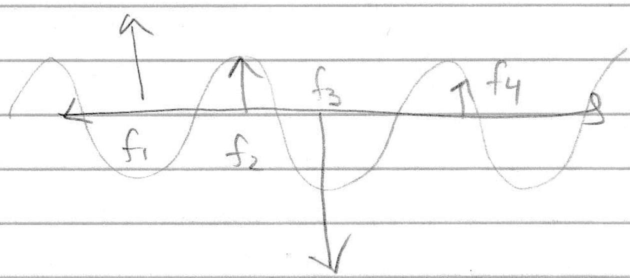
Super resolution

Many optical devices are inherently lowpass

<u>Main Question</u>: can we recover fine-grained structure from course measurements?

[Donoho] introduced the setup



measurement frequency ω

$|\omega| \leq m$

cutoff frequency

$$x(t) = \sum_{j=1}^{k} u_j \underbrace{\delta_{f_j}(t)}_{\text{delta function at } f_j}$$

$$P_\omega = \int_0^1 e^{i2\pi\omega t} x(t)\, dt$$

$$= \sum_{j=1}^{k} u_j e^{i2\pi f_j \omega}$$

Can we recover the parameters (coefficients u_j and frequencies f_j) from low frequency measurements?

<u>Proposition</u> [Prony] when there is no noise, $\exists$ polynomial time algorithm for exact recovery from <u>any</u> $2k+1$ distinct measurements

This theorem seems forgotten in the literature, e.g.

Compressed Sensing: can recover a k-sparse vector $x \in \mathbb{R}^n$ from $O(k \log \frac{n}{k})$ noisy linear measurements

"We can beat Shannon-Nyquist by assuming sparsity"

Applications in MRI, etc

But notice Prony's method:

(1) gets $O(k)$ measurements

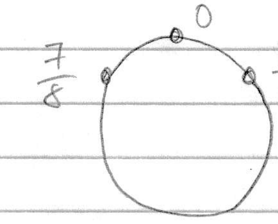
(2) doesn't require points on a grid

But is it stable? More on this later

definition: The wrap around distance d_w is

$$d_w(x,y) = \min(|x-y|, |1+x-y|)$$

e.g.



$d_w(1/8, 7/8) = 1/4$

History, minus Prony

Let $\Delta = \min_{j \neq j'} d_w(f_j, f_{j'})$

[Donoho] Asymptotic stability bounds for $m = 1/\Delta$ on grid

[Candes, Fernandez-Granda] Convex program for $m \geq 2/\Delta$, no noise

[Fernandez-Granda] Same, but with noise

[Liao, Fannjiang] (independent) Algorithm for $m = \frac{1+C(\Delta)}{\Delta}$, with noise

[Moitra]

<u>Theorem</u>. There is a polynomial time algorithm that works with noise, for $m \geq 1/\Delta + 1$ and for any $m \leq \frac{(1-\varepsilon)}{\Delta}$ it is impossible (i.e. noise must be exponentially small)

<u>Matrix Pencil Method</u> <u>(No noise)</u>

<u>definition</u>: Let $\alpha_j = e^{i 2\pi f_j}$ and let

$$(V =) V_m^k = \begin{bmatrix} 1 & 1 & 1 \\ \alpha_1 & \alpha_2 & \alpha_k \\ \vdots & \vdots & \\ \alpha_1^{m-1} & \alpha_2^{m-1} & \alpha_k^{m-1} \end{bmatrix}$$

Now consider the matrix

$$A = V D_u V^H$$

diagonal matrix of u_j's

and B

Claim: The entries of A and B correspond to measurements at integer frequencies $-m+1 \le \omega \le \cancel{m-1}$ m

Proof: By direct computation

$$A_{jj'} = \left[\alpha_1^{j-1}, \alpha_2^{j-1}, \dots \alpha_k^{j-1}\right] \begin{bmatrix} u_1 & & & \\ & u_2 & & \\ & & \ddots & \\ & & & u_k \end{bmatrix} \begin{bmatrix} \alpha_1^{-j'+1} \\ \alpha_2^{-j'+1} \\ \vdots \\ \alpha_k^{-j'+1} \end{bmatrix}$$

$$= p_{j-j'} \quad \blacksquare$$

Similarly define

$$B = V D_u D_\alpha V^H$$

D_α: diagonal matrix of α_j's

Fact: If α_j's are distinct and $m \ge k$, then V has full column rank

Lemma: Under same conditions, and u_j's are nonzero then $\lambda = 1/\alpha_j$ are unique solns to

$$Ax = \lambda Bx$$

Proof: By the fact V^H has a right inverse R s.t.

$$V^H R = I_k$$

Then $r_j = j^{th}$ column of R satisfies

$$Ar_j = V D_u V^H r_j = V D_u e_j$$

$$= u_j V e_j = u_j V_j$$

j^{th} column of V

$$Br_j = V D_u D_\alpha V^H r_j = u_j \alpha_j V_j$$

This is a generalized eigenvalue problem, similar uniqueness. ☒

Thus our algorithm is

Construct A and B

For $Ax = \lambda Bx$, find all values of $\lambda = \frac{1}{\alpha_j}$

Solve $V \begin{bmatrix} u_1 \\ \vdots \\ u_k \end{bmatrix} = \begin{bmatrix} p_1 \\ \vdots \\ p_k \end{bmatrix}$ to find coefficients

<u>Noise stability</u>

When is the ~~MPM~~ generalized eigenvalue problem robust to noise?

i.e. can we get accuracy guarantees that are polynomial in magnitude of noise?

We'll show a sharp phase transition in the condition number of the complex Vandermonde

Theorem $\|V_m^k u\|^2 = (m-1 \pm \frac{1}{\Delta})\|u\|^2$

Theorem: If $m = \frac{(1-\varepsilon)}{\Delta}$, $\exists$ α_j's and u_j's s.t.

$$\|V_m^k u\|^2 \le e^{-\varepsilon k}\|u\|^2$$

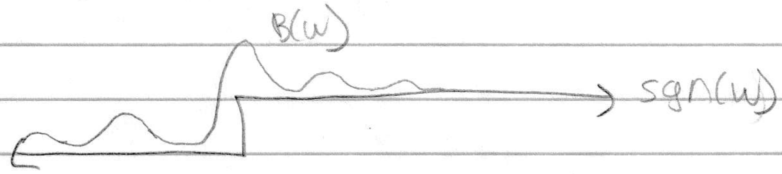
Thus the noise needs to be exponentially small to superresolve

Extremal Functions

Theorem: There is a function $B(w)$ satisfying

(1) $\operatorname{sgn}(w) \le B(w)$

i.e. it majorizes



(2) $\hat{B}(x)$ = fourier transform of $B(w)$

is supported on $[-1, 1]$

i.e. it is smooth

(3) $$\int_{-\infty}^{\infty} B(w) - \operatorname{sgn}(w)\, dw = 1$$

Explicitly we have

$$B(w) = \left(\frac{\sin(\pi w)}{\pi}\right)^2 \left(\sum_{j=0}^{\infty} (w-j)^{-2} - \sum_{j=1}^{\infty} (w+j)^{-2} + \frac{2}{w} \right)$$

This comes from reconstruction formula for functions whose Fourier transform is supported on $[-1, 1]$

Similarly there is a minorant $b(w)$

(1) $b(w) \leq \mathrm{sgn}(w)$

(2) $\hat{b}(w)$ supported in $[-1, 1]$

(3) $\int_{-\infty}^{\infty} \mathrm{sgn}(w) - b(w)\, dw = 1$

and these are sharp

Relatedly we can sandwich the indicator of an interval $E \subseteq \mathbb{R}$

Corollary There are functions $C_E^u(w)$ and $c_E^l(w)$ with $E = [0, m-1]$ that satisfy

(1) $c_E^l(w) \leq \mathbb{1}_E(w) \leq C_E^u(w)$

(2) $\hat{c}_E^l(x)$ and $\hat{C}_E^u(x)$ are supported on $[-\Delta, \Delta]$

(3) $\int_{-\infty}^{\infty} C_E^u(w) - \mathbb{1}_E(w)\, dw = \int_{-\infty}^{\infty} \mathbb{1}_E(w) - c_E^l(w)\, dw$

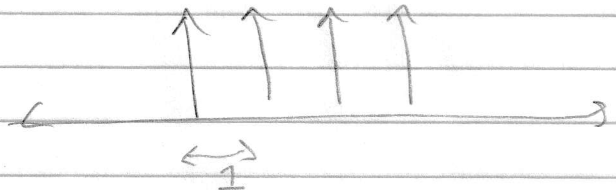
$= \frac{1}{\Delta}$

Now let's prove the condition number bound

Proof: we have

$$\|V_m^k u\|^2 = \sum_{w=0}^{m-1} |p_w|^2 = (*)$$

Now let $h(w)$ = Dirac comb function, i.e



i.e. $h(w) \doteq \sum_{t=-\infty}^{\infty} \delta_t(w)$

then $$(*) = \int_{-\infty}^{\infty} h(w)\, I_E(w)\, |p_w|^2 \, dw$$

$$\leq \int_{-\infty}^{\infty} h(w)\, C_E^u(w)\, |p_w|^2 \, dw$$

1

Now the fourier transform of a comb is a comb so

$$h(w) = \sum_{t=-\infty}^{\infty} e^{i 2\pi t w}$$

2

we get that

Now since $|P_w|^2 = P_w^* P_w$ and

$$P_w = \sum_{j=1}^{k} u_j e^{i2\pi f_j w}$$

we get that the RHS is

$$\sum_{j=1}^{k}\sum_{j'=1}^{k} u_j u_{j'}^* \int_{-\infty}^{\infty} h(w)\, C_E^u(w)\, e^{i2\pi(f_j - f_{j'})w}\, dw$$

1

$$= \sum_{t=-\infty}^{\infty}\sum_{j=1}^{k}\sum_{j'=1}^{k} u_j u_{j'}^* \underbrace{\int_{-\infty}^{\infty} e^{i2\pi t w}\, C_E^u(w)\, e^{i2\pi(f_j - f_{j'})w}\, dw}_{\hat{C}_E^u(f_j - f_{j'} + t)}$$

2

But the f_j's are Δ separated and $\hat{C}_E^u$ is supported on $[-\Delta, \Delta]$ that the cross terms are zero!

$$= \sum_{j=1}^{k} |u_j|^2 \underbrace{\hat{C}_E(0)}_{\ell_1 \text{ norm of } C_E^u \,=\, |E| + \frac{1}{\Delta}}$$

$$= \sum_{j=1}^{k} |u_j|^2 \left(|E| + \frac{1}{\Delta}\right) \quad \square$$

The same proof works for minorant. $\boxtimes$

Thus we have

extremal functions $\Rightarrow$ sharp bounds for condition # $\Rightarrow$ stability of MPM

$\Rightarrow$ sharp bounds for superresolution