Stochastic Systems Group

# Consistent and Efficient Reconstruction of Latent Tree Models

Myung Jin Choi

Joint work with
Vincent Tan, Anima Anandkumar, and Alan S. Willsky

Laboratory for Information and Decision Systems
Massachusetts Institute of Technology

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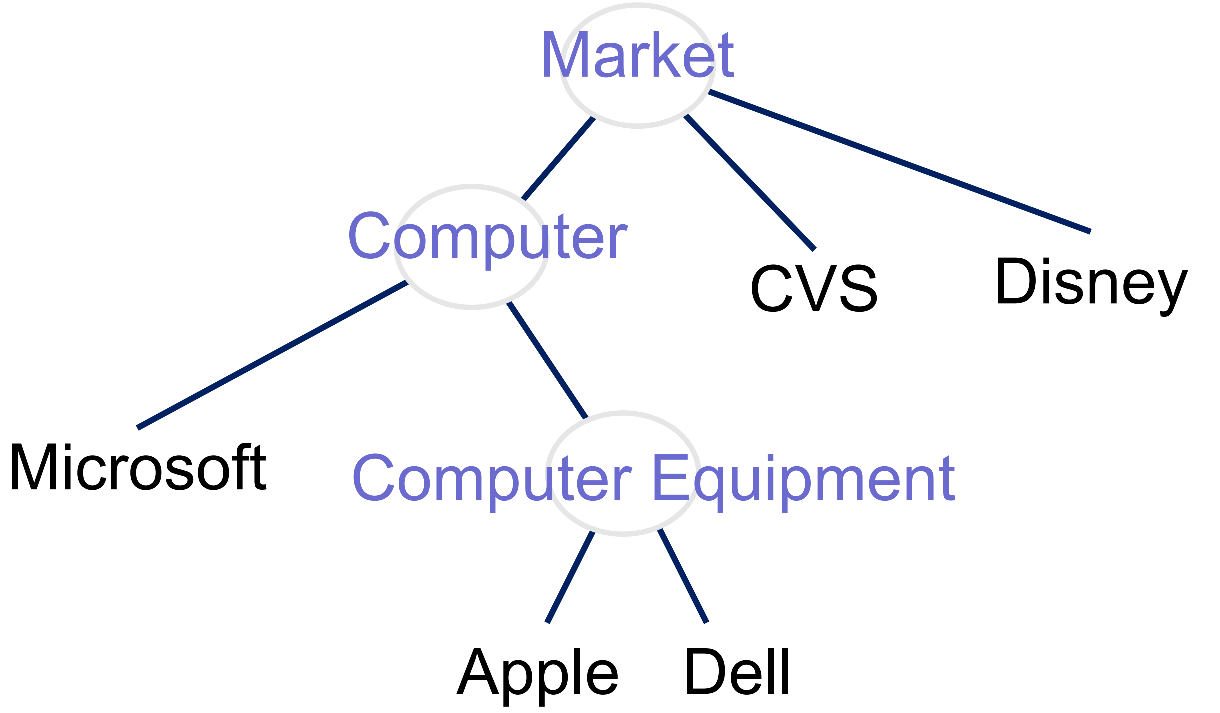
# Latent Tree Graphical Models

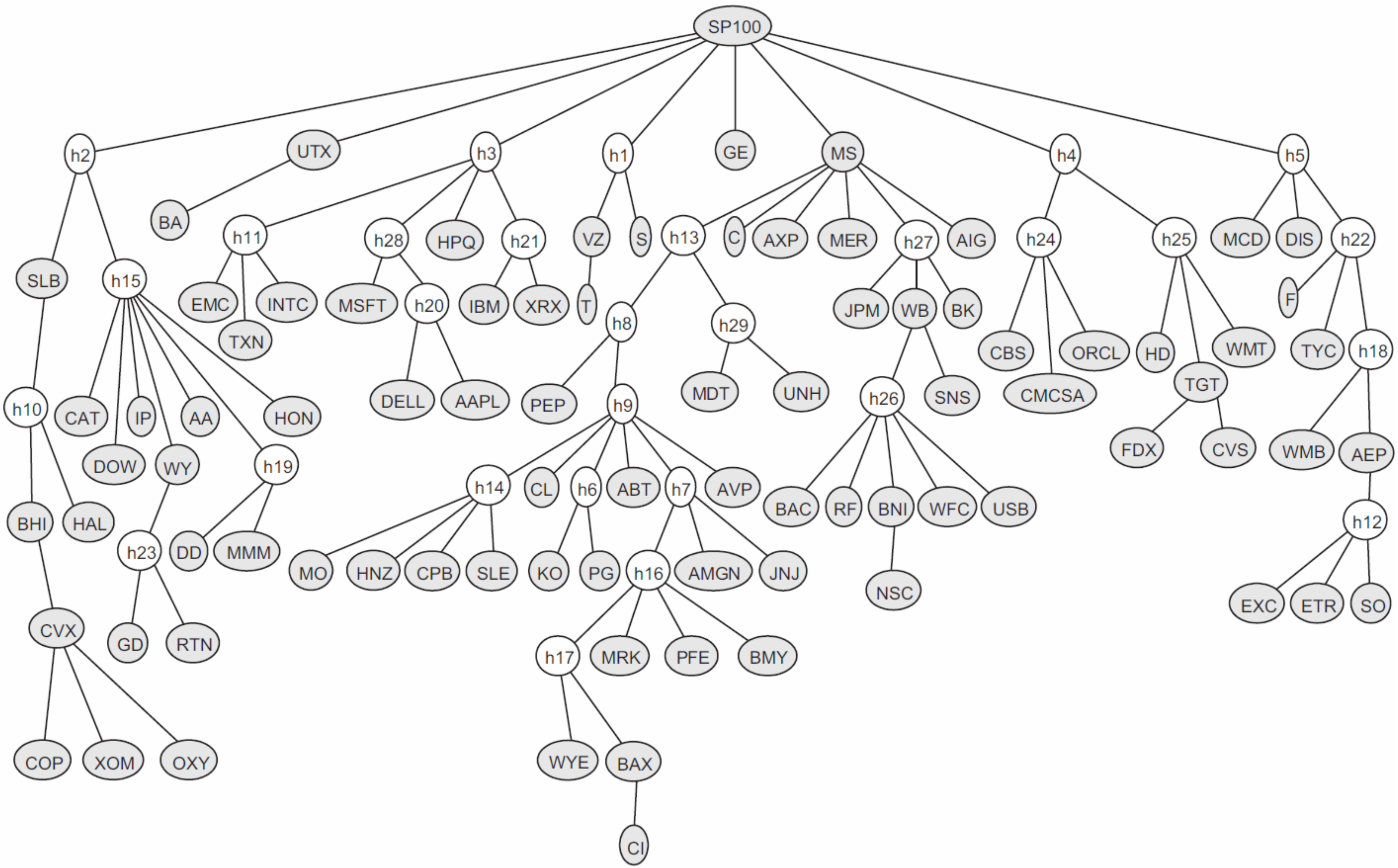
CVS Disney

Microsoft

Apple Dell

# Latent Tree Graphical Models

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# Outline

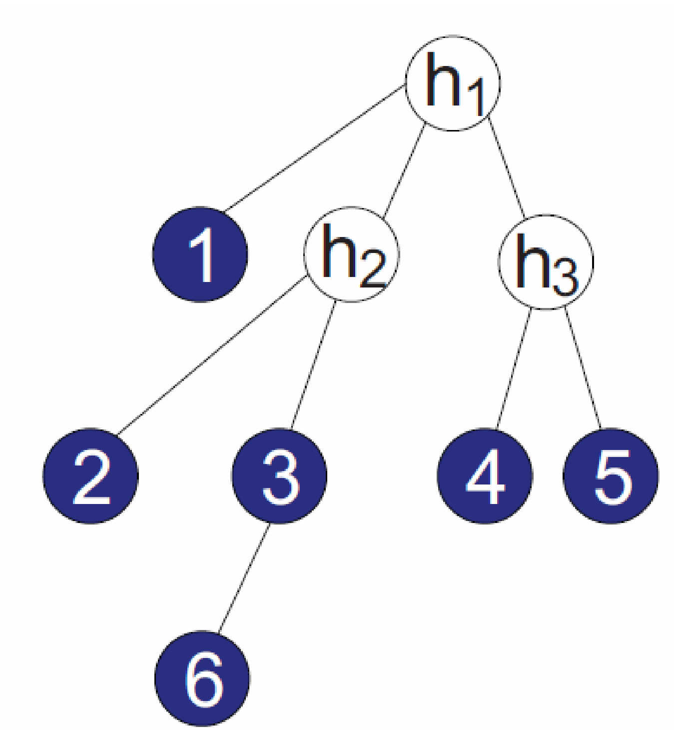
– Reconstruction of a latent tree

– Algorithm 1: Recursive Grouping

– Algorithm 2: CLGrouping

– Experimental results

# Reconstruction of a Latent Tree




- Gaussian model:

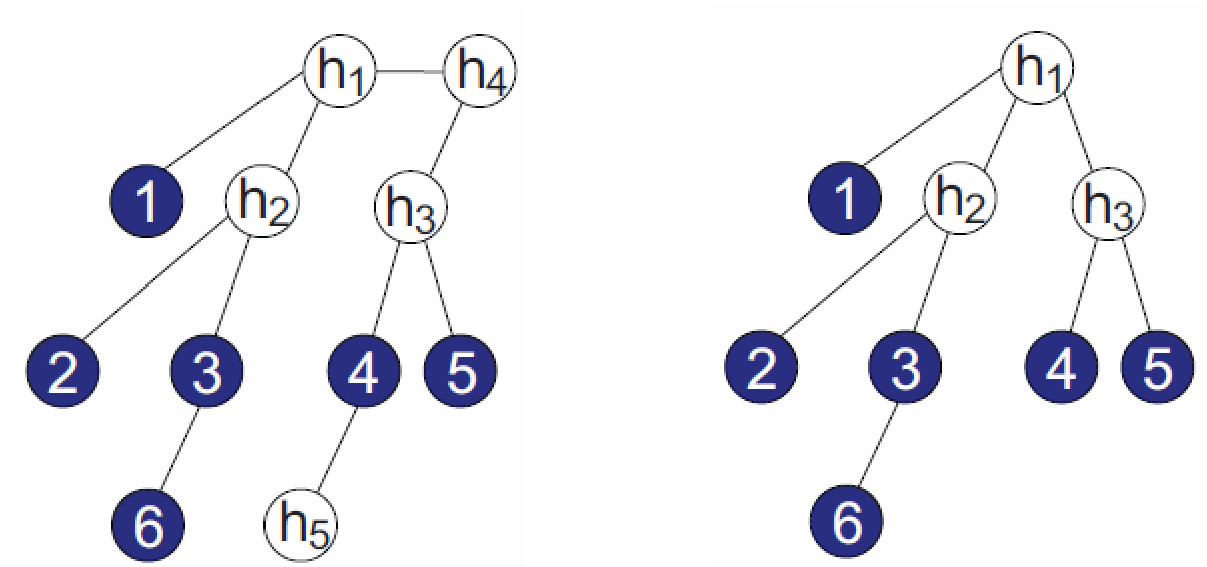
  each node – a scalar Gaussian variable

- Discrete model:

  each node – a discrete variable with K states

Reconstruct a latent tree using samples of the observed nodes.

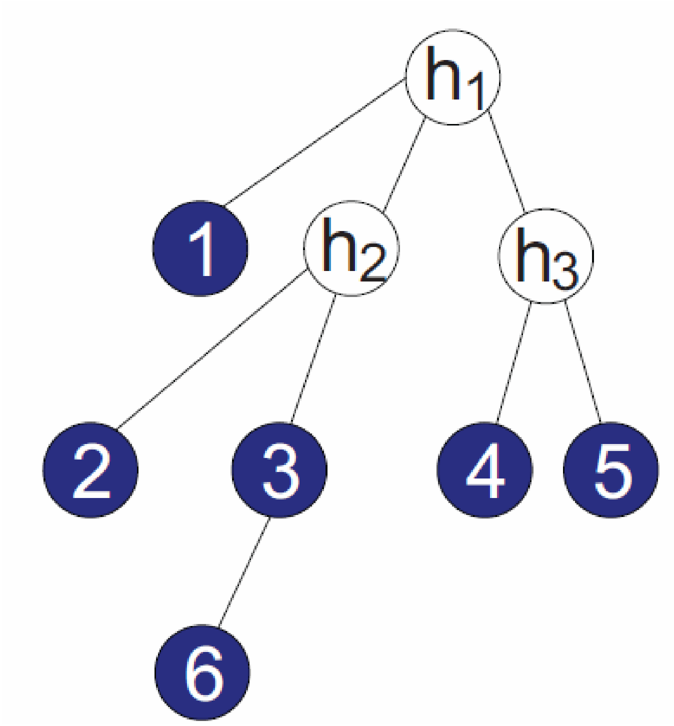
# Minimal Latent Trees (Pearl, 1988)




Conditions for Minimal Latent Trees

- Each hidden node should have at least three neighbors.
- Any two variables are neither perfectly dependent nor independent.

# Desired Properties for Algorithms




1. Consistent for minimal latent trees

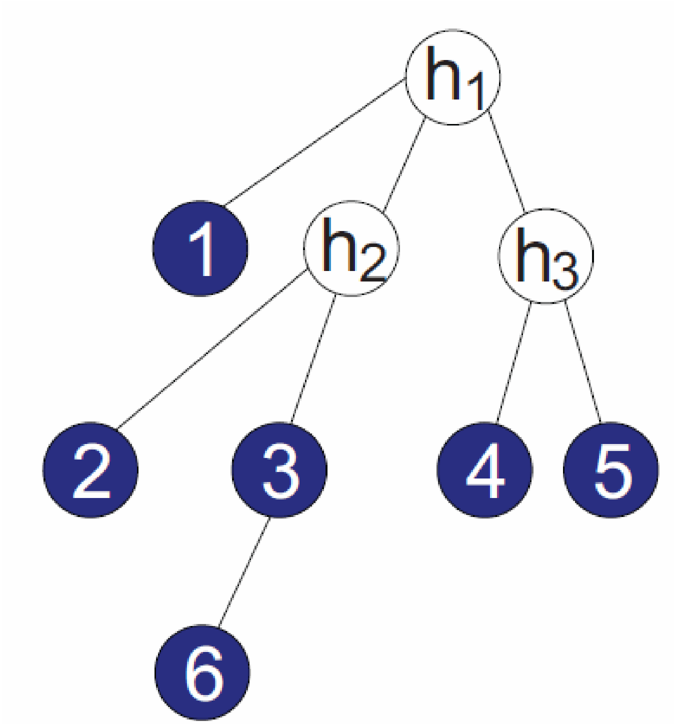
   ⇨ Correct recovery given exact distributions.

2. Computationally efficient

3. Low sample complexity

4. Good empirical performance

# Desired Properties for Algorithms




1. Consistent for minimal latent trees

   ⇨ Correct recovery given exact distributions.

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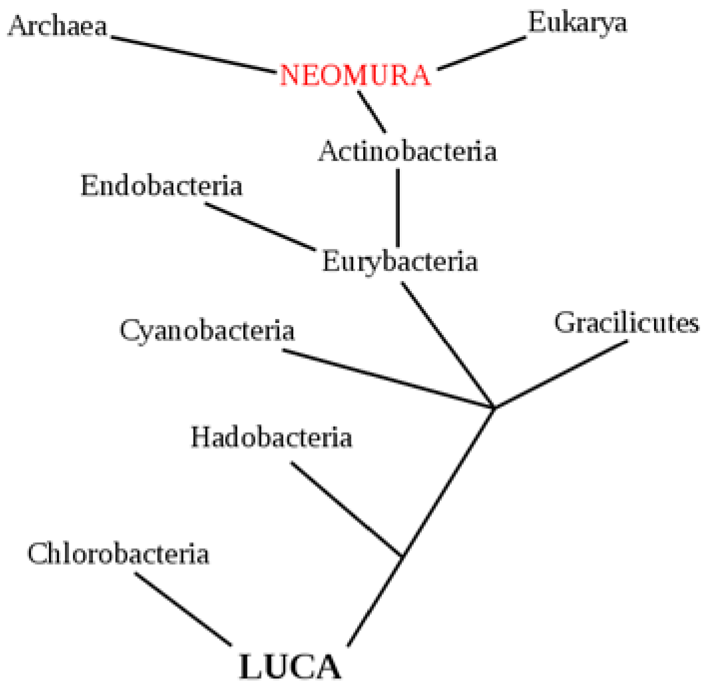
4. Good empirical performance

# Related Work

- EM-based approaches
  - ZhangKocka04, HarmelingWilliams10, ElidanFriedman05
  - No consistency guarantees
  - Computationally expensive
- Phylogenetic trees
  - Neighbor-joining (NJ) method (SaitouNei87)



# Information Distance

- Gaussian distributions

$$d_{ij} := -\log|\rho_{ij}|$$

$$\rho_{ij} := \frac{\mathrm{Cov}(X_i, X_j)}{\sqrt{\mathrm{Var}(X_i)\mathrm{Var}(X_j)}}$$

# Information Distance

- Gaussian distributions

$$d_{ij} := -\log|\rho_{ij}| \qquad \rho_{ij} := \frac{\mathrm{Cov}(X_i, X_j)}{\sqrt{\mathrm{Var}(X_i)\mathrm{Var}(X_j)}}$$

- Discrete distributions

$$d_{ij} := -\log \frac{|\det \mathbf{J}^{ij}|}{\sqrt{\det \mathbf{M}^i \det \mathbf{M}^j}}$$

$\mathbf{J}^{ij}$ Joint probability matrix $\mathbf{M}^i$ Marginal probability matrix

ex)
$$\mathbf{J}^{ij} = \begin{pmatrix} p(x_i = 0, x_j = 0) & p(x_i = 0, x_j = 1) \\ p(x_i = 1, x_j = 0) & p(x_i = 1, x_j = 1) \end{pmatrix}$$

$$\mathbf{M}^i = \begin{pmatrix} p(x_i = 0) & 0 \\ 0 & p(x_i = 1) \end{pmatrix}$$

# Information Distance

- Gaussian distributions

$$d_{ij} := -\log |\rho_{ij}|$$

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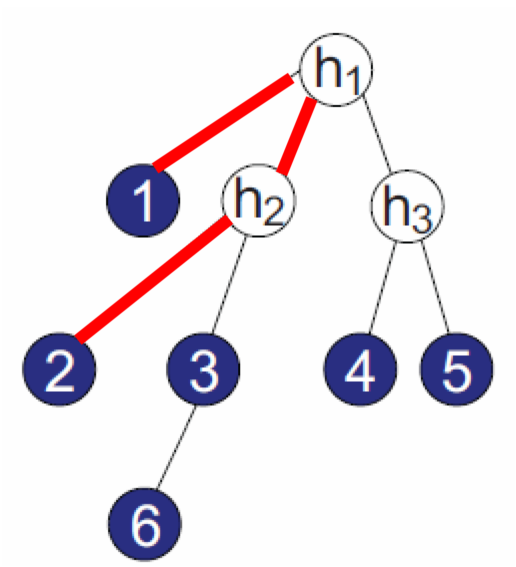
$\mathbf{J}^{ij}$ Joint probability matrix $\mathbf{M}^i$ Marginal probability matrix

- Algorithms use information distances of observed variables.
- Assume first that the exact information distances are given.

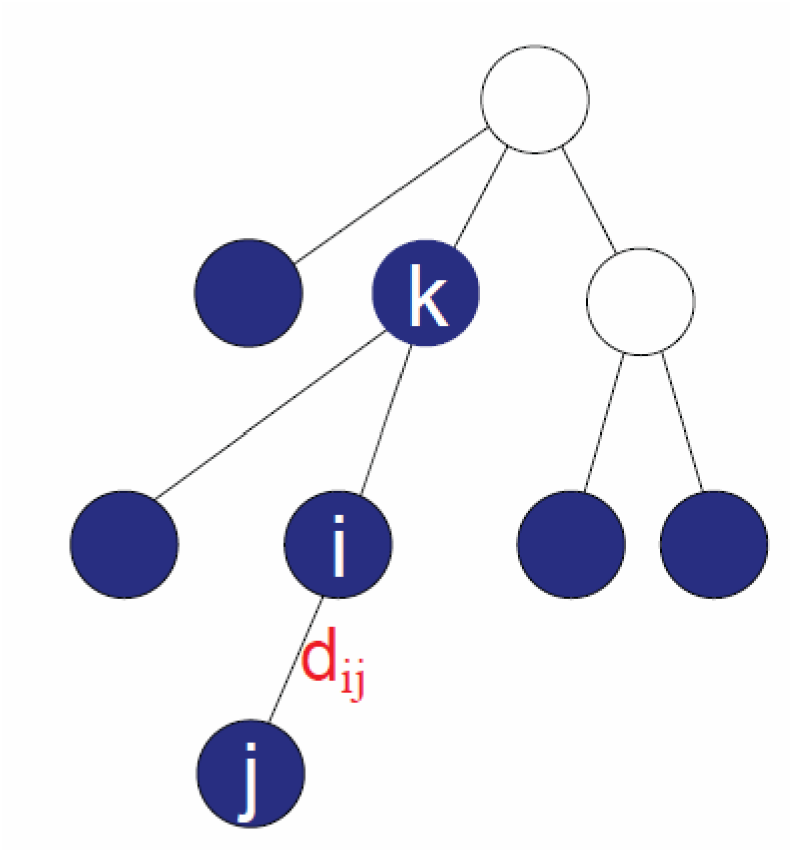
# Additivity of Information Distances on Trees

$$d_{k,l} = \sum_{(i,j) \in \mathrm{Path}((k,l);E_p)} d_{i,j}$$




$$d_{12} = d_{1h_1} + d_{h_1h_2} + d_{2h_2}$$

# Testing Node Relationships




Node j – a leaf node

Node i – parent of j

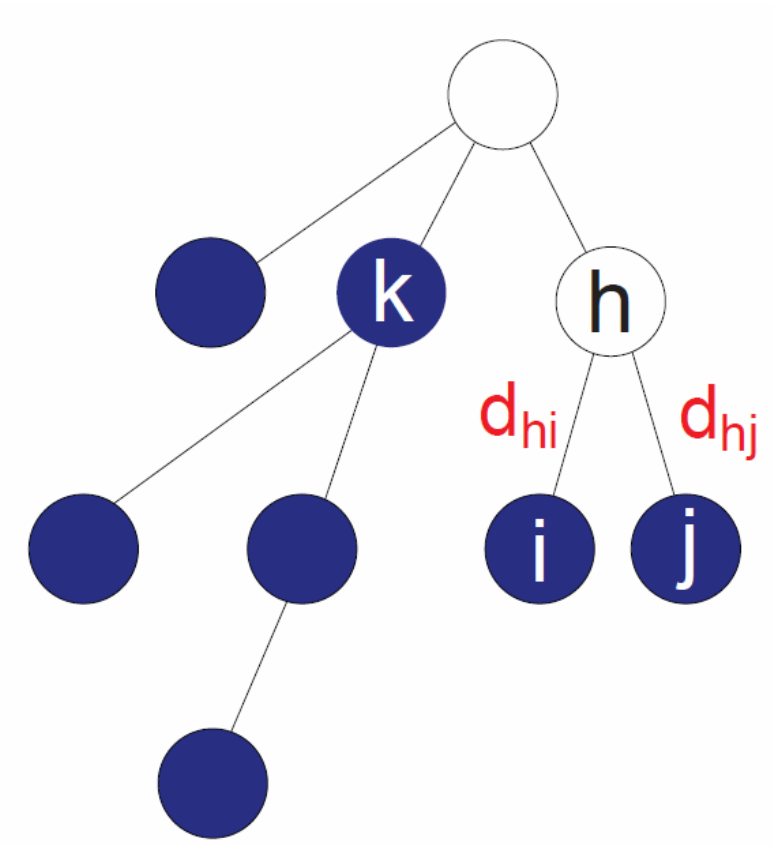
$$\Leftrightarrow d_{jk} - d_{ik} = d_{ij}$$

for all k $\neq$ i, j .

Can identify

(parent, leaf child) pair

# Testing Node Relationships



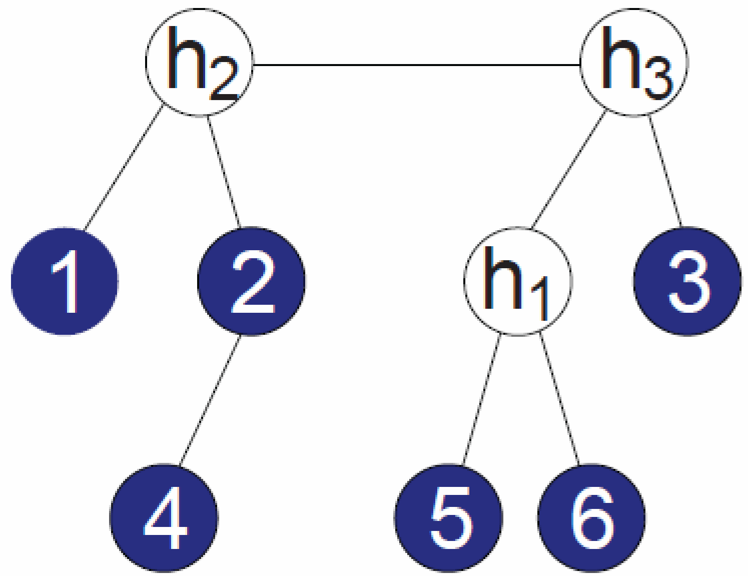

Node i and j – leaf nodes and share the same parent (sibling nodes)

$$\Longleftrightarrow d_{jk} - d_{ik} = d_{hj} - d_{hi}$$

for all k ≠ i, j .

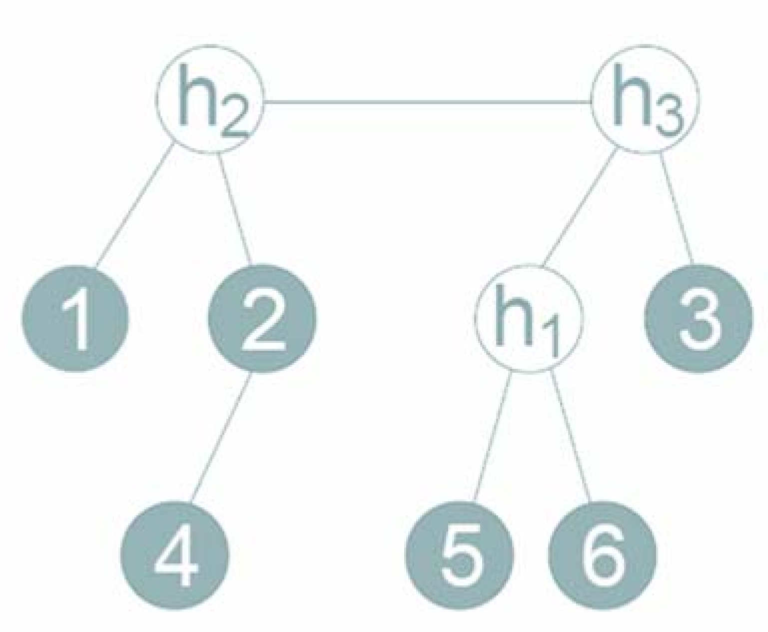
Can identify leaf-sibling pairs.

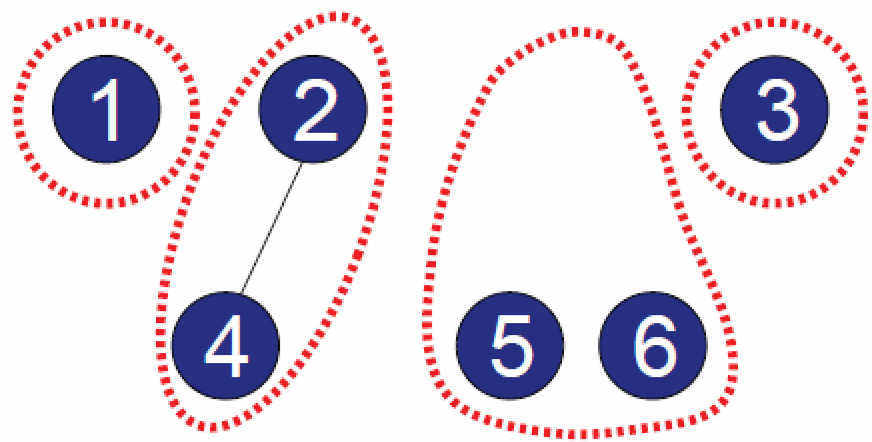
# Recursive Grouping




Step 1. Compute $d_{jk} - d_{ik}$ for all observed nodes (i, j, k).

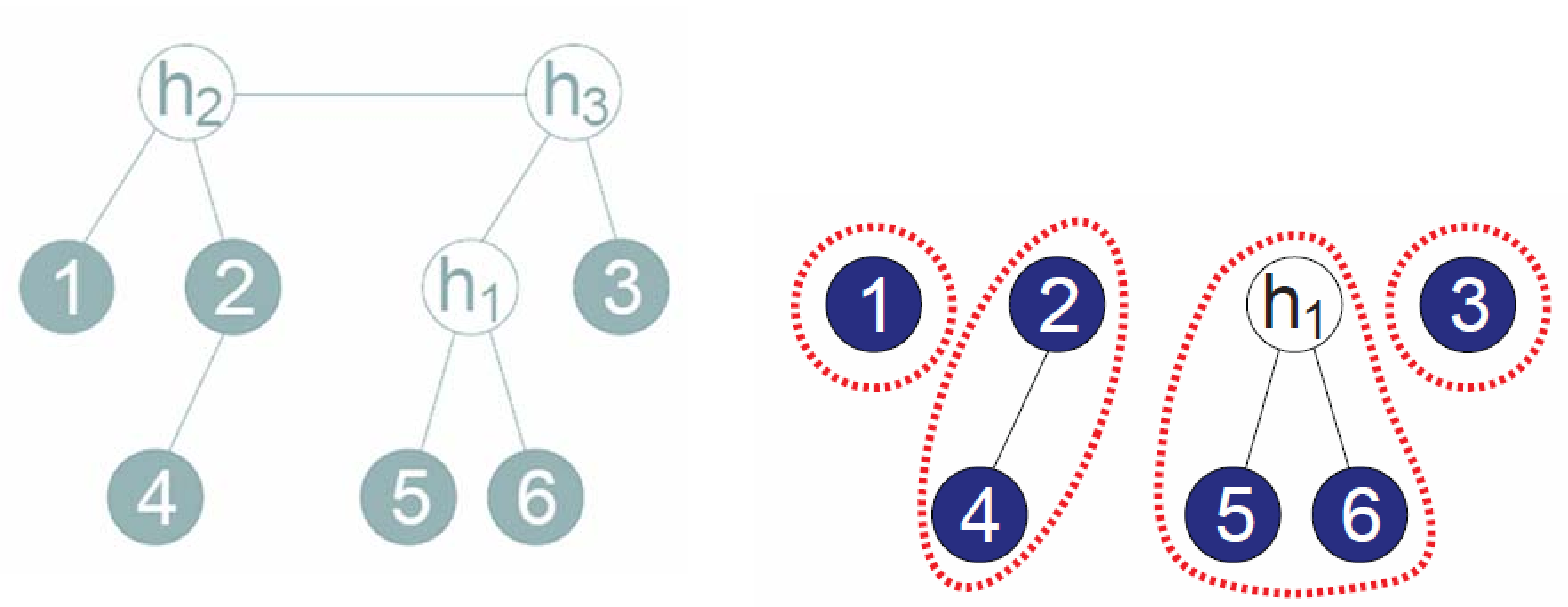
# Recursive Grouping



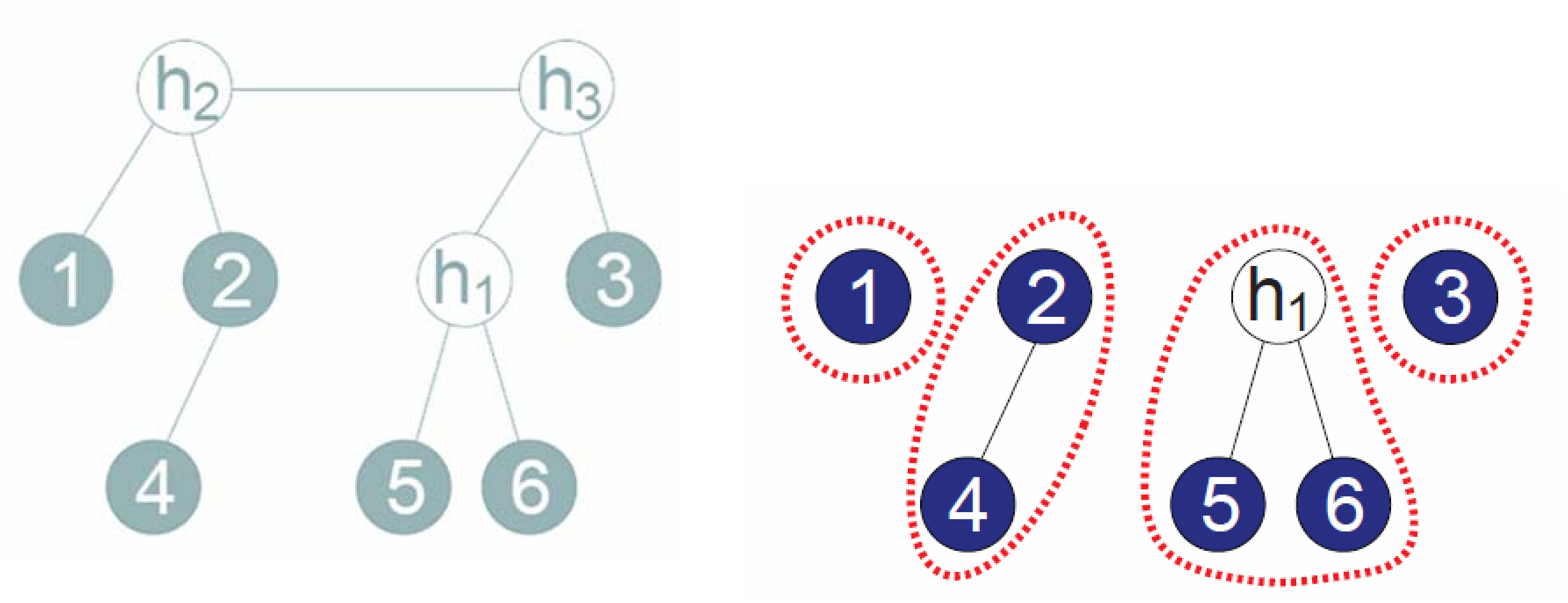

Step 2. Identify (parent, leaf child) or (leaf siblings) pairs.

# Recursive Grouping




Step 3. Introduce a hidden parent node for each sibling group without a parent.

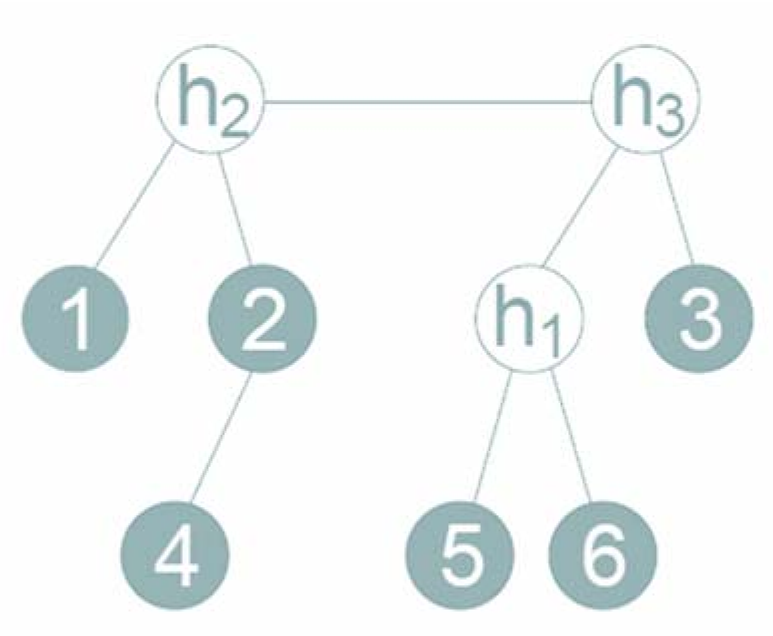
# Recursive Grouping

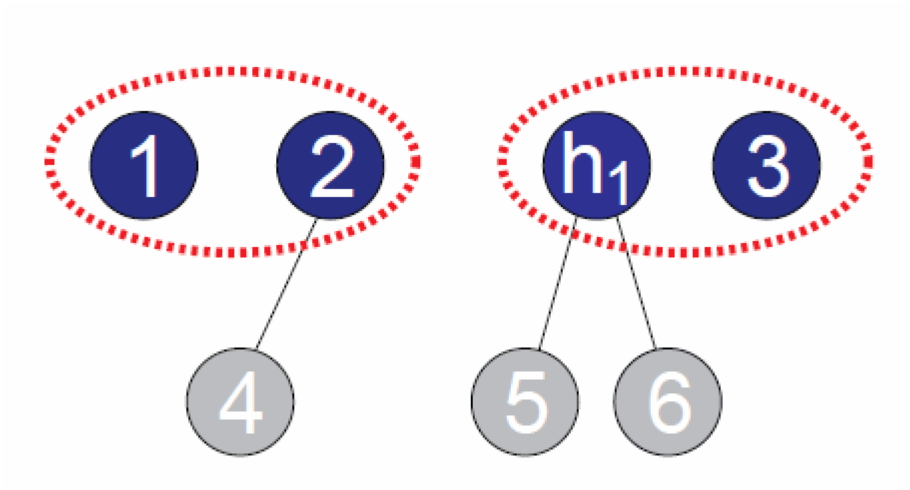



Step 4. Compute the information distance for new hidden nodes.

e.g.) $$d_{5h_1} = \frac{1}{2}(d_{56} + d_{53} - d_{63})$$

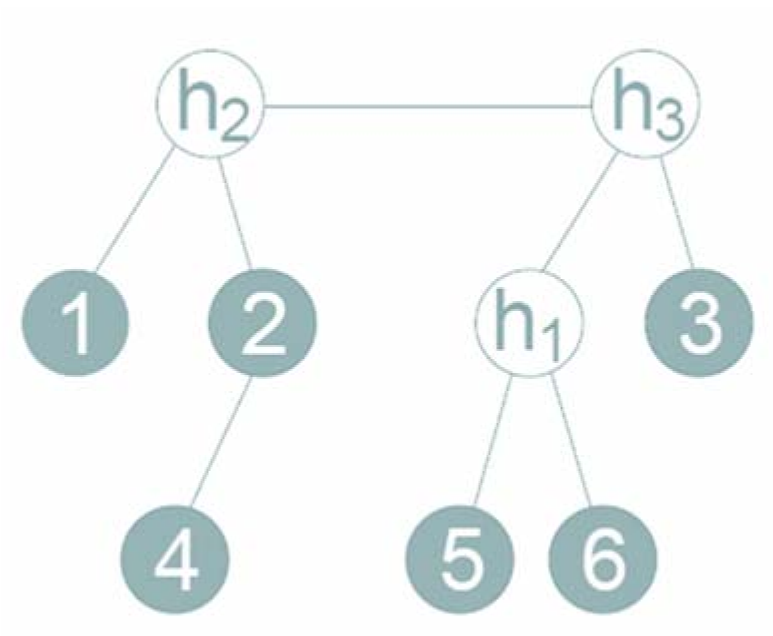
# Recursive Grouping

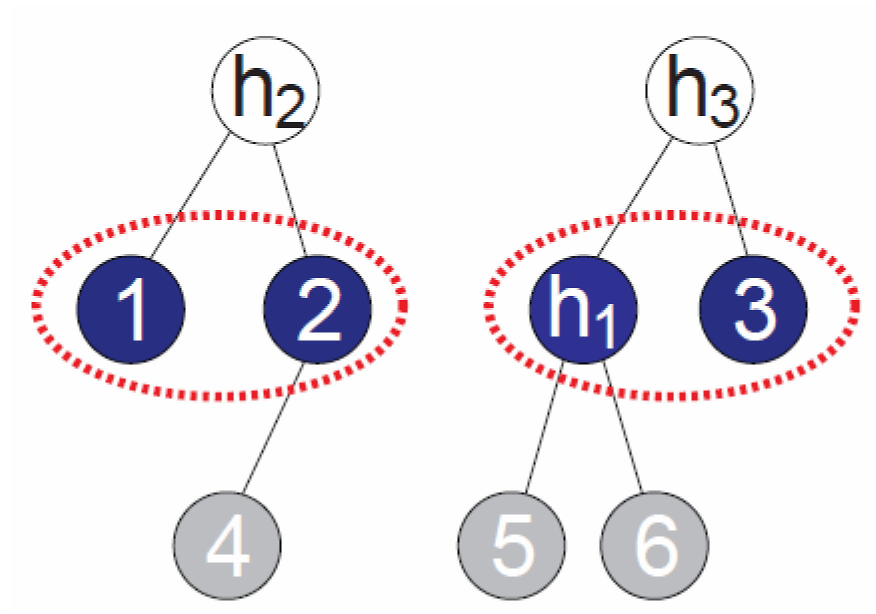



Step 5. Remove the identified child nodes and repeat Steps 2-4.

# Recursive Grouping

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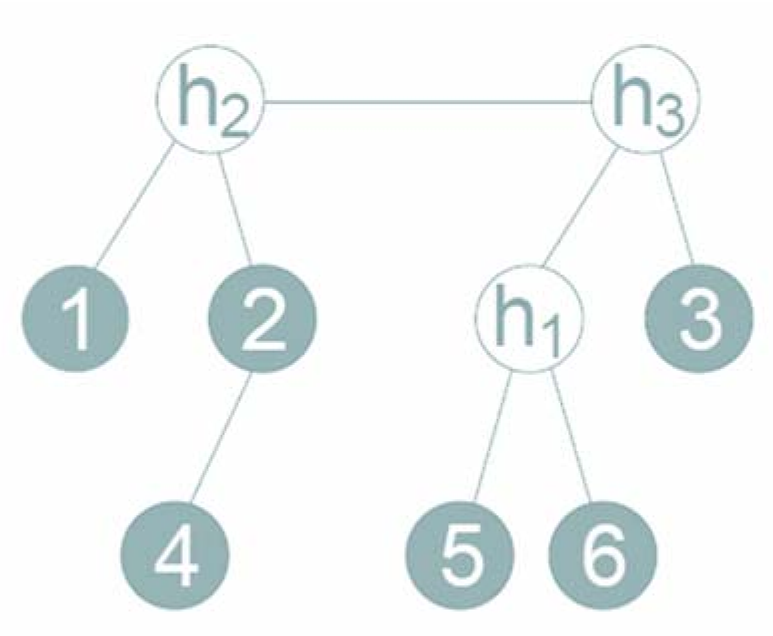
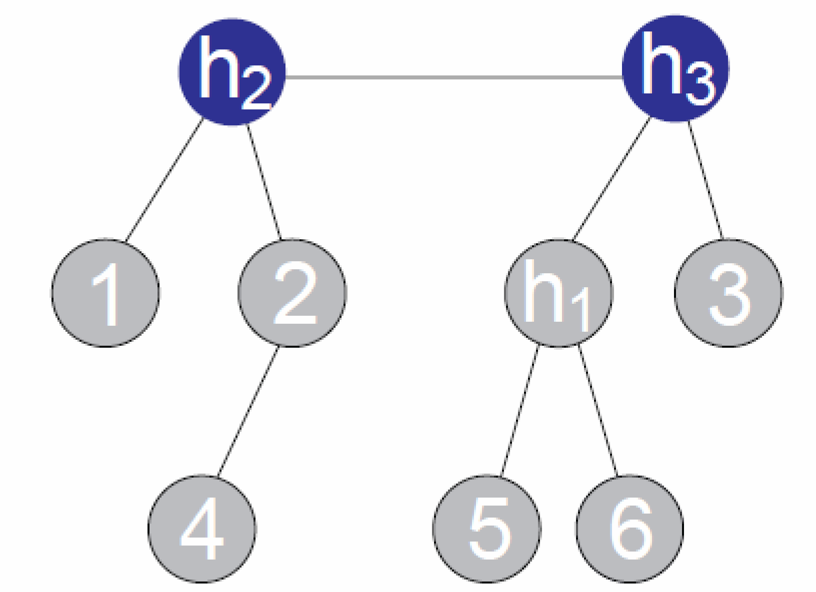
# Recursive Grouping



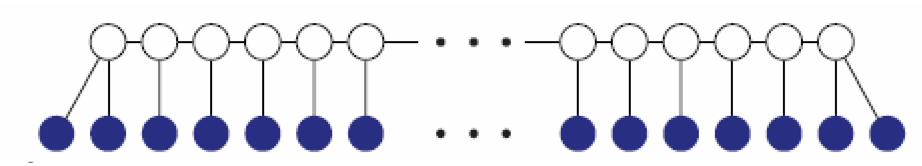




Step 5. Remove the identified child nodes and repeat Steps 2-4.

# Recursive Grouping

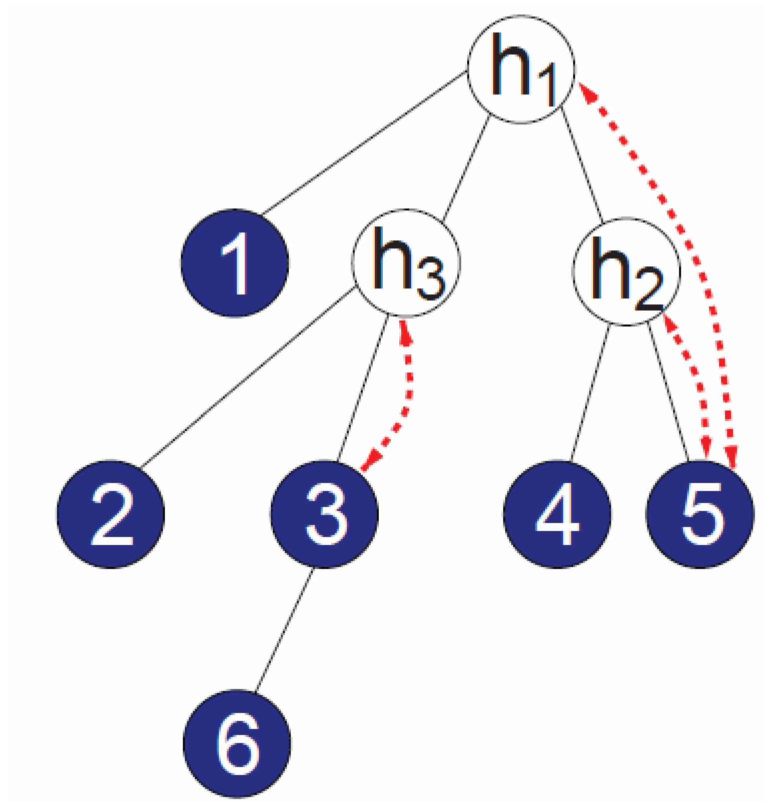
- Identifies a group of family nodes at each step.
- Introduces hidden nodes recursively.
- Correctly recovers all minimal latent trees.

- Computational complexity O(diam(T) $m^3$).



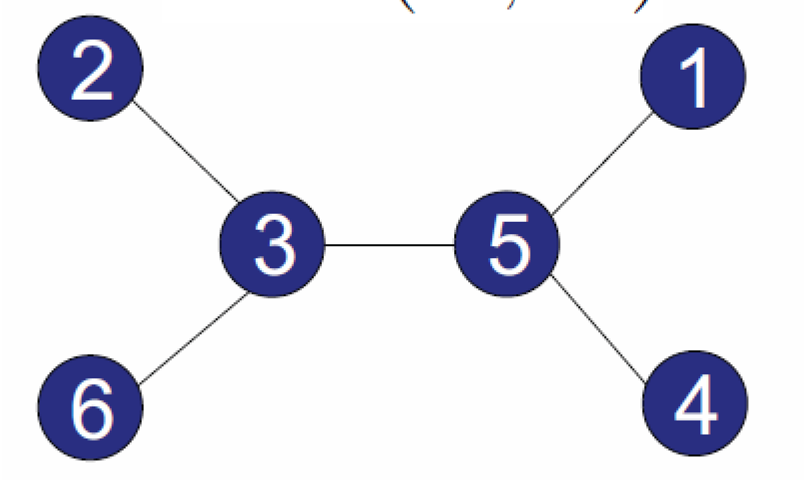
Worst case O($m^4$)

# CLGrouping Algorithm



# Chow-Liu Tree

$$\mathrm{MST}(V; \mathbf{D})$$




Minimum spanning tree of V

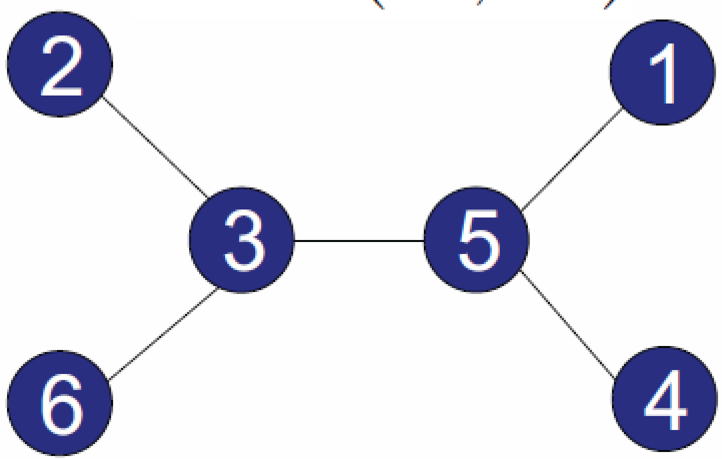
using D as edge weights

V = set of observed nodes

D = information distances

# Chow-Liu Tree

$\mathrm{MST}(V; \mathbf{D})$



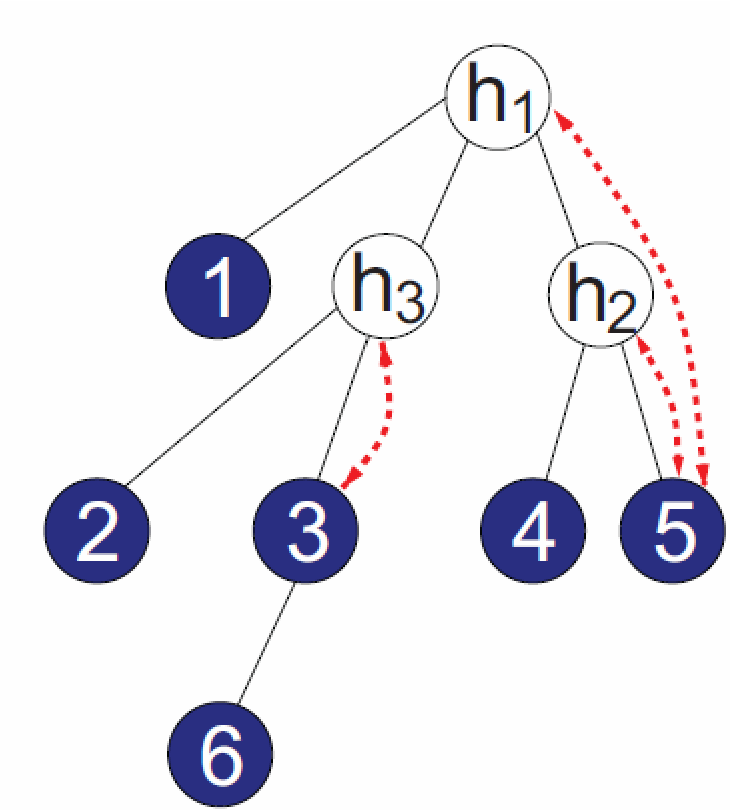

Minimum spanning tree of V using D as edge weights

V = set of observed nodes

D = information distances

- Computational complexity $O(m^2 \log m)$
- For Gaussian models, MST(V; D) = Chow-Liu tree (minimizes KL-divergence to the distribution given by D).

# Surrogate Node



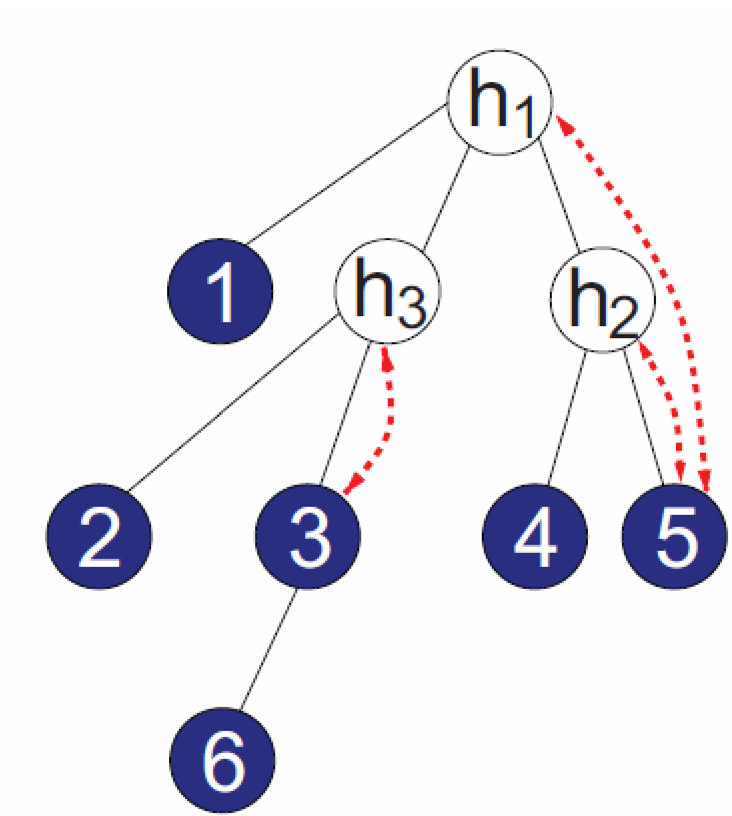

V = set of observed nodes

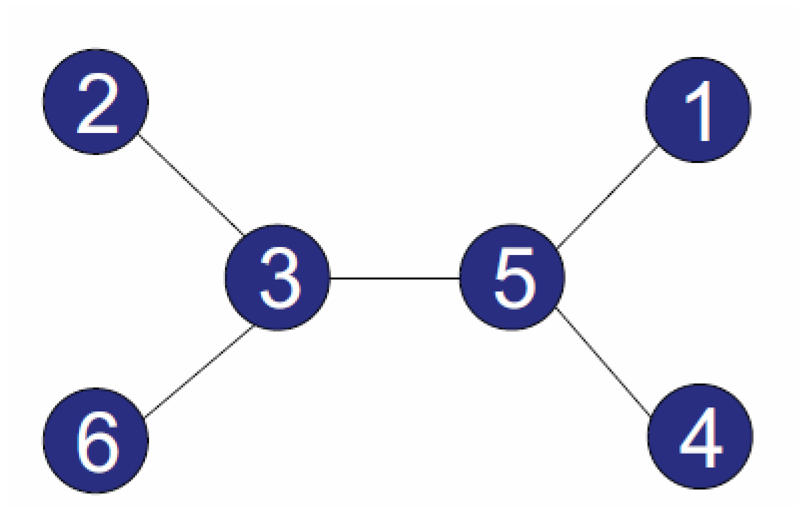
Surrogate node of i

$$\mathrm{Sg}(i) := \operatorname*{argmin}_{j \in V} d_{ij}$$

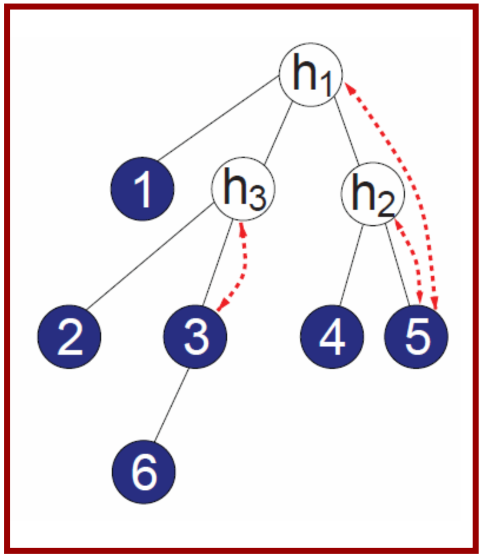
# Property of the Chow-Liu Tree

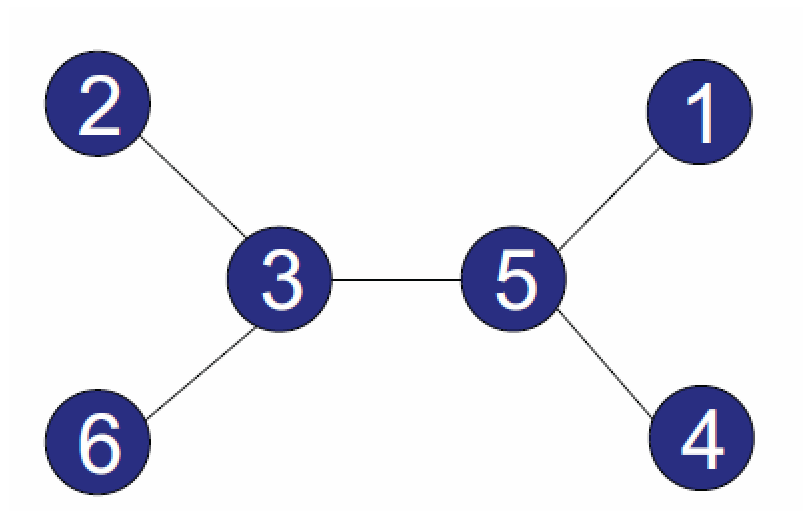
$$\boxed{(i,j) \in E_p \Rightarrow (\mathrm{Sg}(i), \mathrm{Sg}(j)) \in \mathrm{MST}(V; \mathbf{d})}$$



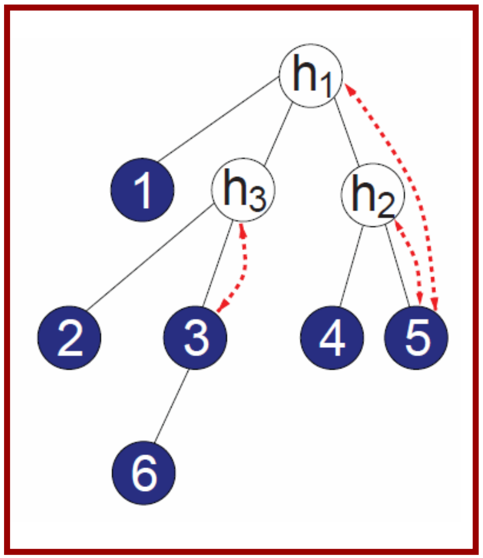
# CLGrouping Algorithm

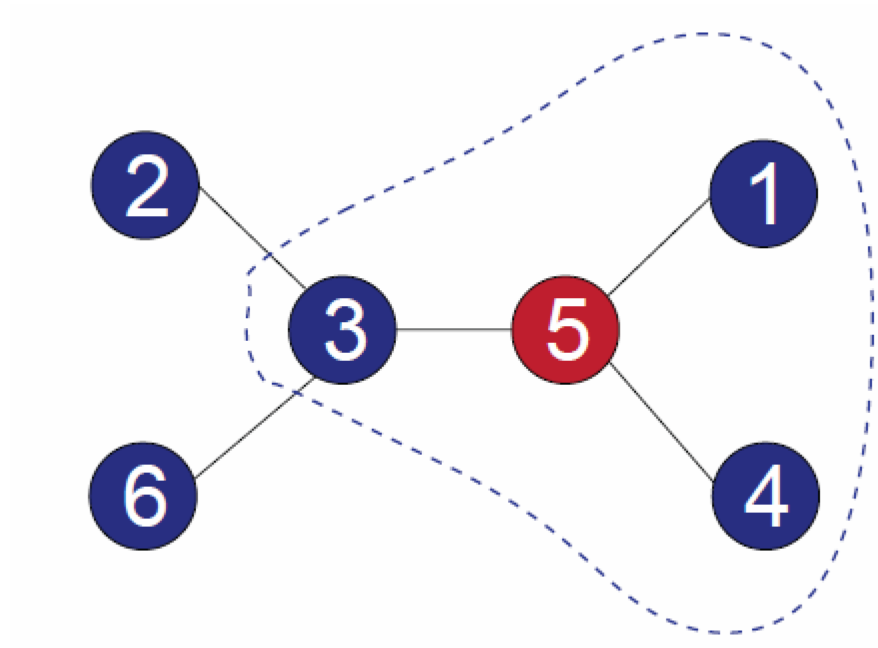



Step 1. Using information distances of observed nodes, construct the Chow-Liu tree, MST(V; D). Identify the set of internal nodes {3, 5}.

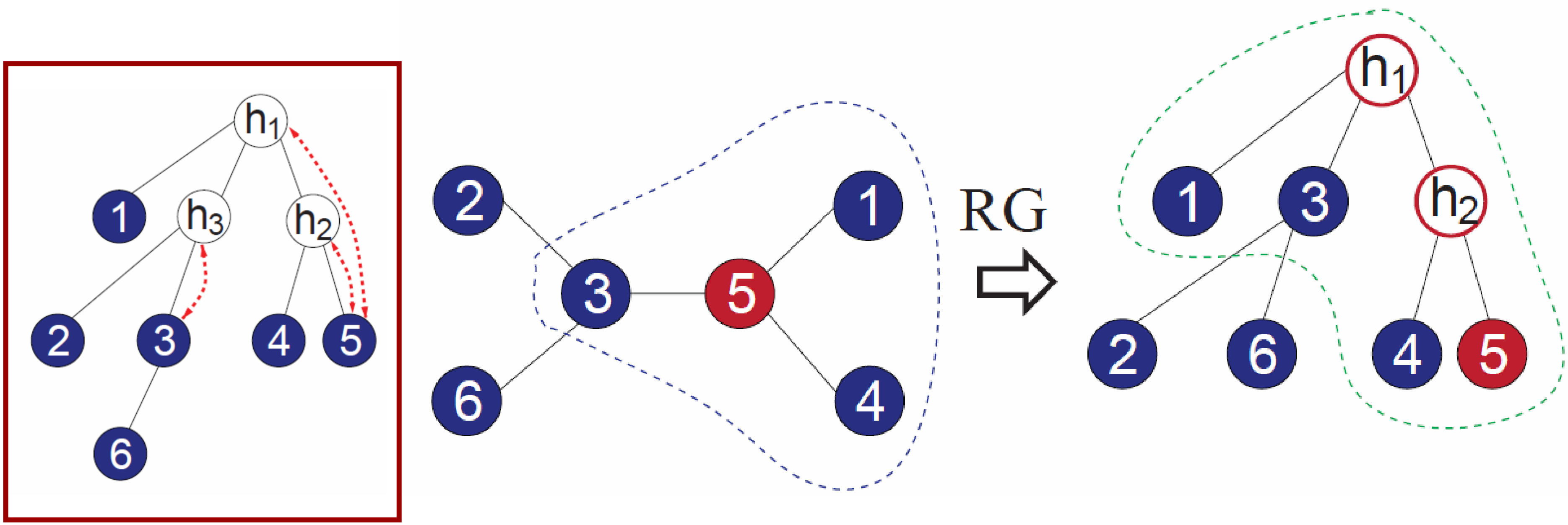
# CLGrouping Algorithm



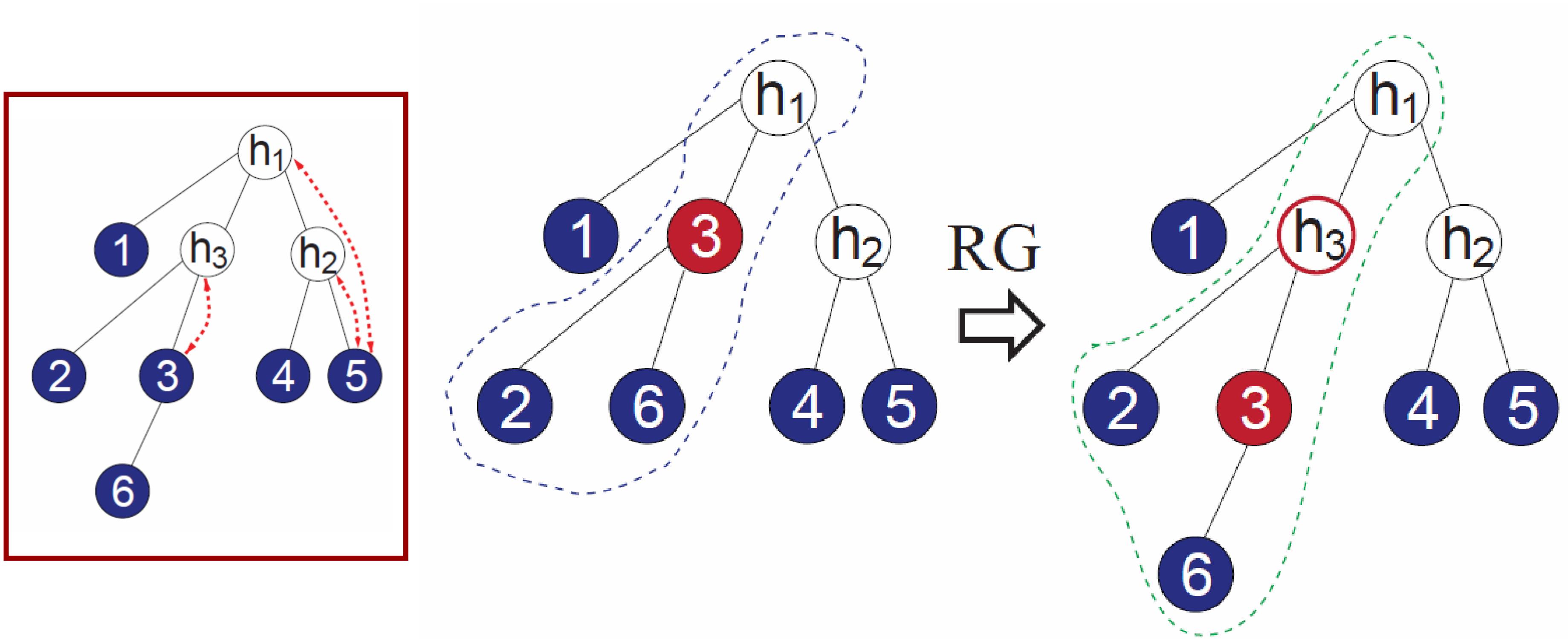

Step 2. Select an internal node and its neighbors, and apply the recursive-grouping (RG) algorithm.

# CLGrouping Algorithm




Step 3. Replace the output of RG with the sub-tree spanning the neighborhood.

# CLGrouping Algorithm

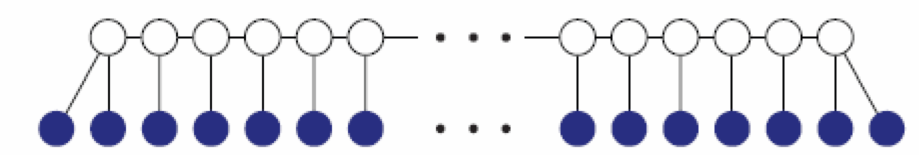



Repeat Steps 2-3 until all internal nodes are operated on.

# CLGrouping

- Step 1: Constructs the Chow-Liu tree, MST(V; D).
- Step 2: For each internal node and its neighbors, applies latent-tree-learning subroutines (RG or NJ).
- Correctly recovers all minimal latent trees.
- Computational complexity

  $O(m^2 \log m + (\#\text{internal nodes})\,(\text{maximum degree})^3)$.



$O(m^2 \log m)$

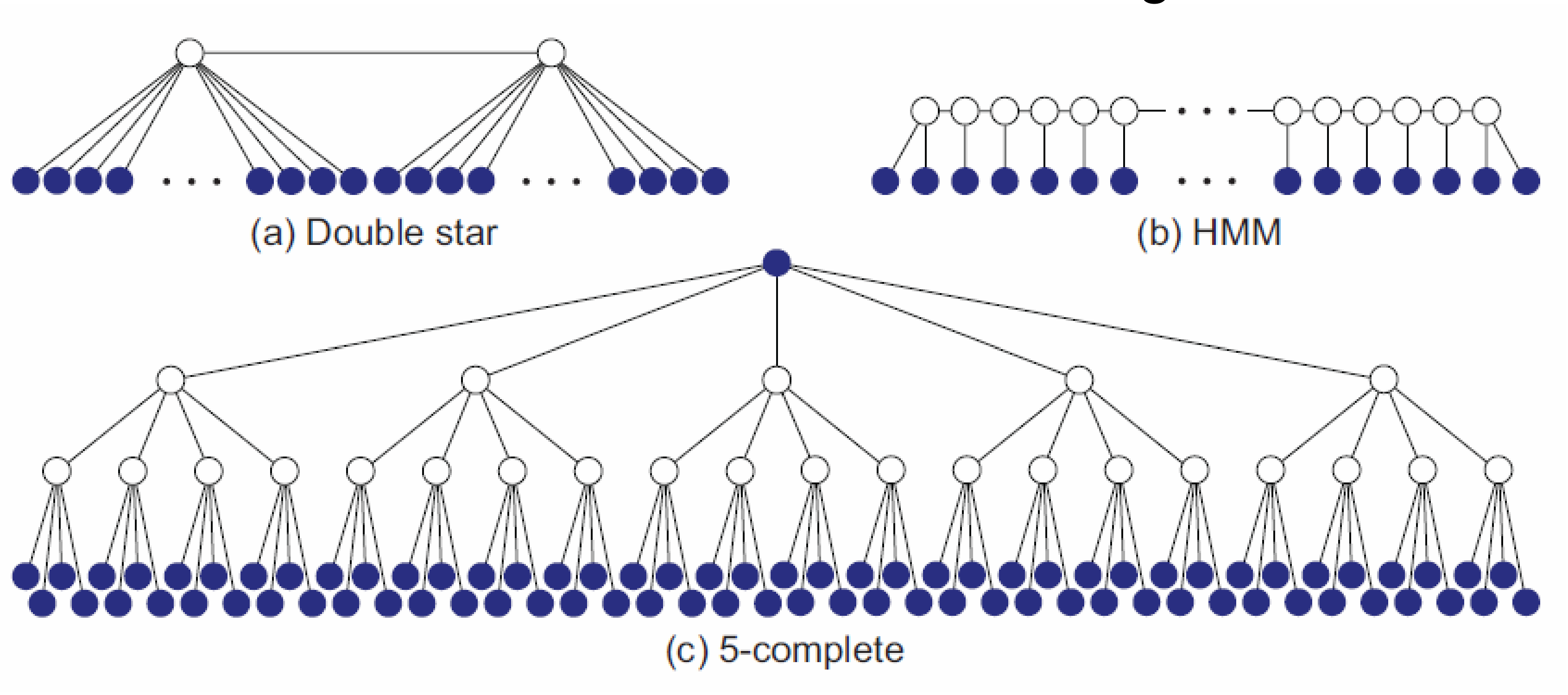
# Sample-based Algorithms

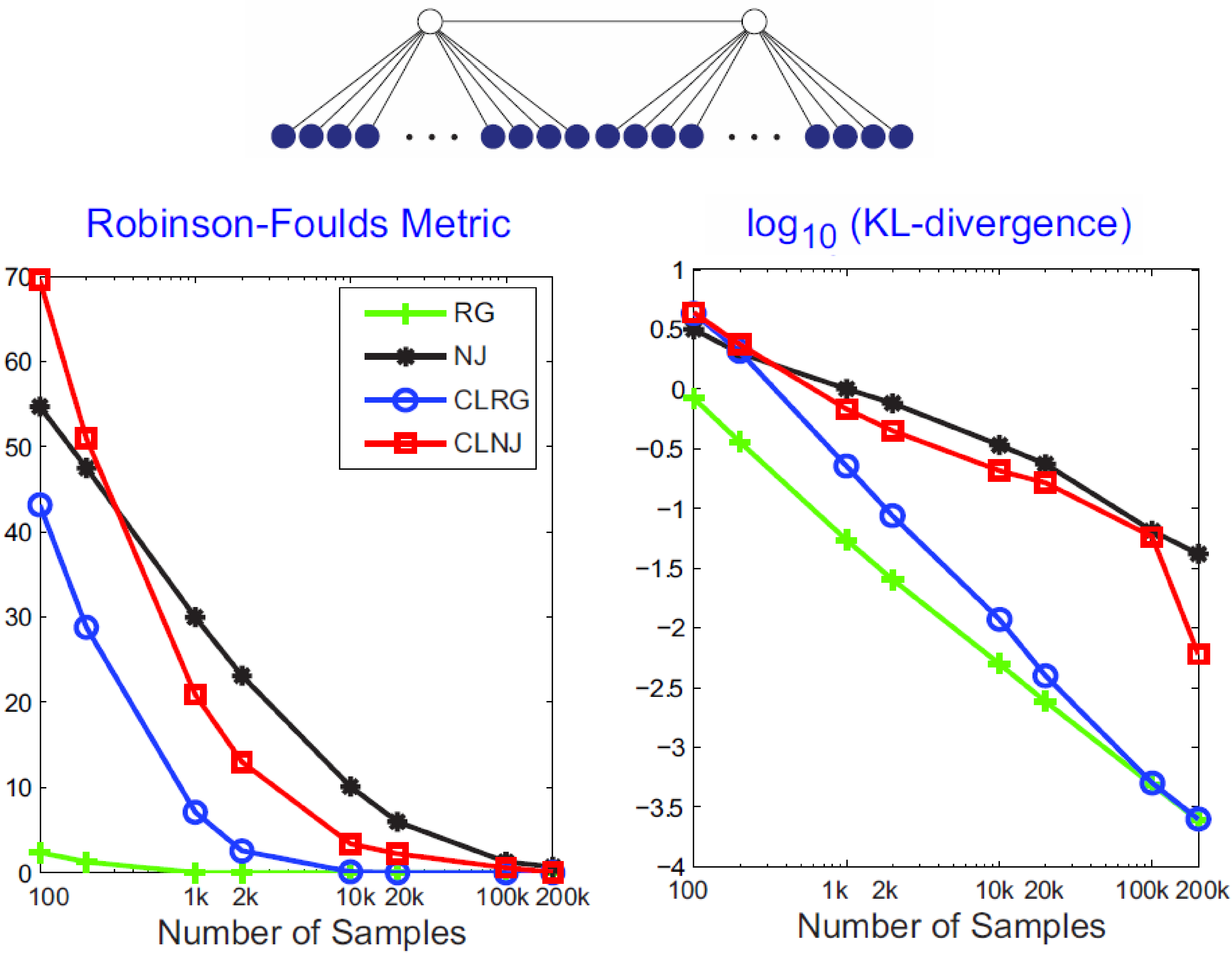
- Compute the ML estimates of information distances.
- Relaxed constraints for testing node relationships.

- Consistent.
- More details in the paper

# Experimental Results

- Simulations using Synthetic Datasets
  - Compares RG, NJ, CLRG, and CLNJ.
  - Robinson-Foulds Metric and KL-divergence.





Robinson-Foulds Metric
log10 (KL-divergence)
RG
NJ
CLRG
CLNJ
70
60
50
40
30
20
10
0
1
0.5
0
−0.5
−1
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100
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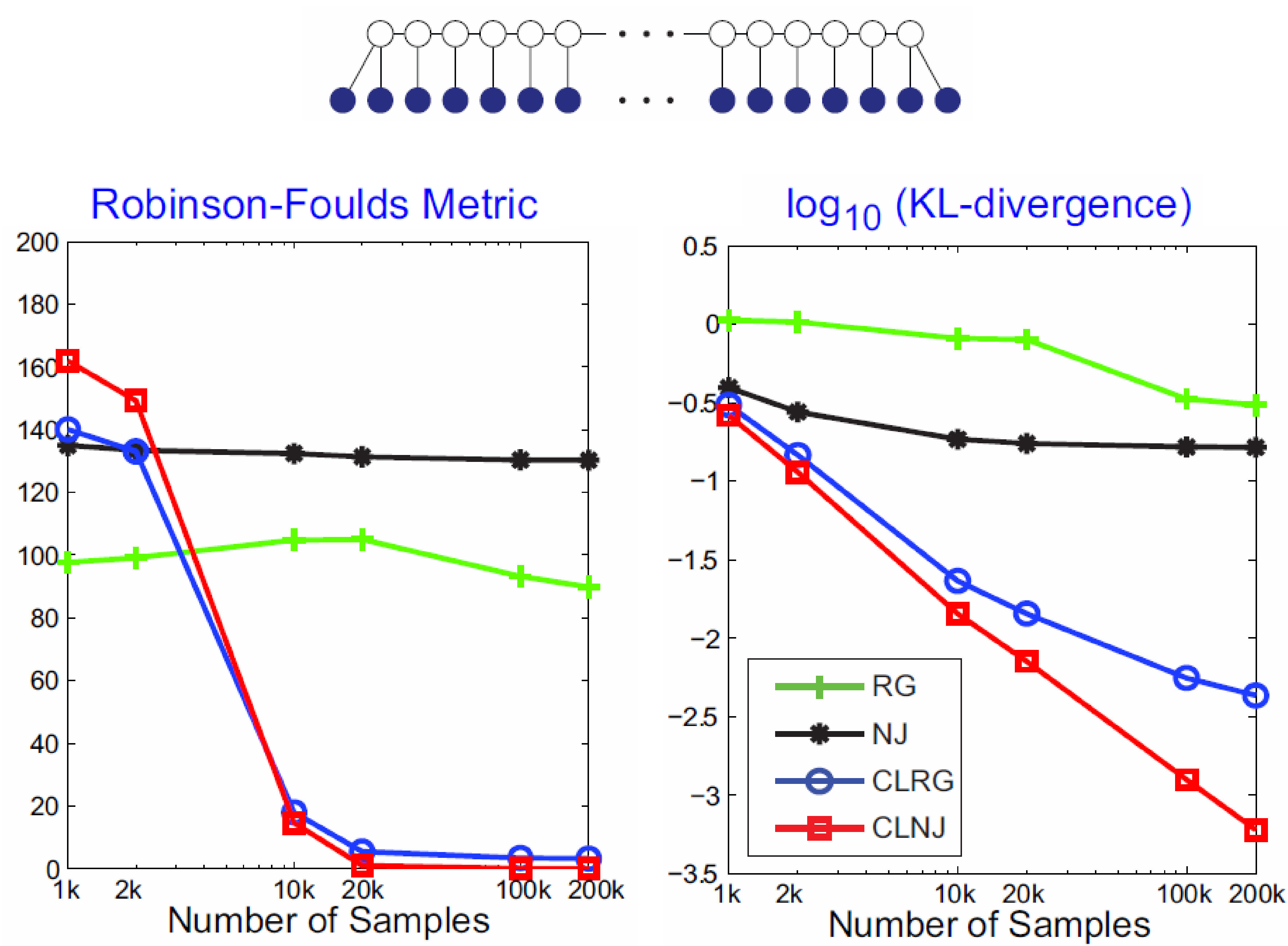
10k
20k
100k
200k
Number of Samples
Number of Samples


Robinson-Foulds Metric
log10 (KL-divergence)
200
180
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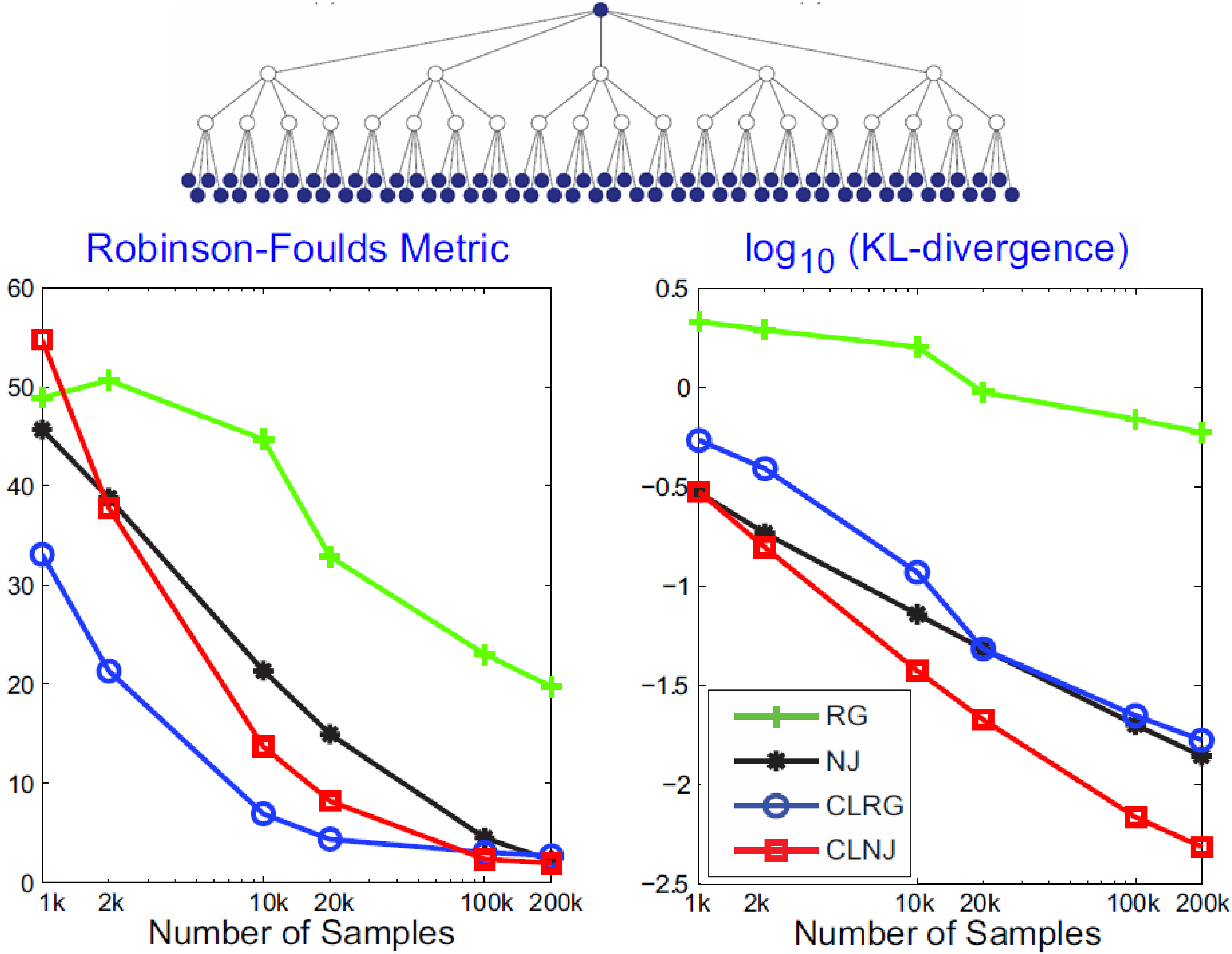
1k
2k
10k
20k
100k
200k
Number of Samples
RG
NJ
CLRG
CLNJ


Robinson-Foulds Metric
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−1
−1.5
−2
−2.5
1k
2k
10k
20k
100k
200k
Number of Samples
RG
NJ
CLRG
CLNJ

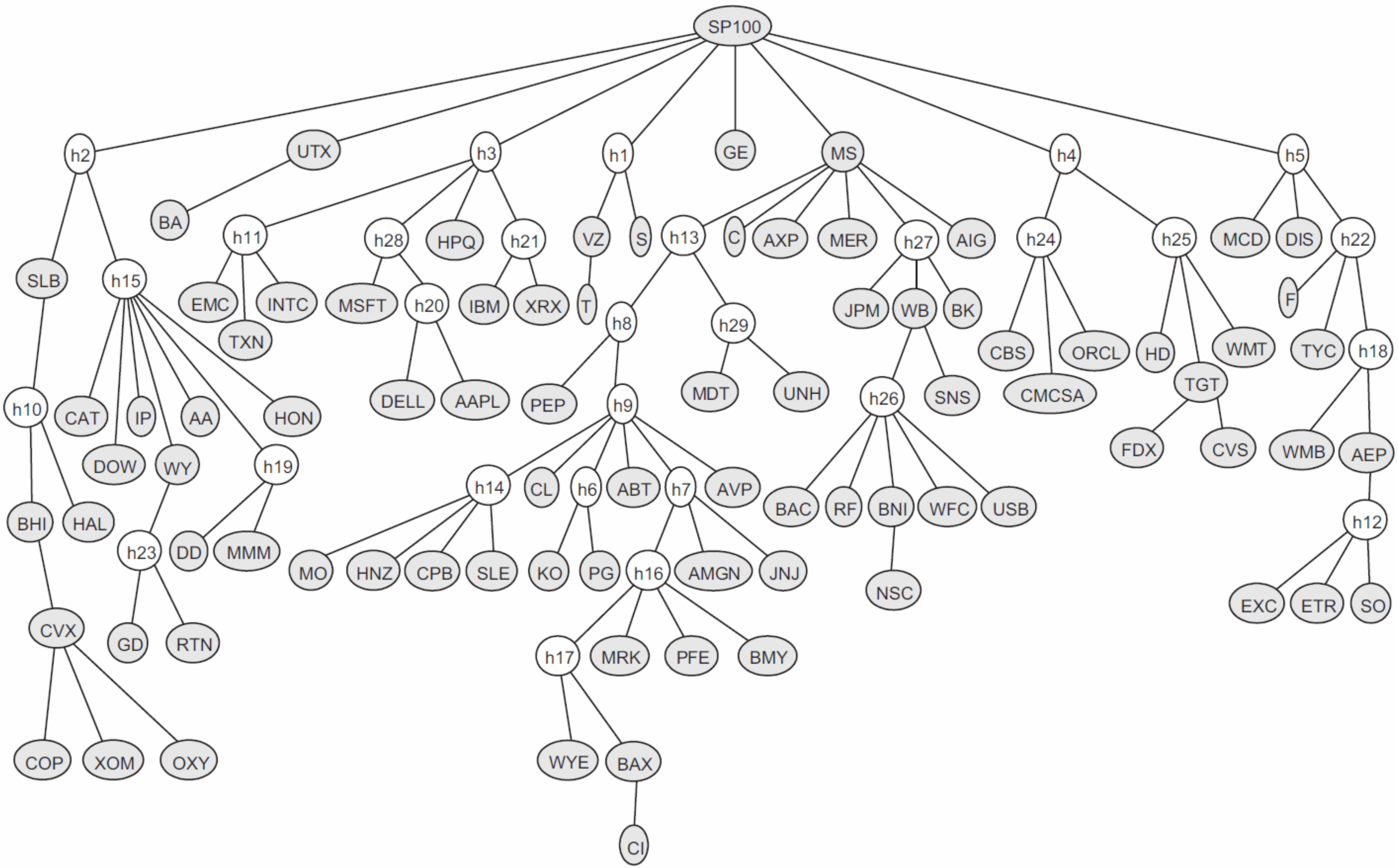
# Performance Comparisons

- For a double star, RG is clearly the best.
- NJ is poor in recovering HMM.
- CLGrouping performs well in all three structures.

- Average running time for CLGrouping < 1 second.

# Monthly Stock Returns

- Monthly returns of 84 companies in S&P 100.
- Samples from 1990 to 2007.
- Latent tree learned using CLNJ.



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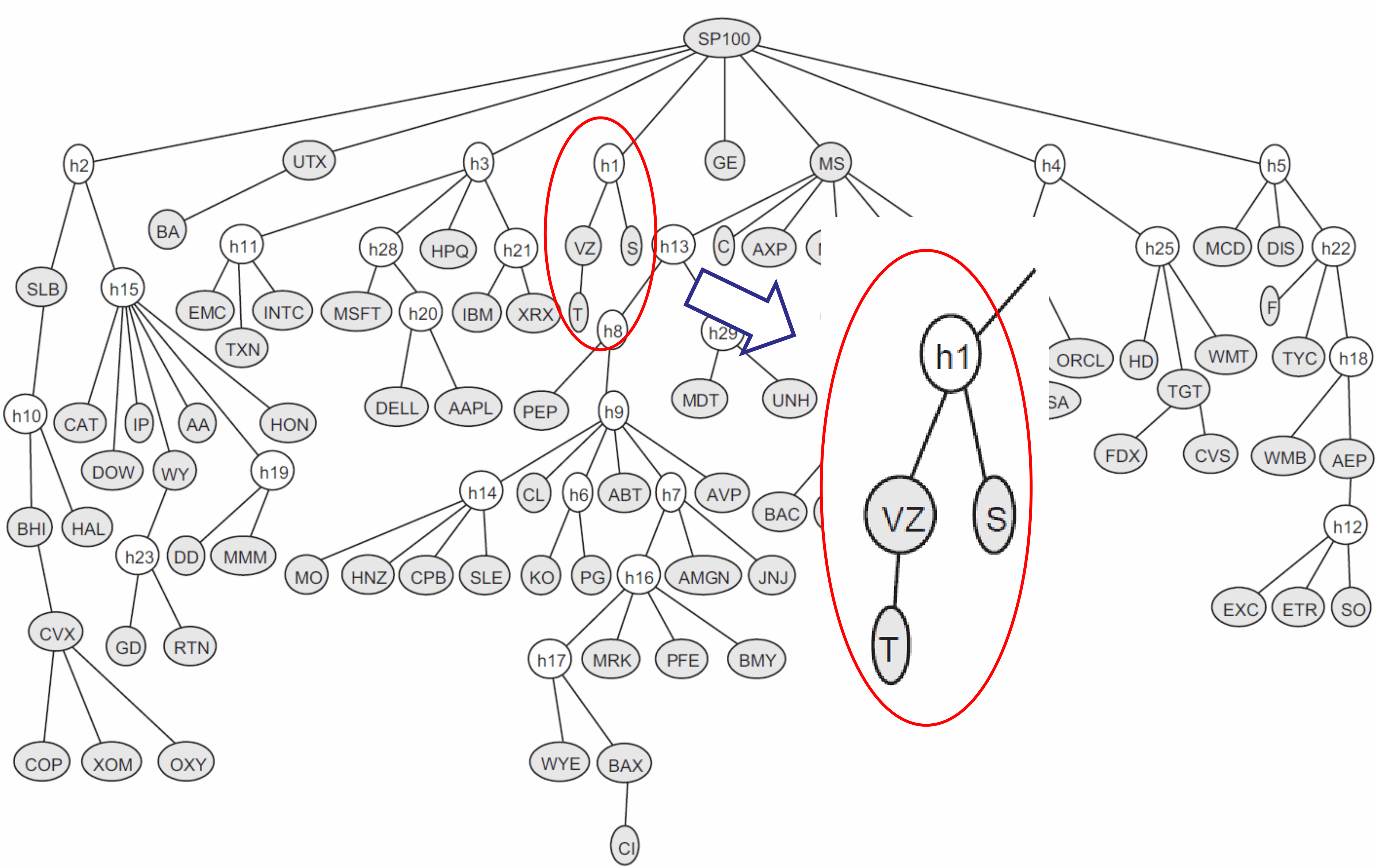
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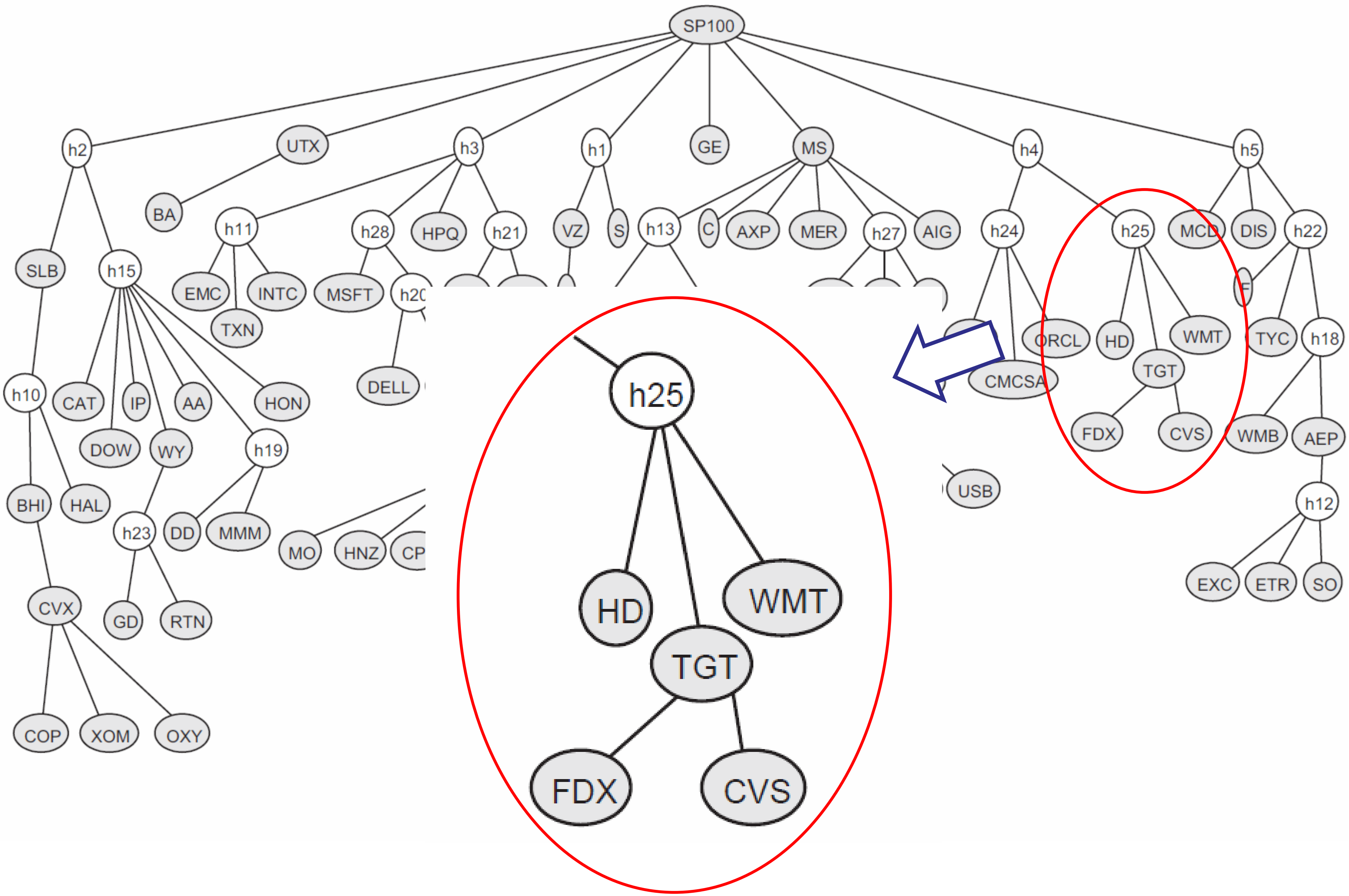

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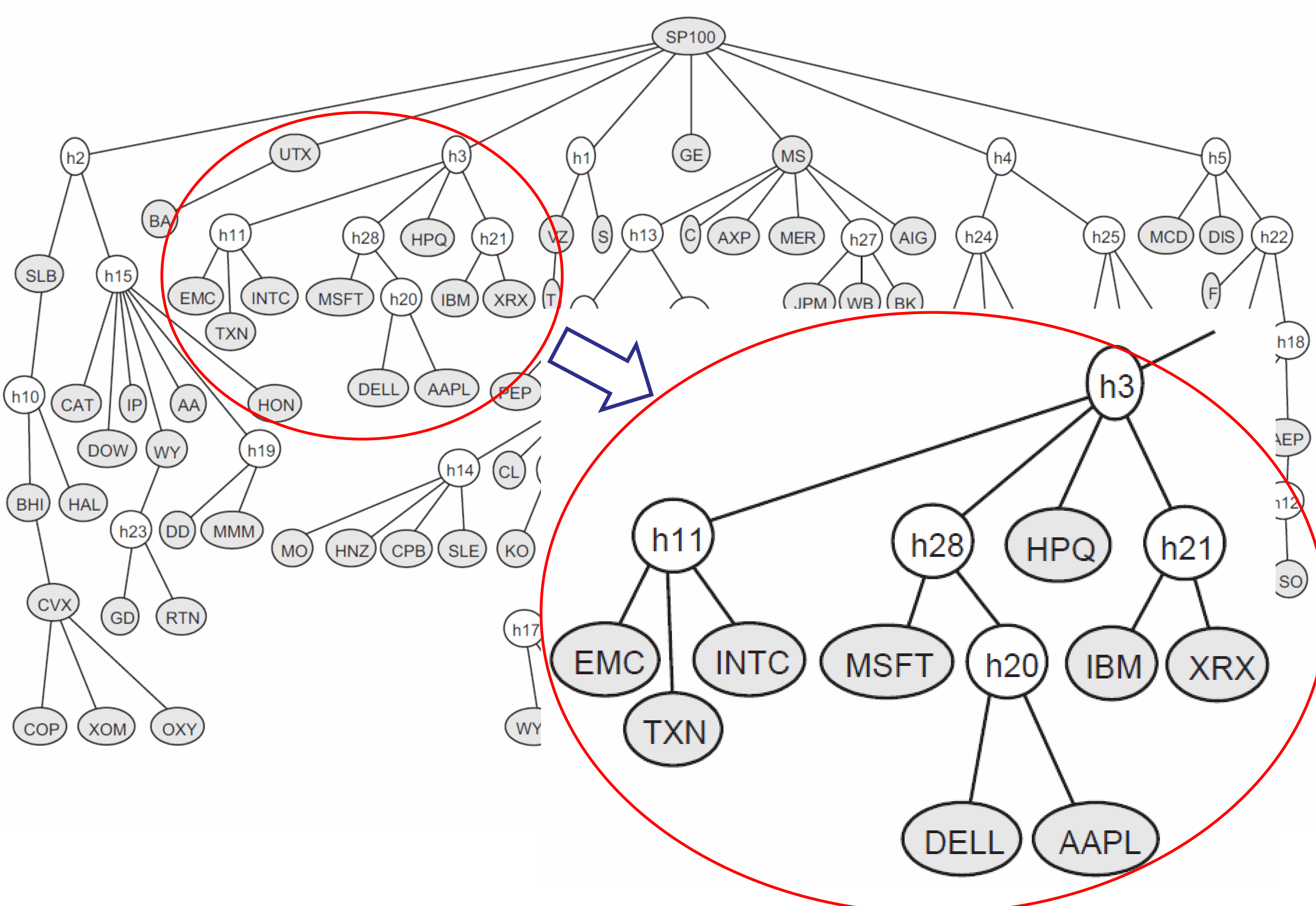
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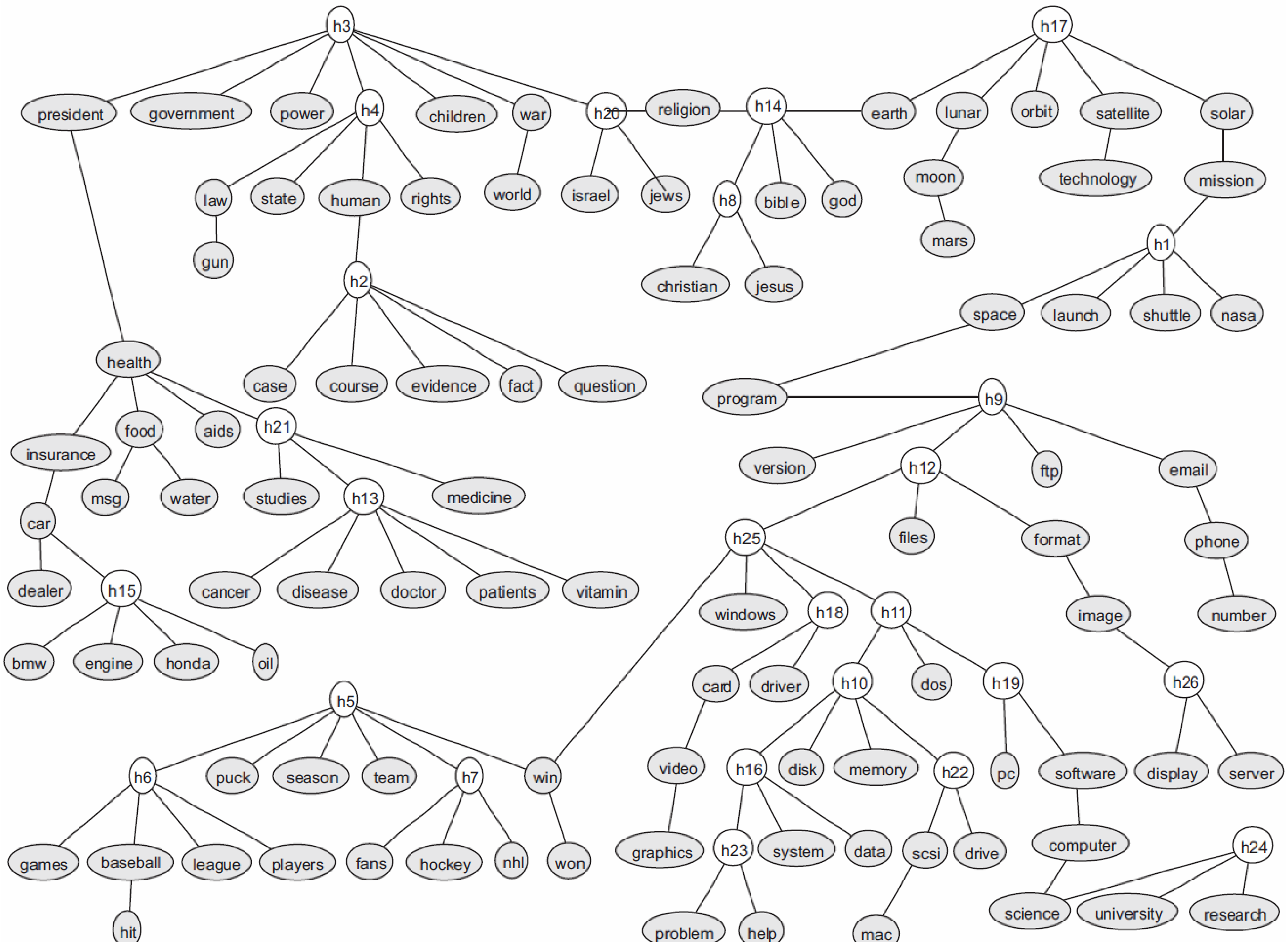
# 20 Newsgroups with 100 Words

- 16,242 binary samples of 100 words
- Latent tree learned using regCLRG.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jesus
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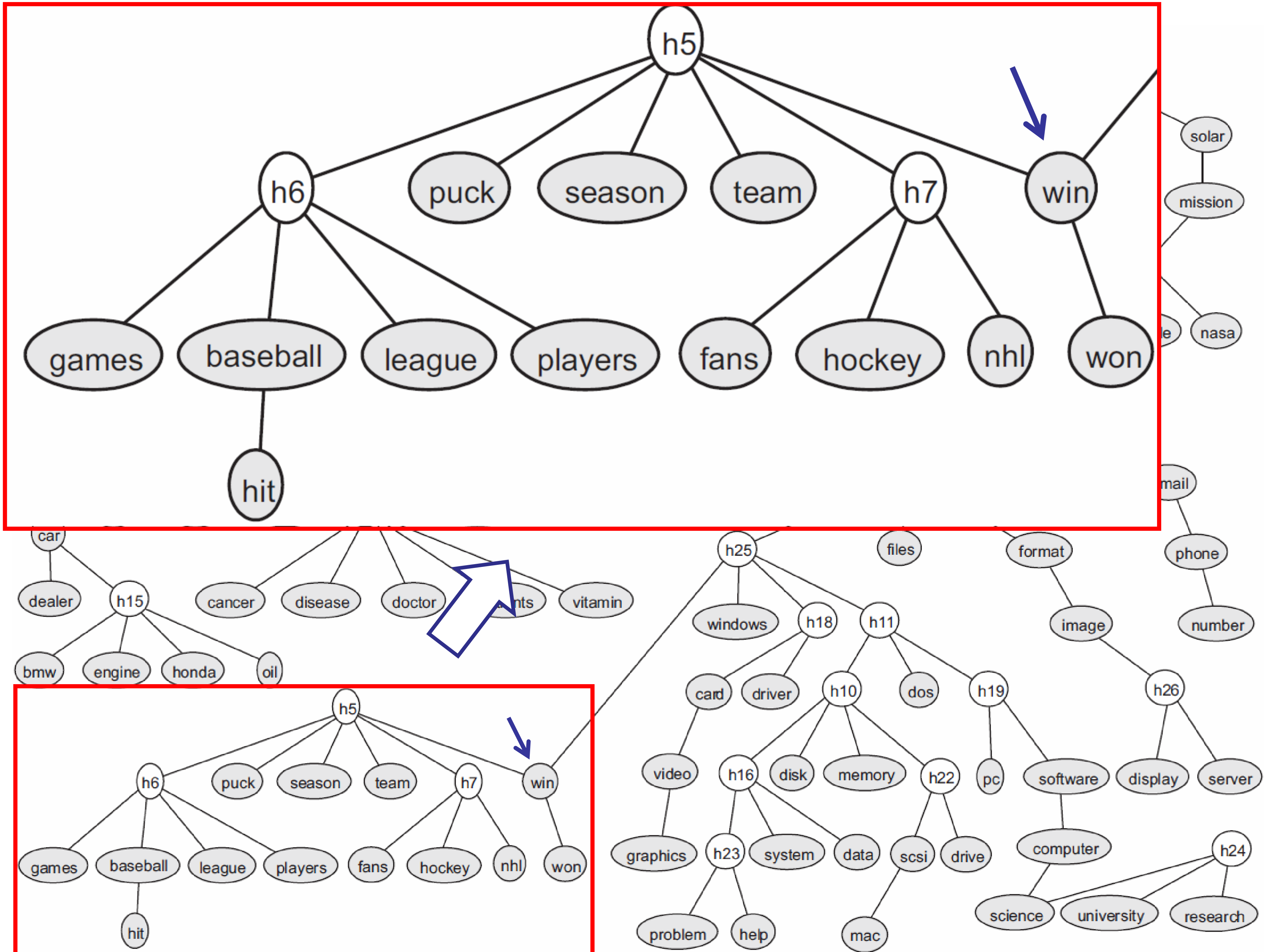
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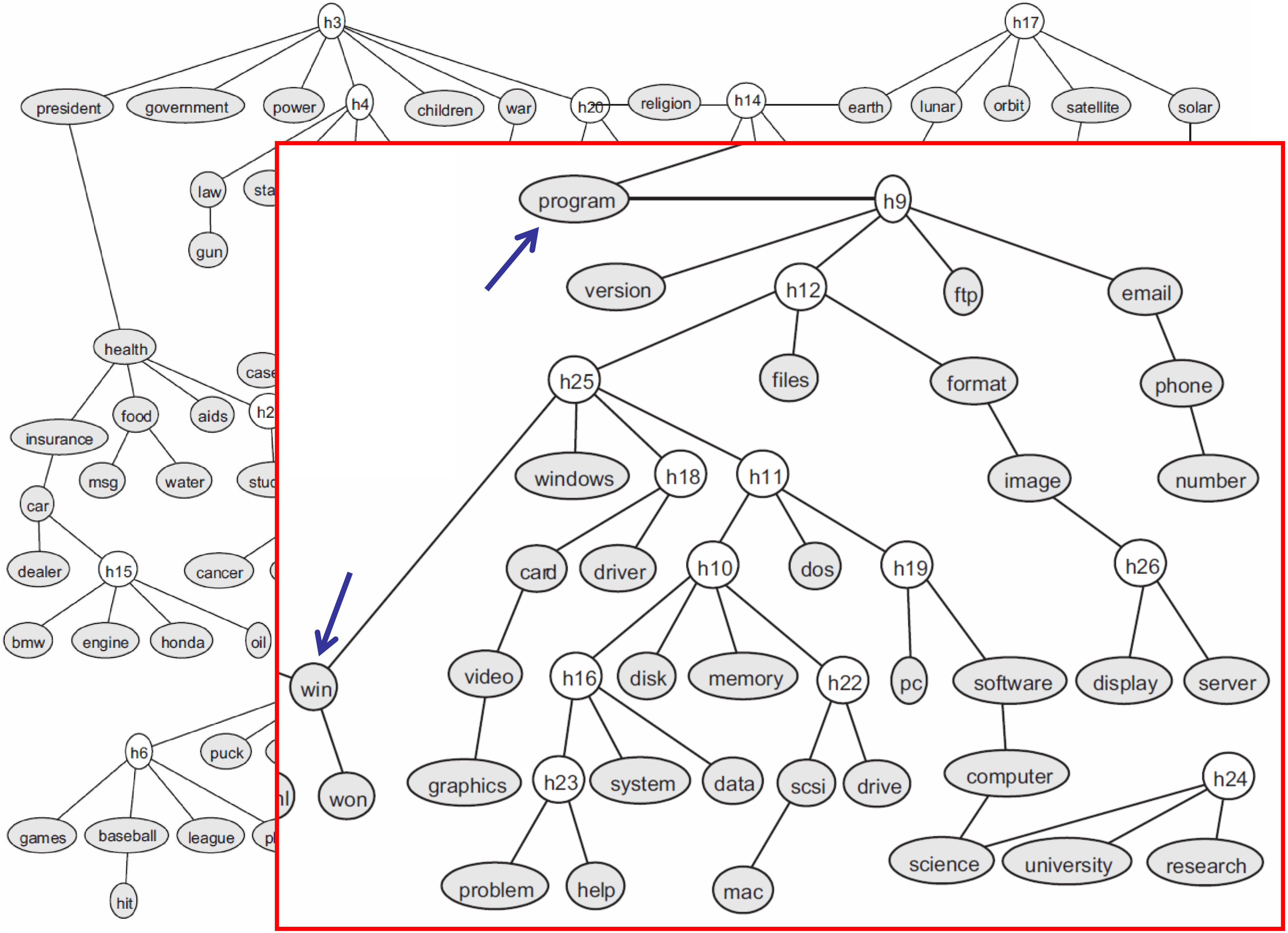
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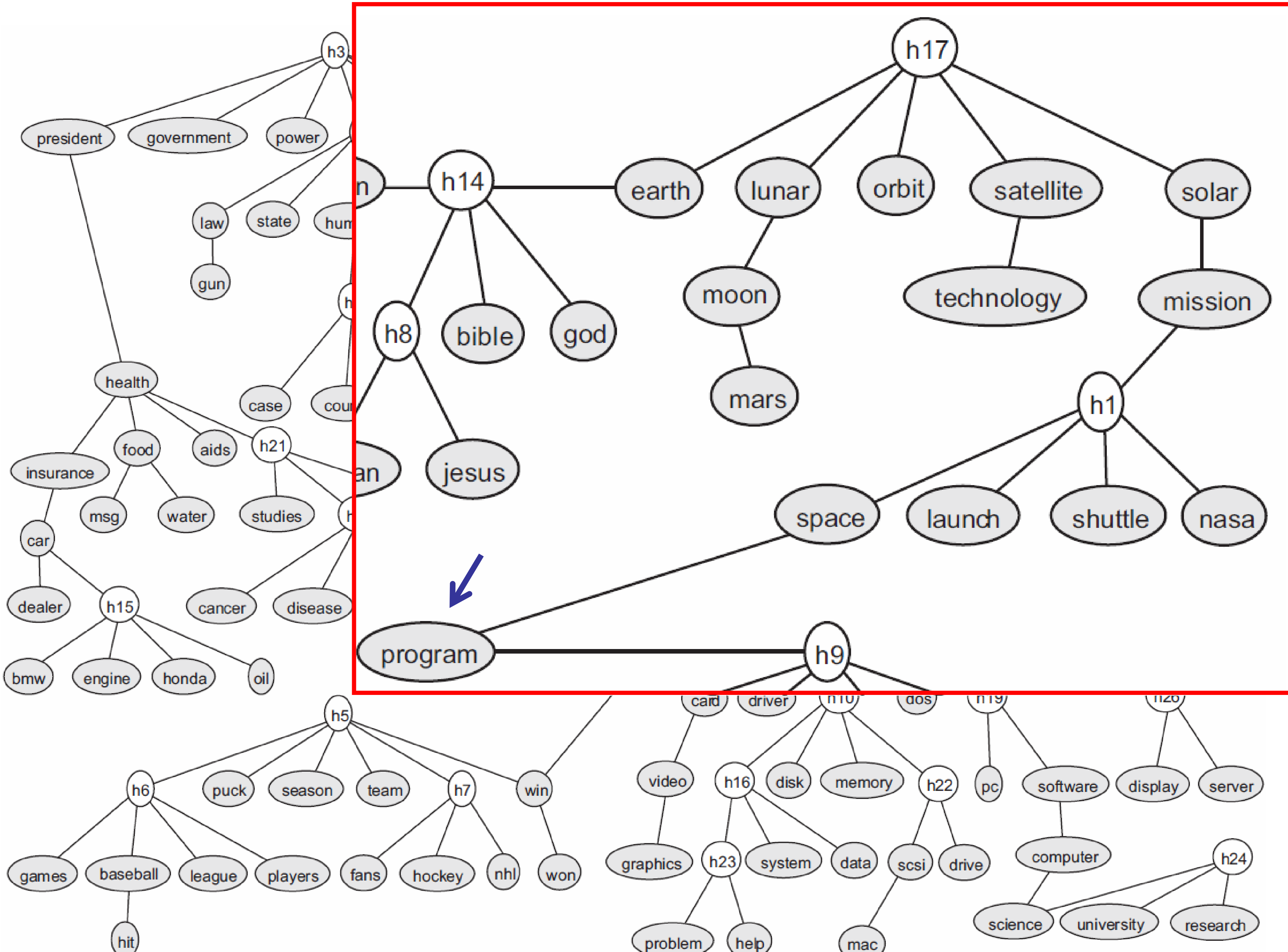
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# Contributions

- Recursive-grouping
  - Identifies families and introduces hidden nodes recursively.
- CLGrouping
  - First learns the Chow-Liu tree
  - Then applies latent-tree-learning subroutines locally.

# Contributions

- Recursive-grouping
- CLGrouping

- Consistent.
- CLGrouping - superior experimental results in both accuracy and computational efficiency.

- Longer version of the paper and MATLAB implementation available at the project webpage.

  http://people.csail.mit.edu/myungjin/latentTree.html