{\epsilon}{2} \log n$

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(3) Can always renormalize pseudo-expectation so $\widetilde{\mathbb{E}}[1] = 1$

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(4) Similar bound holds (again by standard concentration) for

$$\widetilde{\mathbb{E}}[\sum_i x_i] = \omega(1 \pm n^{-\Omega(\epsilon)})$$

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This is why we use $|V(T) \cup A| \leq \tau$ for truncation

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Lemma: Let $f_G(x) = \sum_{|S| \leq 2d} c_A(G) x_A$ where $\deg(c_A) \leq \tau$, then

$$\mathop{\mathbb{E}}_{G \leftarrow G(n,1/2)}[\widetilde{\mathbb{E}}[f_G(x)]] = \mathop{\mathbb{E}}_{(G,x) \leftarrow G(n,1/2,\omega)}[f_G(x)]$$

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- Planted Clique and its Applications
- The Sum-of-Squares Hierarchy
- Our Results

Part II: Fooling SOS

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- Kelner's Polynomial, and Corrections at d = 4
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As is standard, it amounts to proving a certain matrix is PSD, whose entries are:

$$\mathcal{M}(I,J) = \sum_{\substack{T \subseteq \binom{[n]}{2} \\ |V(T) \cup I \cup J| \leq \tau}} \left(\frac{\omega}{n}\right)^{|V(T) \cup I \cup J|} \chi_T(G)$$

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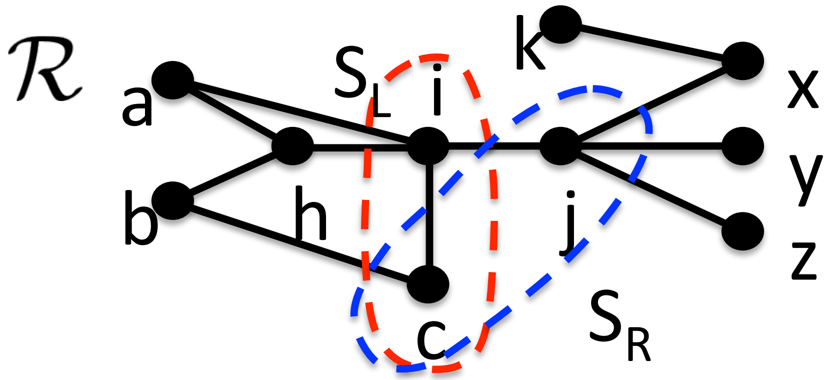
Goal: Write $\mathcal{M}$ as:

$$\mathcal{M} \approx \sum_{k} \mathcal{L}_k \mathcal{Q}_k \mathcal{L}_k^{+}$$

size of minimum vertex separator of T, btwn I and J

RIBBON DECOMPOSITION

We call such graphs **(I,J)-Ribbons**, e.g.



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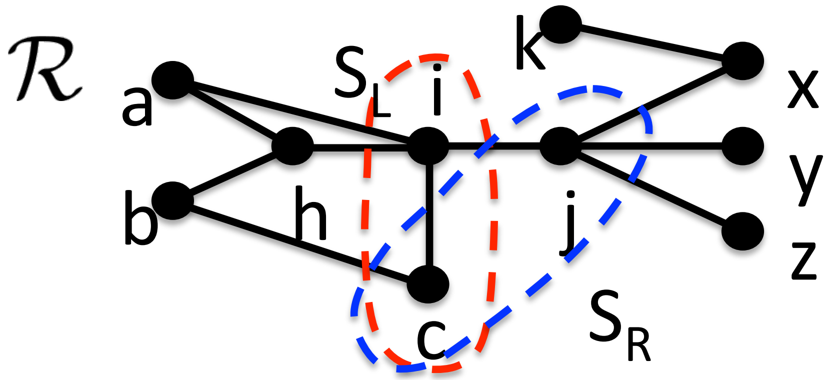

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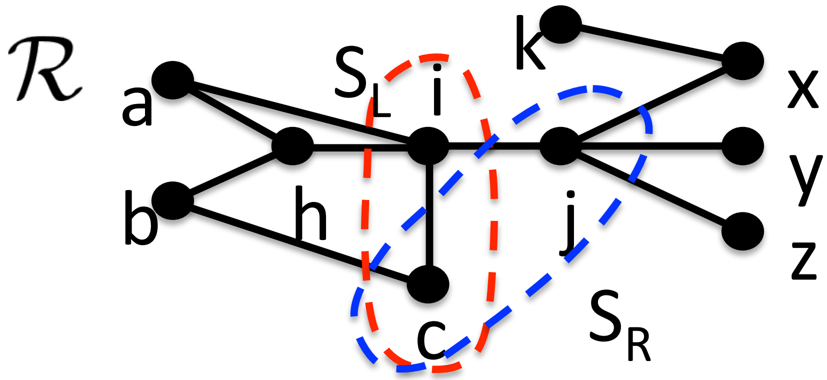

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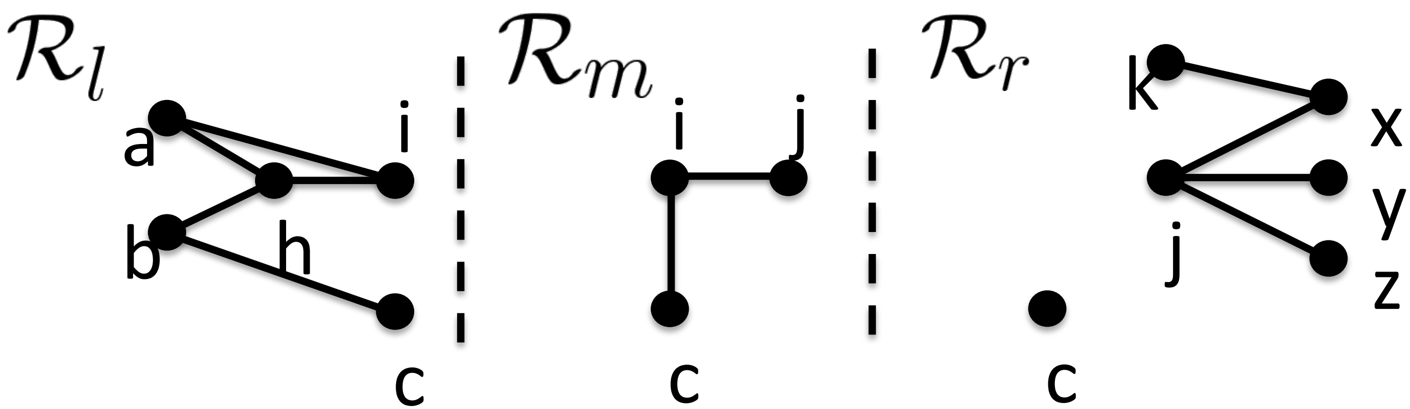

SYMBOLIC FACTORIZATION

Now we can write:

$$\mathcal{M}(I,J) \approx \text{ sum over k of}$$

$$\underbrace{\Big(\sum_{\text{valid }\mathcal{R}_l}\left(\frac{\omega}{n}\right)^{|V(\mathcal{R}_l)|}\Big)}_{\mathcal{L}_k}\underbrace{\Big(\sum_{\text{valid }\mathcal{R}_m}\left(\frac{\omega}{n}\right)^{|V(\mathcal{R}_m)|-2k}\Big)}_{\mathcal{Q}_k}\underbrace{\Big(\sum_{\text{valid }\mathcal{R}_r}\left(\frac{\omega}{n}\right)^{|V(\mathcal{R}_r)|}\Big)}_{\mathcal{L}_k^T}$$

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Major issue: $\mathcal{R}_l, \mathcal{R}_m, \mathcal{R}_r$ were assumed to be **disjoint** except at S_L, S_R, $I \cap J$ which leads to substantial **error terms**

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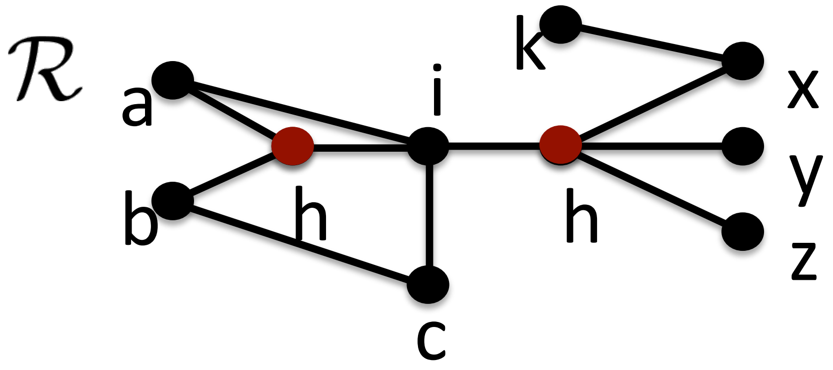
Idea: Keep iterating the decomposition, carefully charging

ITERATING THE DECOMPOSITION

Suppose h = j



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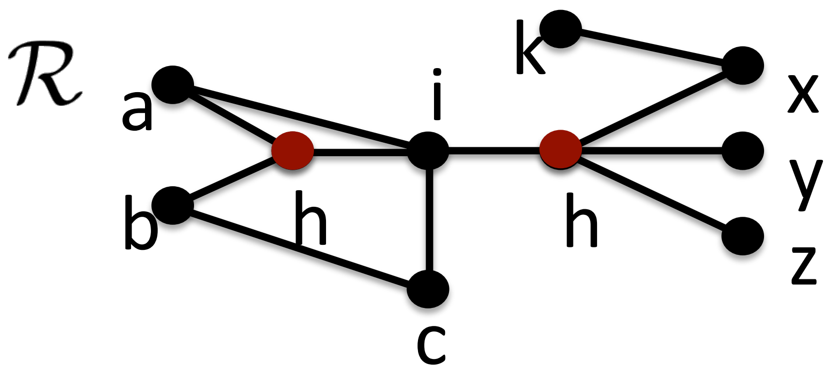
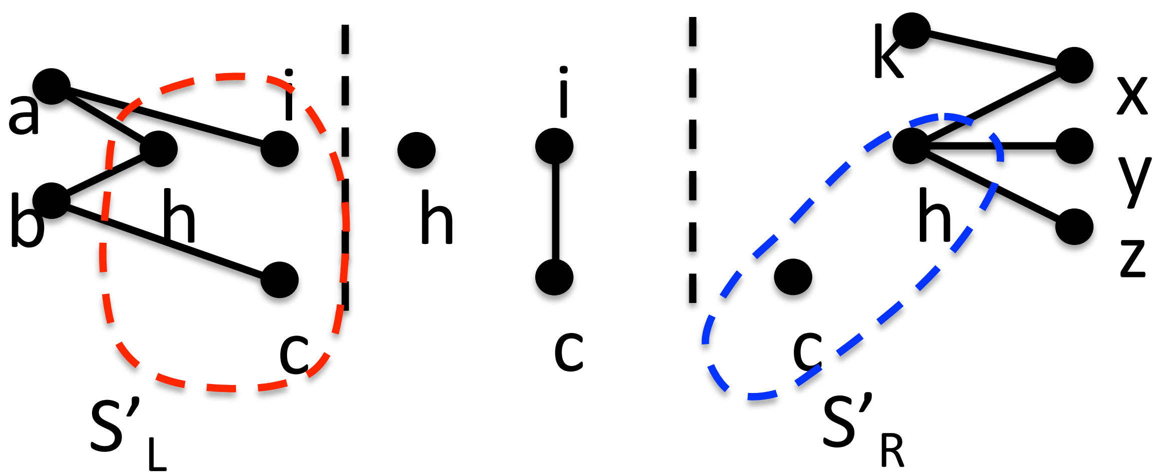


Look for new leftmost, rightmost separators that separate I from J and intersection vertices



THE MAIN CHARGING ARGUMENT

Complications:

(1) Vertices can become isolated

(2) Separators not necessarily equal size

(3) Need to sum over all pre-images of ribbons, their contributions

Main Tradeoff Lemma: There is a way to tradeoff all these parameters, to charge error terms

Summary:

- Nearly optimal lower bounds against SoS, for the planted clique problem
- **Pseudo-calibration** as a recipe for constructing good pseudo-moments
- When the recipe fails, are there algorithms?
- Connections between **SoS-evidence** and **BP-evidence**?

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Thanks! Any Questions?