The Harshman-Jennrich Algorithm

This algorithm has been rediscovered many times, originates in psychometrics

Setup: We are given T, assumed to be of the form

$$T = \sum_{i=1}^{r} u^{(i)} \otimes v^{(i)} \otimes w^{(i)}$$

definition We say two sets of factors

$$\{(u^{(i)}, v^{(i)}, w^{(i)})\} \text{ and } \{(\hat{u}^{(i)}, \hat{v}^{(i)}, \hat{w}^{(i)})\}$$

are equivalent if there is a permutation π s.t.

$$u^{(i)} \otimes v^{(i)} \otimes w^{(i)} = \hat{u}^{\pi(i)} \otimes \hat{v}^{\pi(i)} \otimes \hat{w}^{\pi(i)}$$

i.e. the produce equivalent low rank decompositions

Main Question When are the factors of T determined up to equivalence?

Theorem [Harshman, Jennrich] suppose the following conditions hold

(1) The vectors $\{u^{(i)}\}$ are linearly indep.

(2) same for $\{v^{(i)}\}$

(3) every pair of vectors in $\{w^{(i)}\}$ are linearly indep.

Then the factors are uniquely determined up to equivalence, and there is a polynomial time algorithm to find them.

The algorithm is simple

- Choose $a, b \in S^{n_3}$ uniformly at random, set
$$T_a = \sum_{i=1}^{n_3} a_i T_{(\cdot,\cdot,i)} \;;\; T_b = \sum_{i=1}^{n_3} b_i T_{(\cdot,\cdot,i)}$$

- Compute the eigen decompositions of
$$T_a (T_b)^+ \text{ and } (T_a^+ T_b)^T$$

Let $\hat{u}$ and $\hat{v}$ be eigenvectors with non-zero eigenvalue

Pair up $\hat{u}^{(i)}$ and $\hat{v}^{(j)}$ iff their eigenvalues are reciprocals

- Solve for $\hat{w}^{(i)}$ in the linear system
$$T = \sum_{i=1}^{r} \hat{u}^{(i)} \otimes \hat{v}^{(i)} \otimes \hat{w}^{(i)}$$

The analysis follows by tracking the structure of T through the algorithm, and using standard uniqueness facts about eigendecomp.

Let $D_a \doteq \mathrm{Diag}(a^T w^{(i)})$; $D_b = \mathrm{Diag}(b^T w^{(i)})$

Lemma: we have that

$$T_a = U D_a V^T \text{ and } T_b = U D_b V^T$$

Proof: Since the operation of computing T_a from T is linear, we can do it just for a rank one term:

If $T = u \otimes v \otimes w$ then $T_a = (a^T w)\, u \otimes v$

Thus, in general, we have

$$T_a = \sum_{i=1}^{r} (a^T w^{(i)})\, u^{(i)} \otimes v^{(i)} = U D_a V^T \quad \blacksquare$$

For simplicity, let's assume T_a and T_b are invertible. Then

$$T_a T_b^{-1} \overset{\text{lemma}}{=} U D_a V^T (V^T)^{-1} D_b^{-1} U^{-1}$$

$$= U D_a D_b^{-1} U^{-1}$$

From property (3), almost surely the diag. entries of

$$D_a D_b^{-1}$$

will be distinct. Thus $T_a T_b^{-1}$ has distinct eigenvalues $\Rightarrow$ its eigendecomposition is unique $\Rightarrow$ we can find U, up to a permutation/rescaling of its columns

Similarly we have

$$(T_a^{-1} T_b)^T = \left((V^T)^{-1} D_a^{-1} U^{-1} U D_b V^T\right)^T$$

$$= \left((V^T)^{-1} D_a^{-1} D_b V^T\right)^T$$

$$= V D_a^{-1} D_b V^{-1}$$

Thus we can determine V, up to a permutation/rescaling of its columns, again from uniqueness of the eigendecomp.

Moreover pairing succeeds, again because the diagonal entries of $D_a^{-1} D_b$ are distinct

Finally, we show:

Lemma: The matrices $u^{(i)} \otimes v^{(i)}$ are linearly independent

Proof: Suppose not. Then $\exists\ \alpha_i$'s s.t.

$$\sum_{i=1}^{r} \alpha_i u^{(i)} \otimes v^{(i)} = 0 \qquad \text{suppose WLOG } \alpha_1 \neq 0$$

By condition (1), we know $\exists x$ s.t.

$$x^T u^{(1)} \neq 0, \quad x^T u^{(i)} = 0 \ \ \forall i \neq 1$$

Now using the identity above, we get

$$(\alpha_1 a^T u^{(1)}) v^{(1)} = 0 \Rightarrow v^{(1)} = 0$$

which contradicts condition (2). $\blacksquare$

Why was this algorithm forgotten?

Psychometrics generally cared about uniqueness, and there are better non-algorithmic uniqueness theorems known

Now returning to factor analysis

Given: $T = \sum_{i=1}^{r} u^{(i)} \otimes v^{(i)} \otimes w^{(i)}$, when are the factors determined up to equivalence?

Harshman-Jennrich: when $\{u^{(i)}\}$ and $\{v^{(i)}\}$ are linearly indep, and no pair in $\{w^{(i)}\}$ are scalar multiples of each other