Robust Statistics, Revisited

Ankur Moitra (MIT)

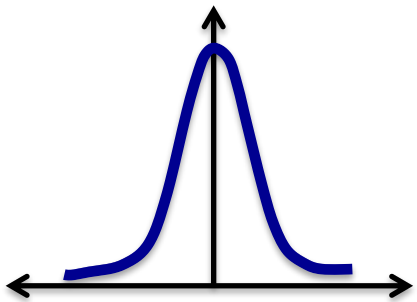
joint work with Ilias Diakonikolas, Jerry Li, Gautam Kamath, Daniel Kane and Alistair Stewart

CLASSIC PARAMETER ESTIMATION

Given samples from an unknown distribution in some *class*

e.g. a 1-D Gaussian

$$\mathcal{N}(\mu, \sigma^2)$$



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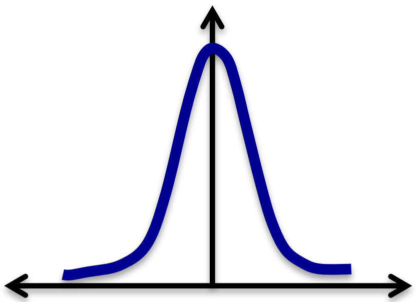

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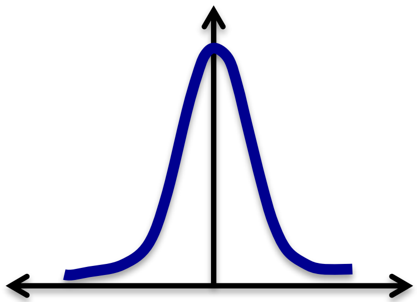

can we accurately estimate its parameters? Yes!

empirical mean:

$$\frac{1}{N}\sum_{i=1}^{N} X_i \to \mu$$

empirical variance:

$$\frac{1}{N}\sum_{i=1}^{N} (X_i - \overline{X})^2 \to \sigma^2$$

R. A. Fisher

The **maximum likelihood estimator** is asymptotically efficient (1910-1920)

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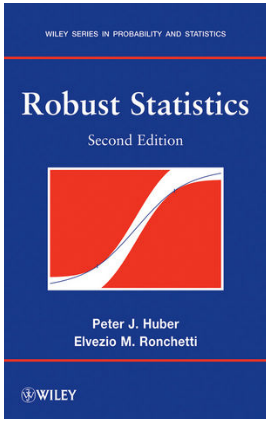
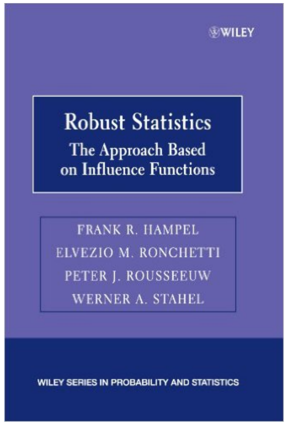
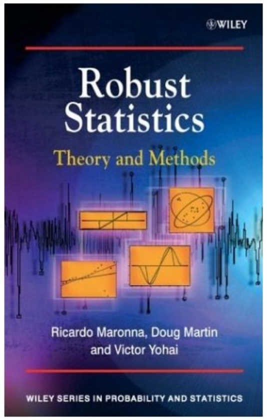
J. W. Tukey

What about **errors** in the model itself? (1960)

ROBUST STATISTICS







What estimators behave well in a **neighborhood** around the model?

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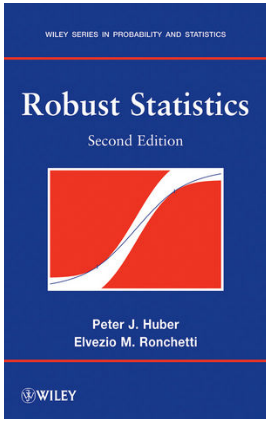
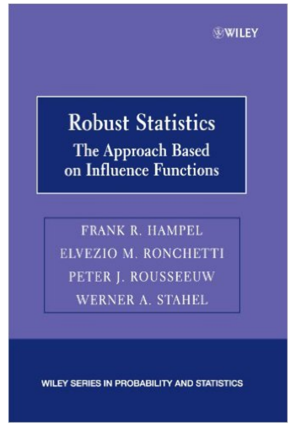
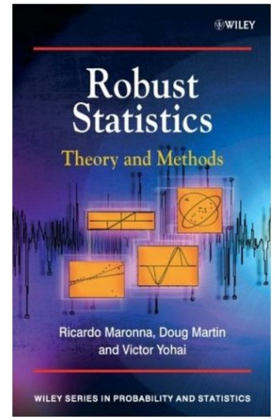




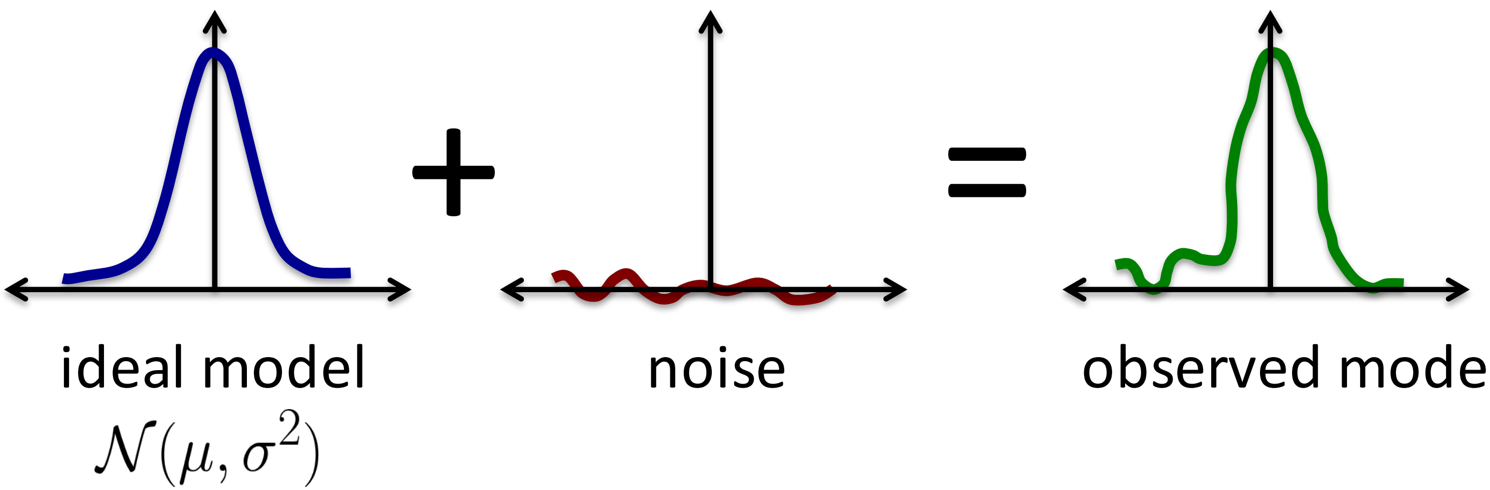


What estimators behave well in a **neighborhood** around the model?

Let's study a simple one-dimensional example….

ROBUST PARAMETER ESTIMATION

Given **corrupted** samples from a 1-D Gaussian:



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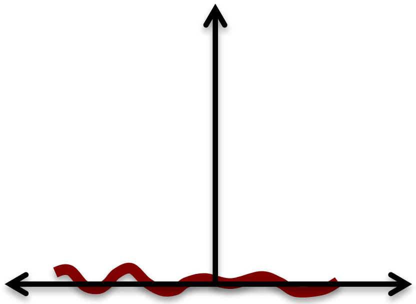
Equivalently:

L_1-norm of noise at most $O(\varepsilon)$



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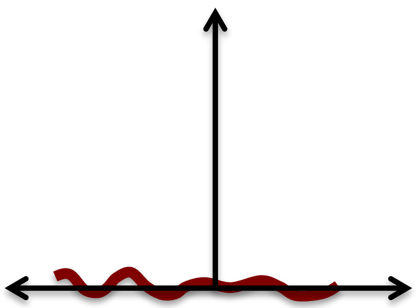
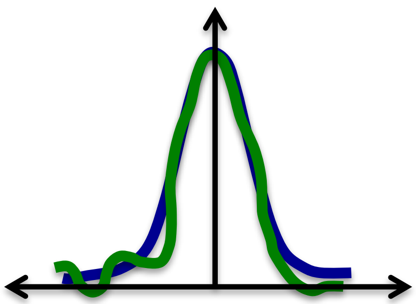


Arbitrarily corrupt $O(\varepsilon)$-fraction of samples (in expectation)



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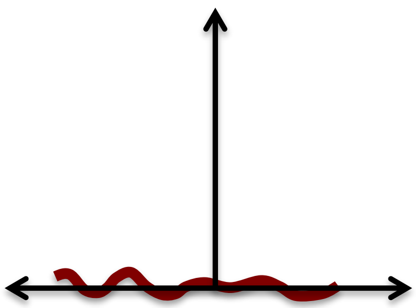

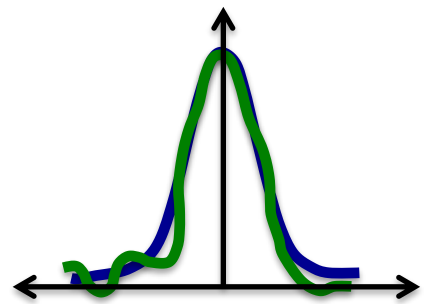

This generalizes **Huber's Contamination Model:** An adversary can add an ε-fraction of samples

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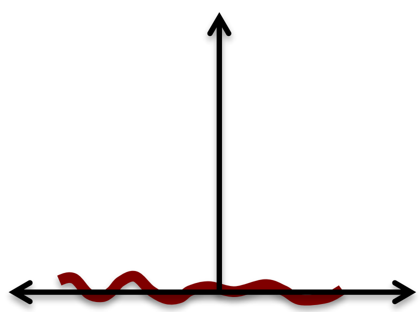

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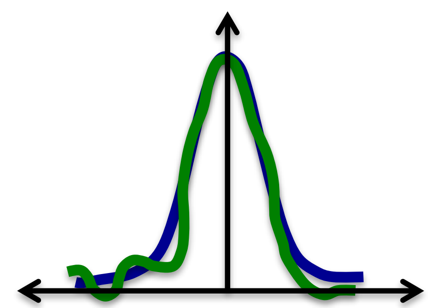

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Outliers: Points adversary has corrupted, **Inliers:** Points he hasn't

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Definition: The total variation distance between two distributions with pdfs f(x) and g(x) is

$$d_{TV}(f(x), g(x)) \triangleq \frac{1}{2} \int_{-\infty}^{\infty} \left| f(x) - g(x) \right| dx$$

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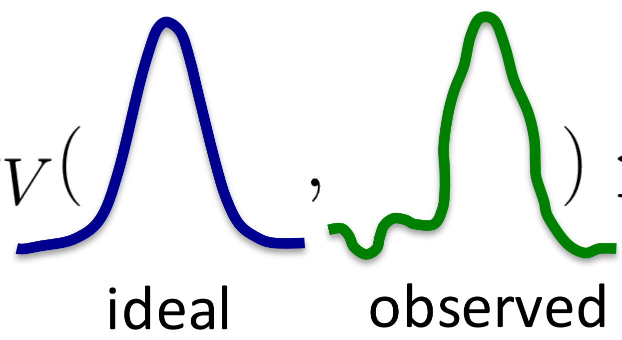
From the bound on the L_1-norm of the noise, we have:

$$d_{TV}(\quad , \quad) \leq O(\epsilon)$$



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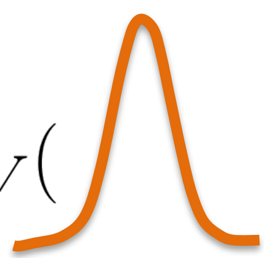
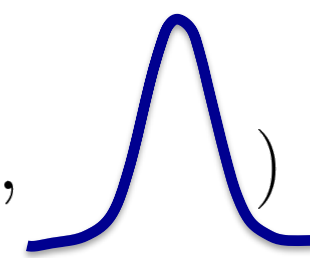
Goal: Find a 1-D Gaussian that satisfies

$d_{TV}($  $,$  $) \leq O(\epsilon)$

estimate

ideal

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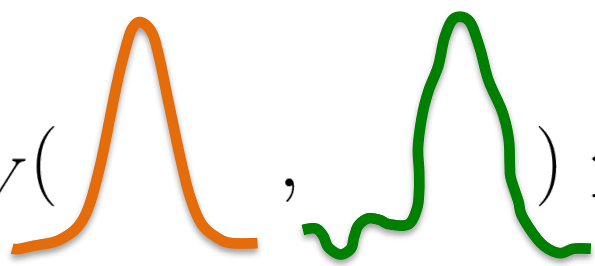
$d_{TV}($  $) \leq O(\epsilon)$

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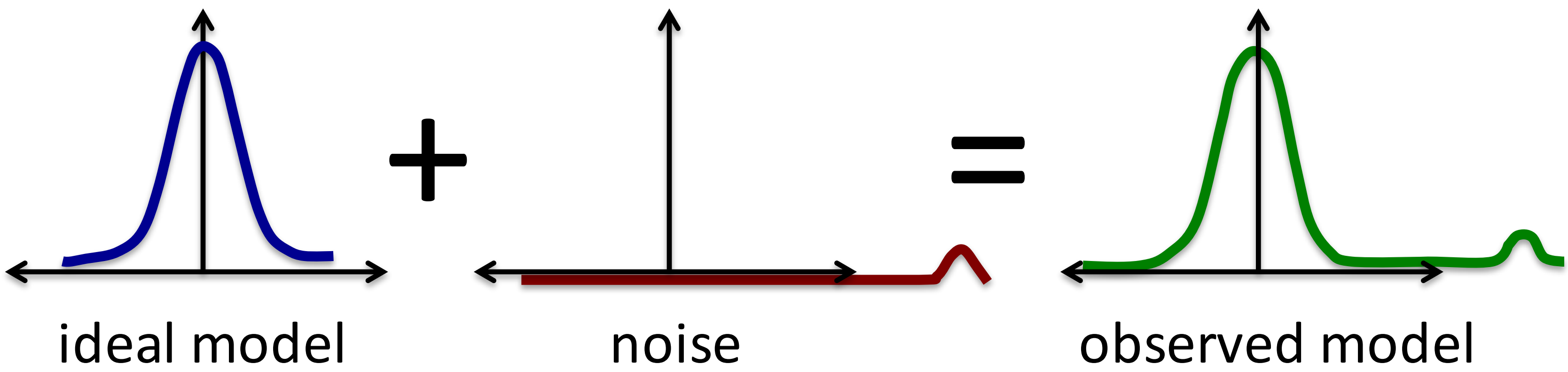

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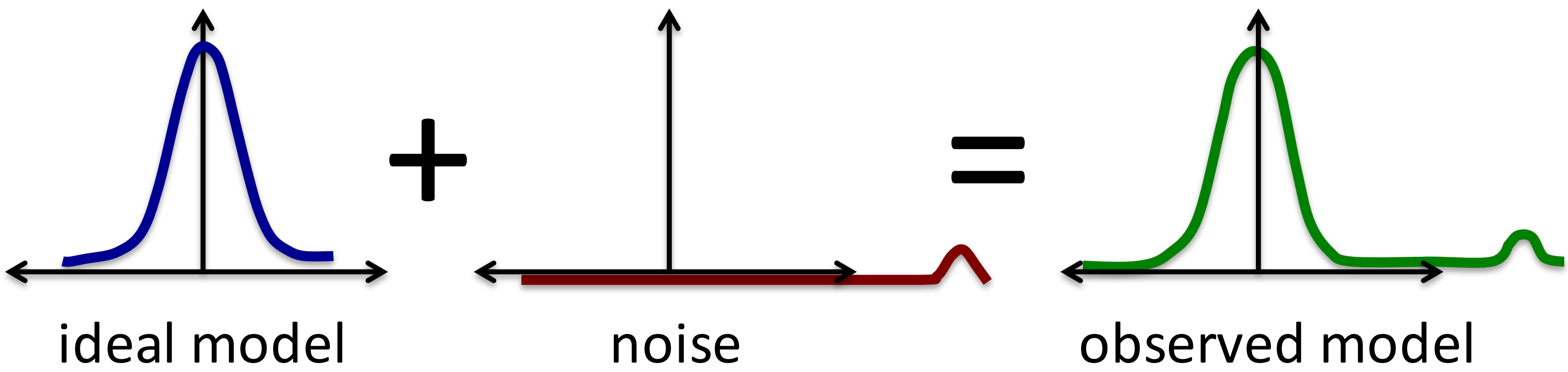

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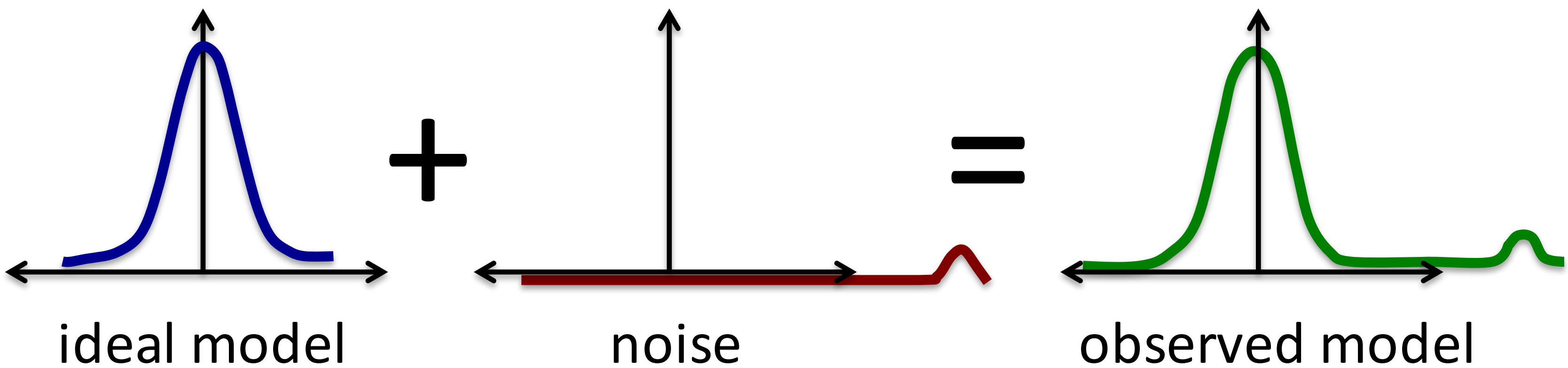

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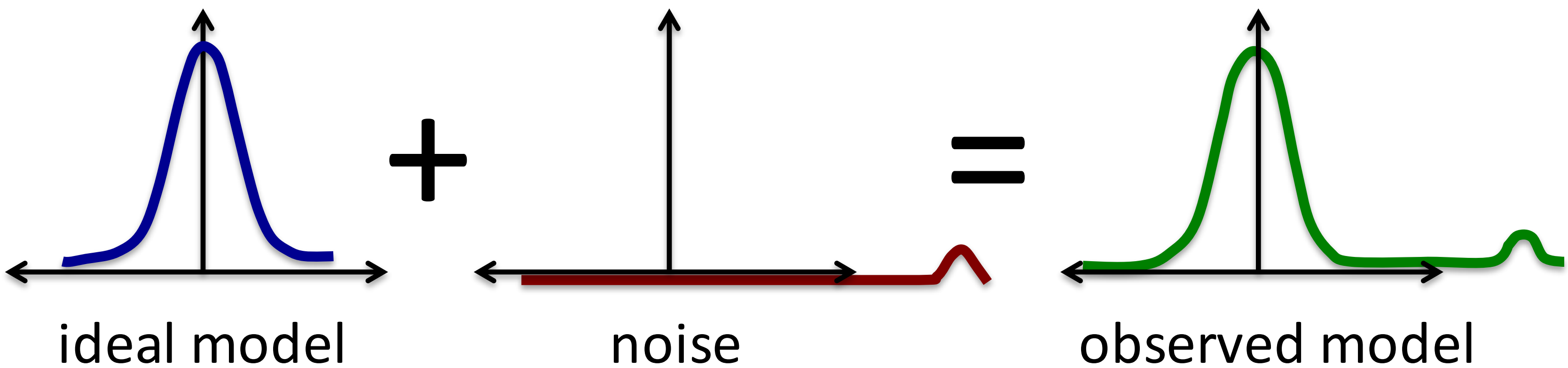

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But the **median** and **median absolute deviation** do work

$$\mathrm{MAD} = \mathrm{median}(|X_i - \mathrm{median}(X_1, X_2, \ldots, X_n)|)$$

Fact [Folklore]: Given samples from a distribution that are ε-close in total variation distance to a 1-D Gaussian

$$\mathcal{N}(\mu, \sigma^2)$$

the median and MAD recover estimates that satisfy

$$d_{TV}(\mathcal{N}(\mu, \sigma^2), \mathcal{N}(\widehat{\mu}, \widehat{\sigma}^2)) \leq O(\epsilon)$$

where $\widehat{\mu} = \mathrm{median}(X), \; \widehat{\sigma} = \dfrac{\mathrm{MAD}}{\Phi^{-1}(3/4)}$

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Also called (properly) **agnostically learning** a 1-D Gaussian

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What about robust estimation in high-dimensions?

e.g. microarrays with 10k genes

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Part II: Agnostically Learning a Gaussian

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- Filtering and Convex Programming
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Main Problem: Given samples from a distribution that are ε-close in total variation distance to a d-dimensional Gaussian

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Special Cases:

(1) Unknown mean $\mathcal{N}(\mu, I)$

(2) Unknown covariance $\mathcal{N}(0, \Sigma)$

A COMPENDIUM OF APPROACHES

Unknown Mean	Error Guarantee	Running Time

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Is robust estimation algorithmically possible in high-dimensions?

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Robust estimation is high-dimensions is algorithmically possible!

Theorem [Diakonikolas, Li, Kamath, Kane, Moitra, Stewart '16]:
There is an algorithm when given $N = \widetilde{O}(d^3/\epsilon^2)$ samples from a distribution that is ε-close in total variation distance to a d-dimensional Gaussian $\mathcal{N}(\mu, \Sigma)$ finds parameters that satisfy

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Alternatively: Can approximate the Tukey median, etc, in interesting semi-random models

Simultaneously **[Lai, Rao, Vempala '16]** gave agnostic algorithms that achieve:

$$\|\mu - \widehat{\mu}\|_2 \leq C\epsilon^{1/2}\|\Sigma\|_2^{1/2}\log^{1/2} d$$

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Many other applications across both papers: product distributions, mixtures of spherical Gaussians, SVD, ICA

A GENERAL RECIPE

Robust estimation in high-dimensions:

- **Step #1:** Find an appropriate parameter distance
- **Step #2:** Detect when the naïve estimator has been compromised
- **Step #3:** Find good parameters, or make progress

  Filtering: Fast and practical

  Convex Programming: Better sample complexity

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Let's see how this works for **unknown mean**…

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This can be proven using Pinsker's Inequality

$$d_{TV}(f,g)^2 \leq \frac{1}{2}\, d_{KL}(f,g)$$

and the well-known formula for KL-divergence between Gaussians

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Our new goal is to be close in **Euclidean distance**

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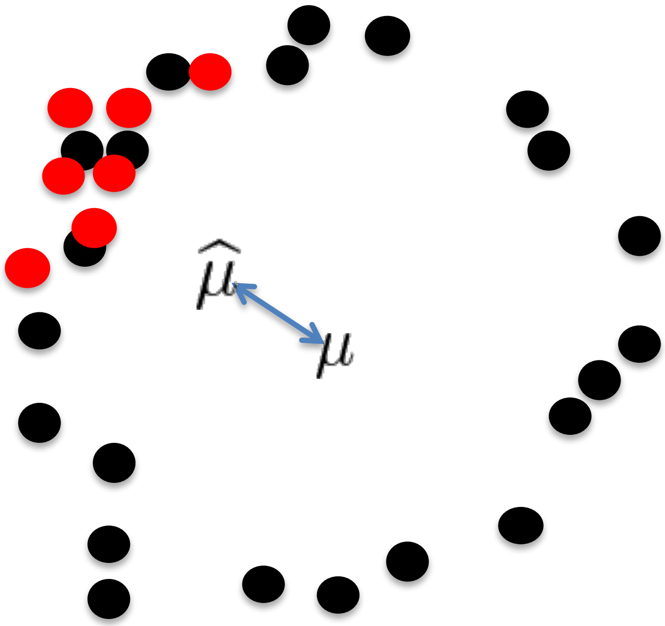
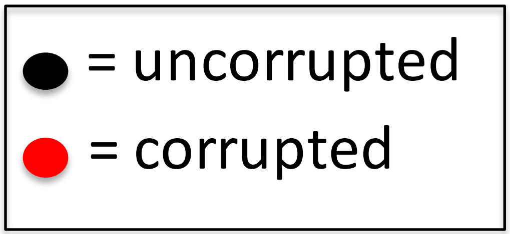
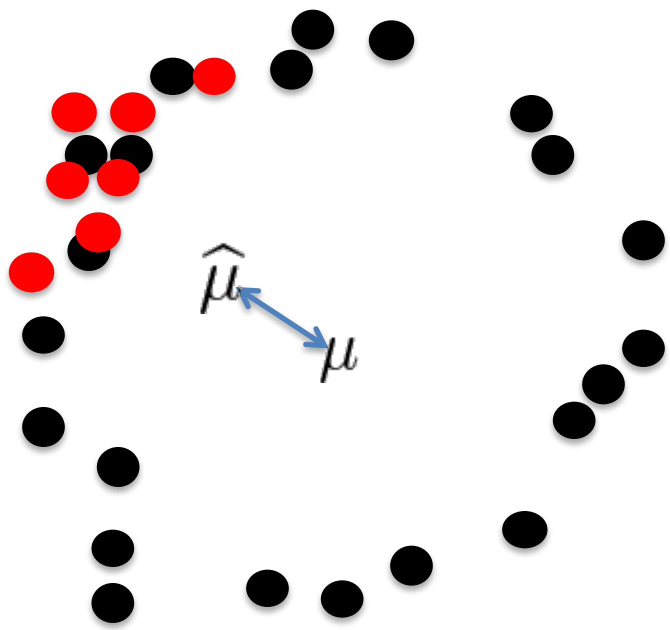


$$\widehat{\mu} \triangleq \frac{1}{N} \sum_{i=1}^{N} X_i$$



DETECTING CORRUPTIONS

Step #2: Detect when the naïve estimator has been compromised



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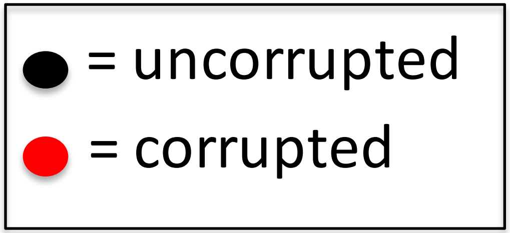

There is a direction of large (> 1) variance

Key Lemma: If X_1, X_2, … X_N come from a distribution that is ε-close to $\mathcal{N}(\mu, I)$ and $N \geq 10(d + \log 1/\delta)/\epsilon^2$ then for

(1) $\widehat{\mu} \triangleq \frac{1}{N}\sum_{i=1}^{N} X_i$ (2) $\widehat{\Sigma} \triangleq \frac{1}{N}\sum_{i=1}^{N}(X_i - \widehat{\mu})(X_i - \widehat{\mu})^T$

with probability at least 1-δ

$$\|\mu - \widehat{\mu}\|_2 \geq C\epsilon\sqrt{\log 1/\epsilon} \longrightarrow \|\widehat{\Sigma} - I\|_2 \geq C'\epsilon \log 1/\epsilon$$

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Take-away: An adversary needs to mess up the second moment in order to corrupt the first moment

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Step #3: Either find good parameters, or remove many outliers

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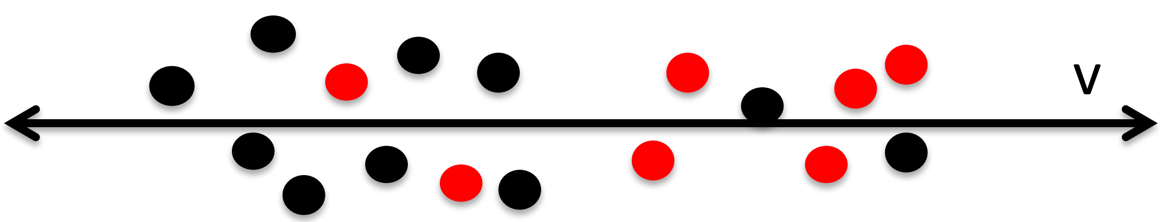

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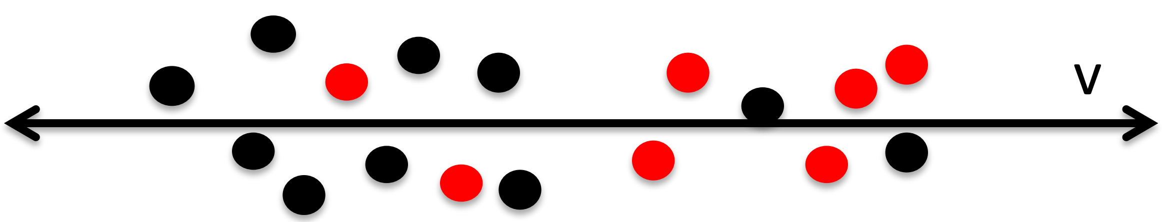

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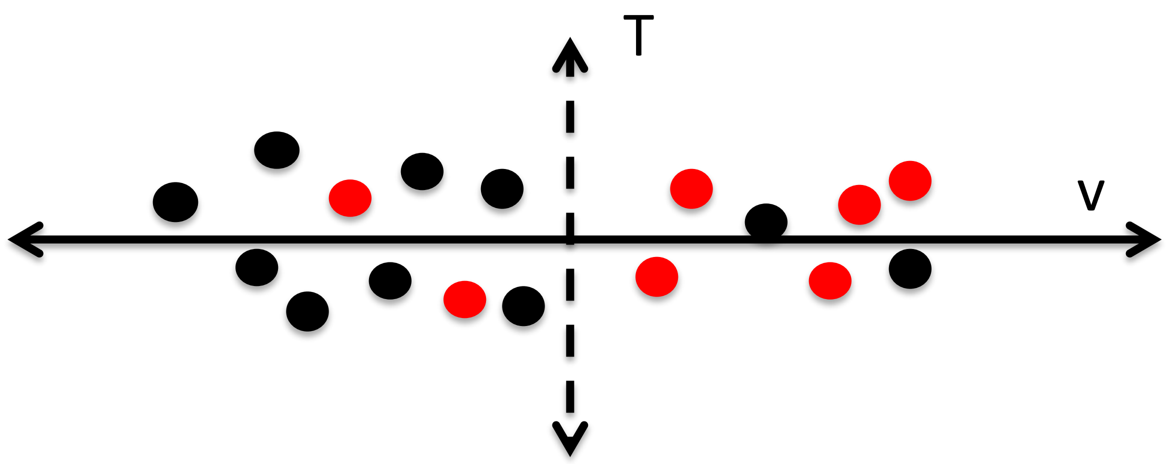

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Concentration of LTFs

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A GENERAL RECIPE

Robust estimation in high-dimensions:

- **Step #1:** Find an appropriate parameter distance
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How about for **unknown covariance**?

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Distance seems strange, but it's the right one to use to bound TV

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Proof uses **Isserlis's Theorem**

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Take-away: An adversary needs to mess up the (restricted) **fourth** moment in order to corrupt the **second** moment

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Mixtures of k Spherical Gaussians:

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Error rates on synthetic data (**unknown mean**):

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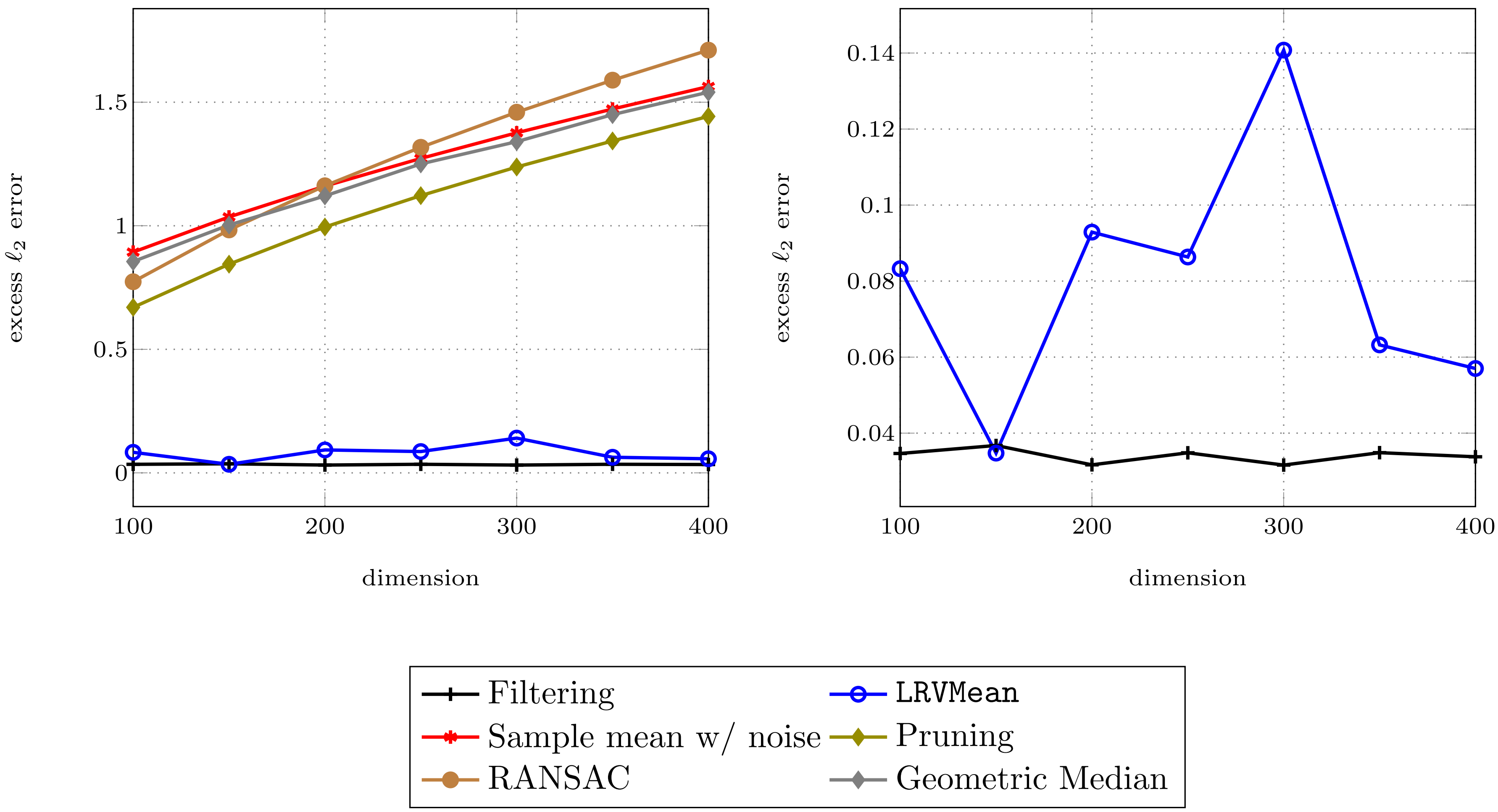

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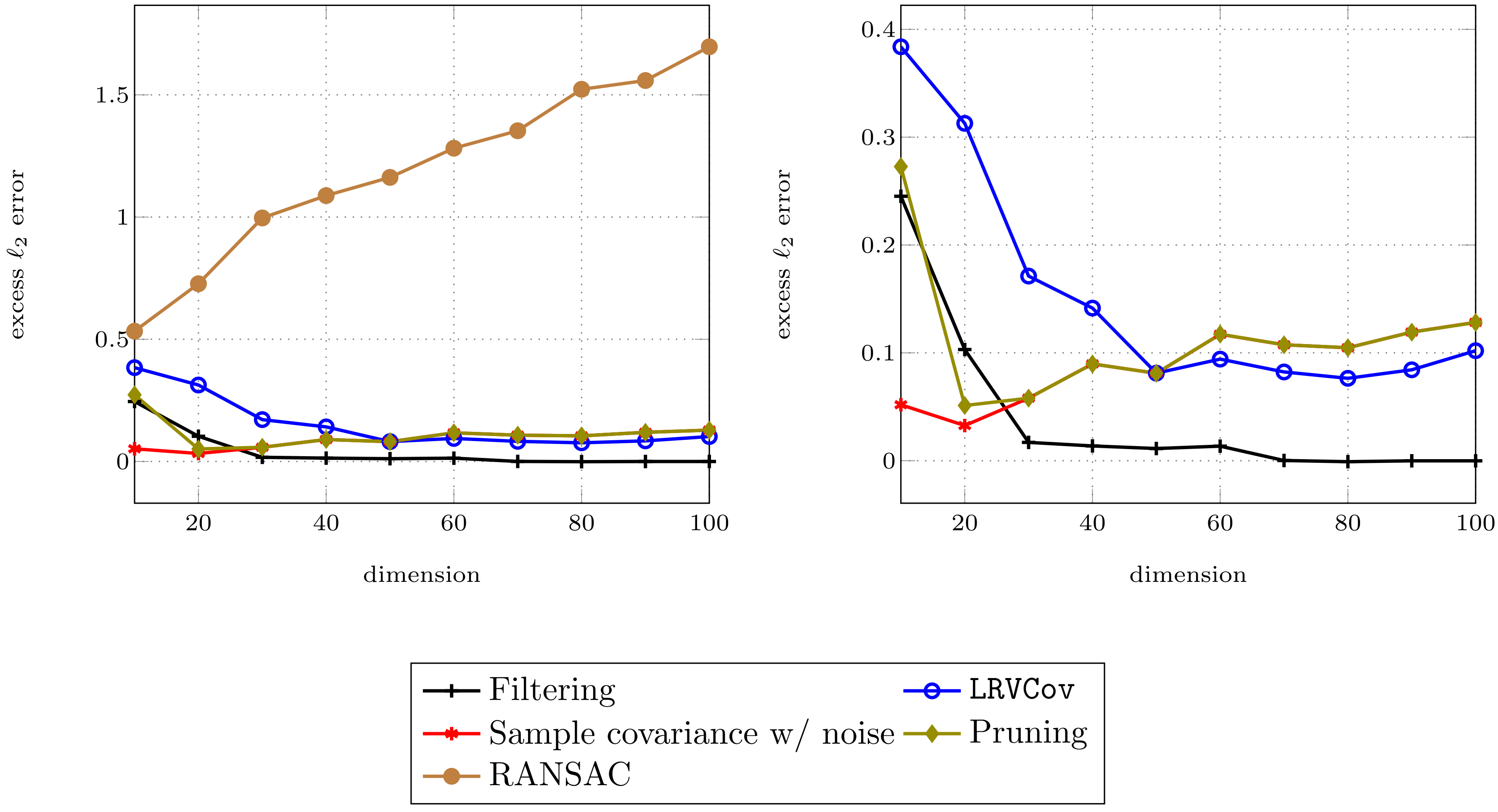

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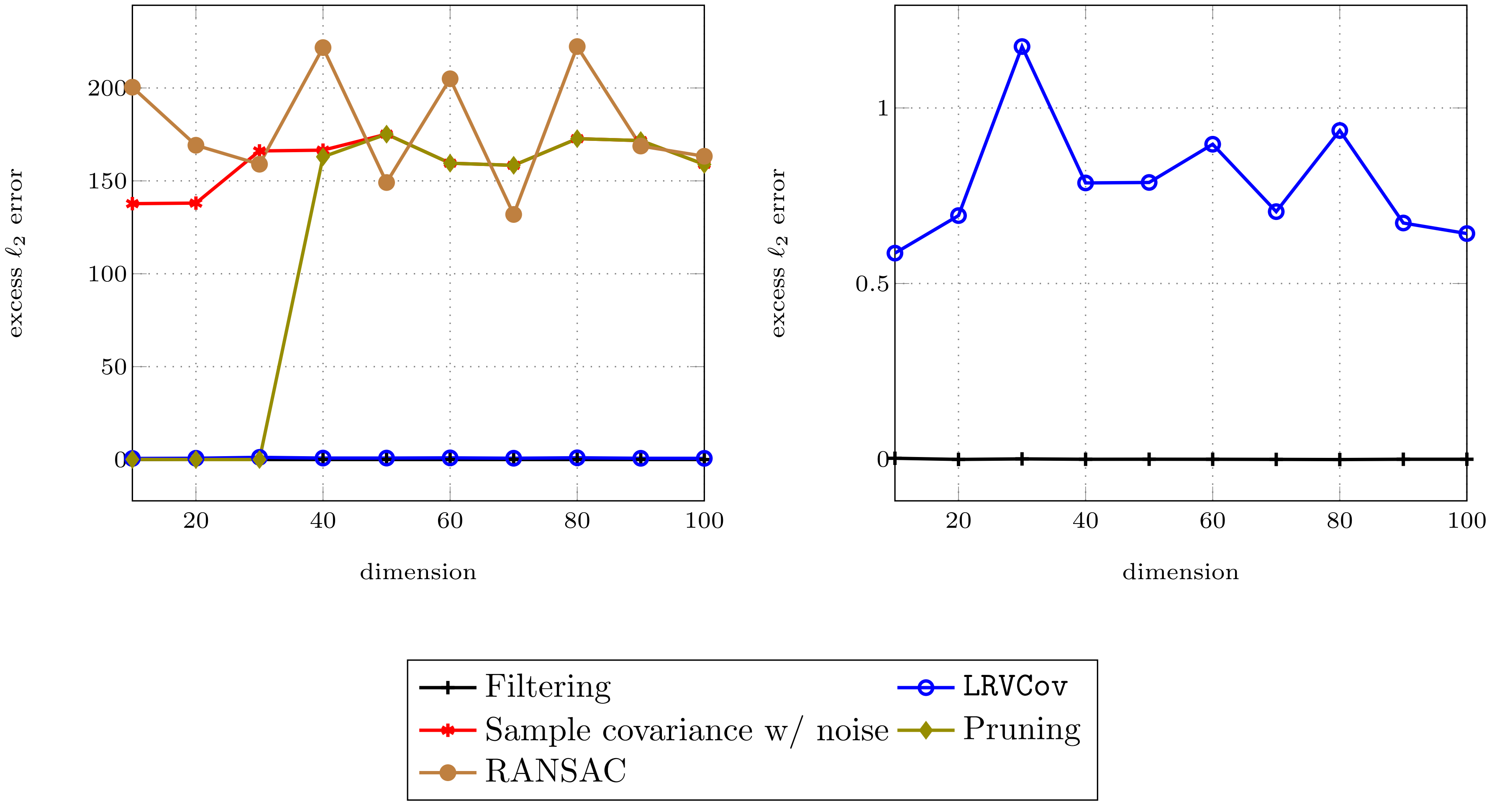

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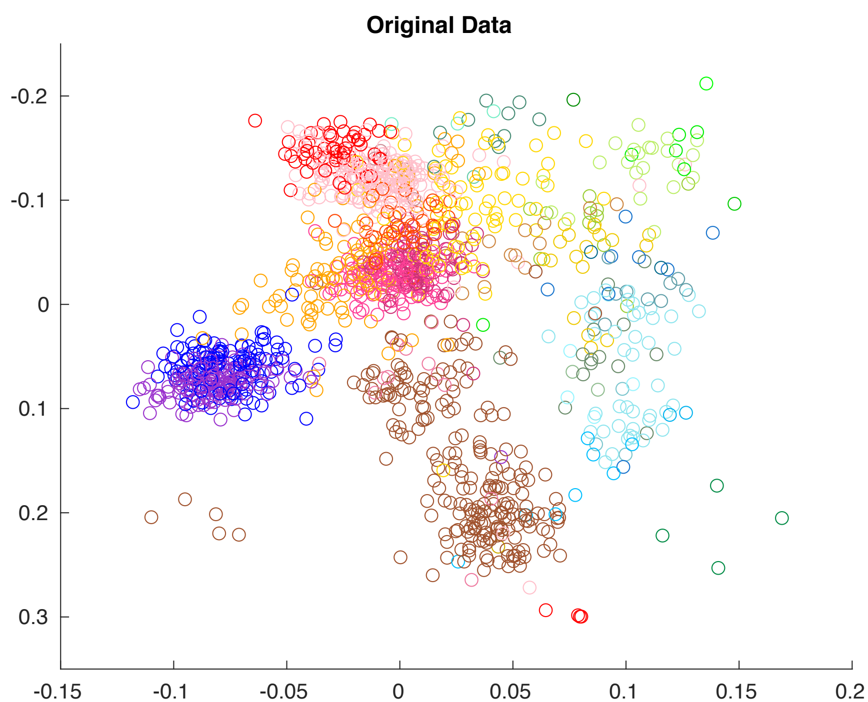

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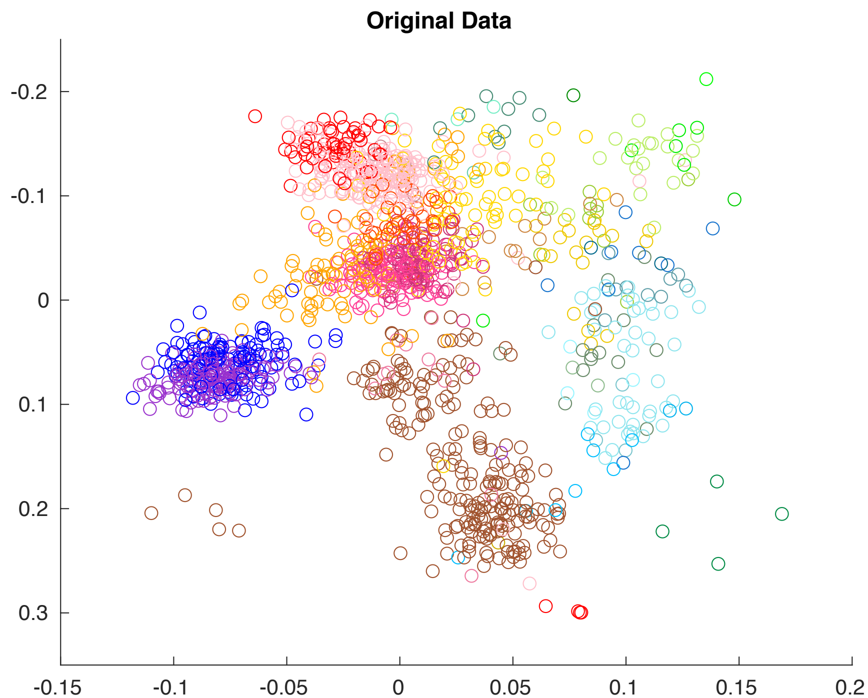

"Genes Mirror Geography in Europe"

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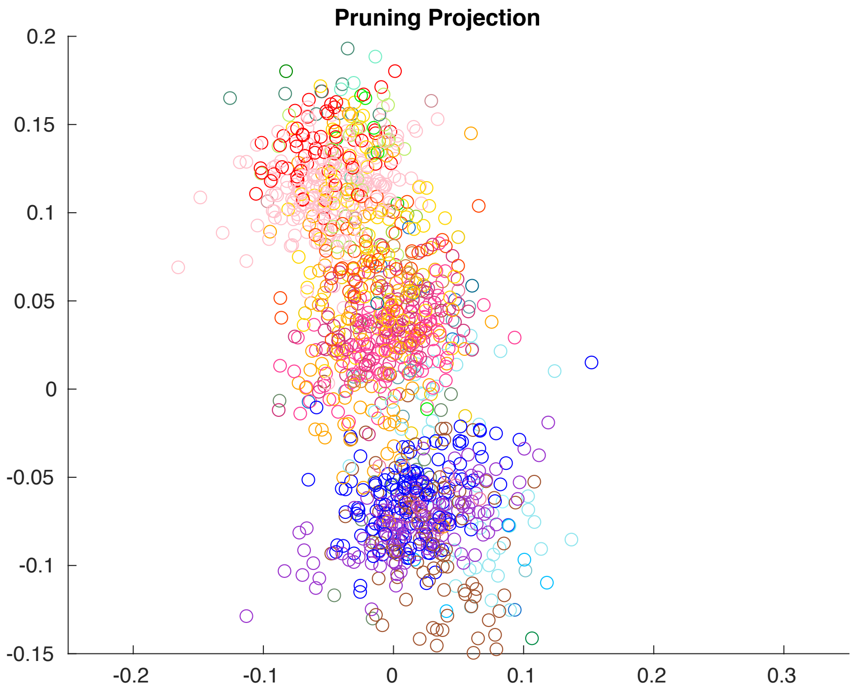

What PCA finds

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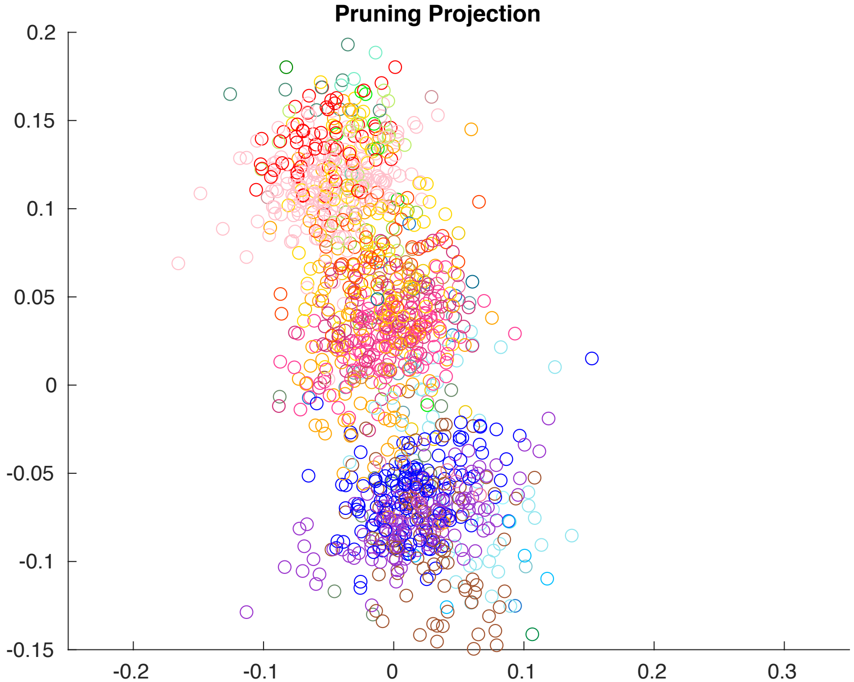

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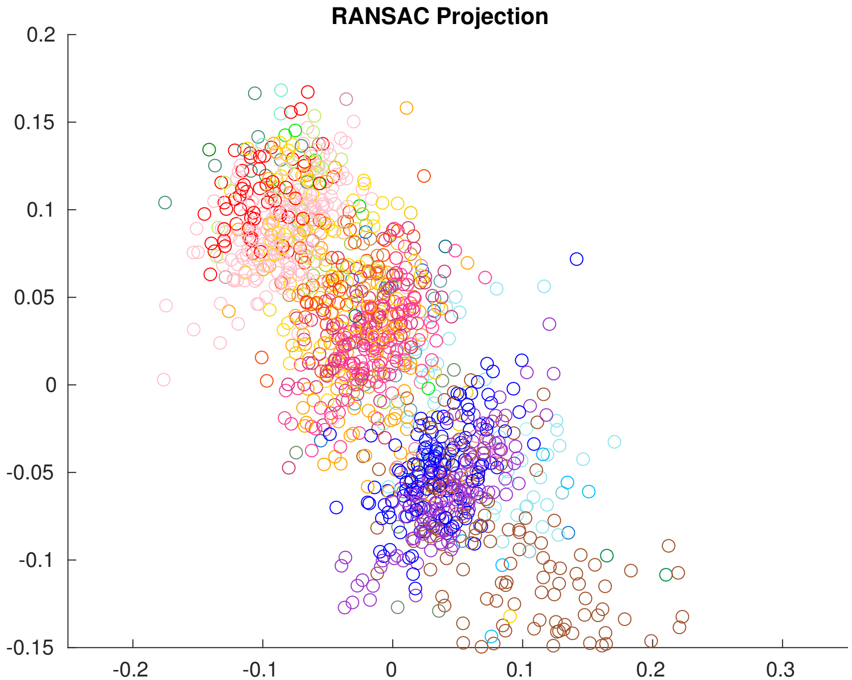

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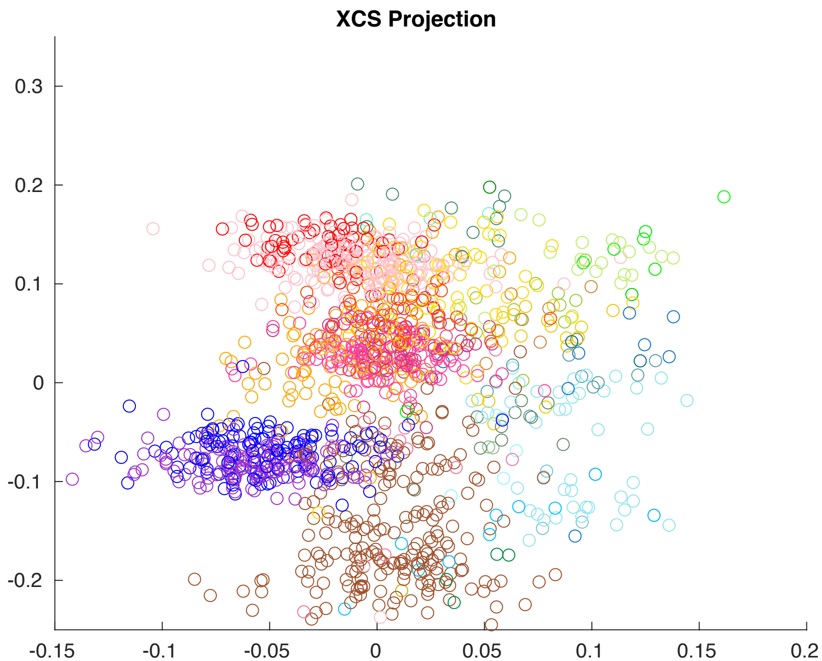

What robust PCA (via SDPs) finds

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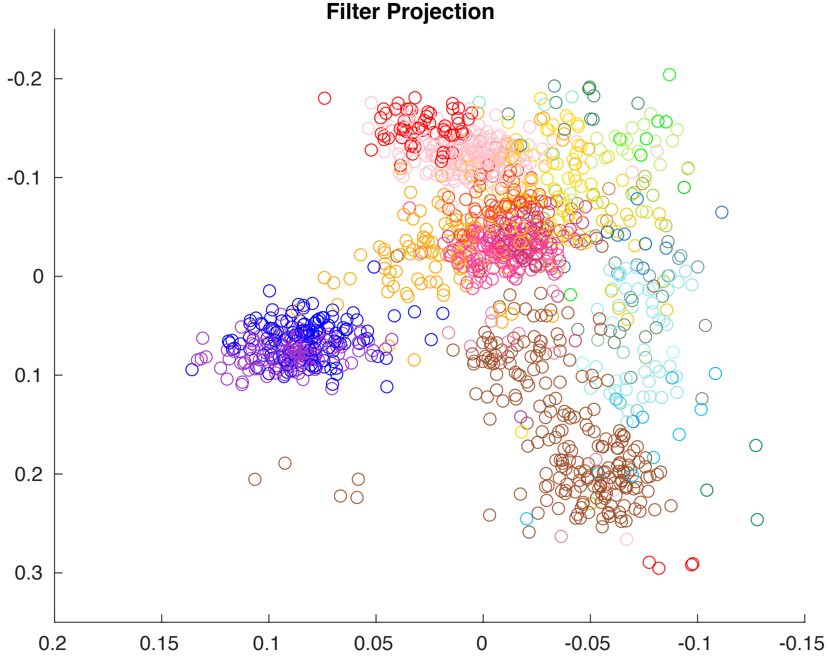
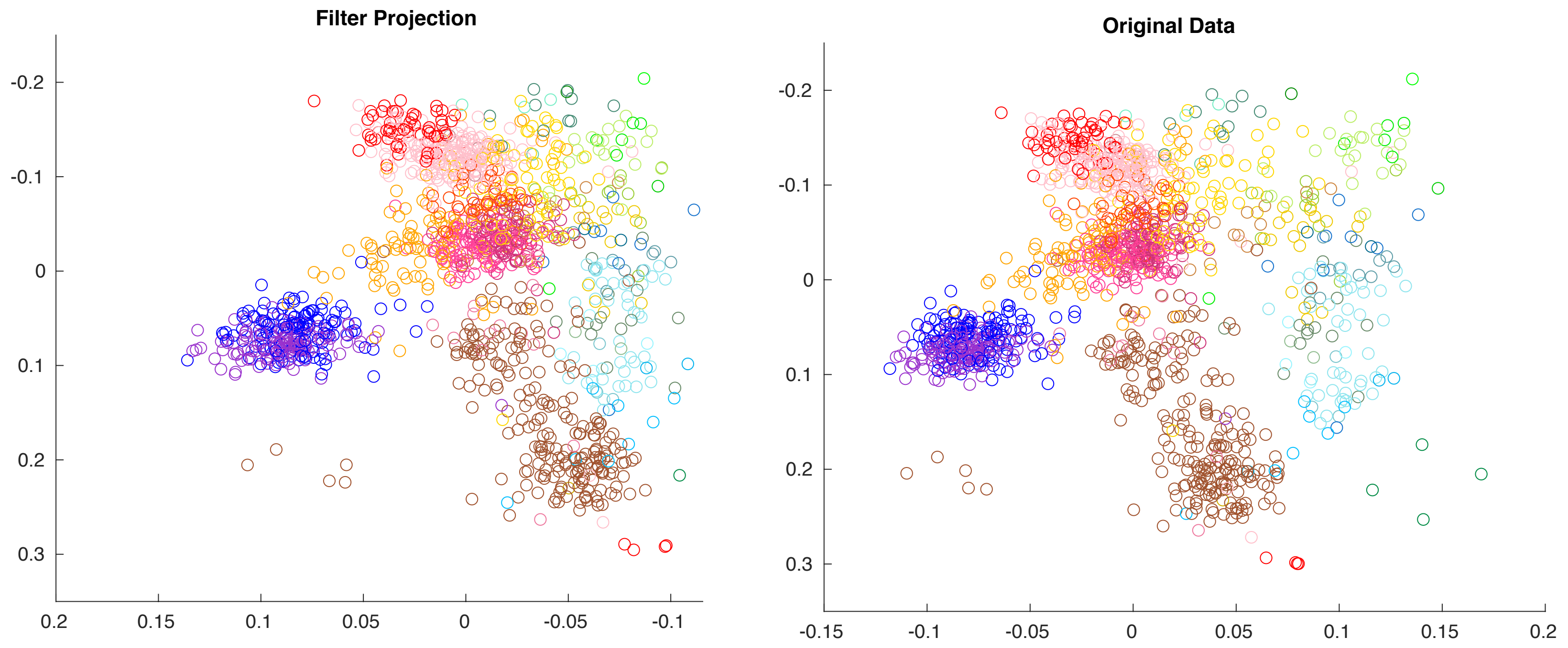


What our methods find

REAL DATA EXPERIMENTS

The power of provably robust estimation:

10% noise **no noise**



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Can algorithms for agnostically learning a Gaussian help in **exploratory data analysis** in high-dimensions?

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Can algorithms for agnostically learning a Gaussian help in **exploratory data analysis** in high-dimensions?

Isn't this what we would have been doing with robust statistical estimators, if we had them all along?

Summary:

- Nearly optimal algorithm for agnostically learning a high-dimensional Gaussian
- General recipe using restricted eigenvalue problems
- Further applications to other mixture models
- **Is practical, robust statistics within reach?**

Thanks! Any Questions?