The Minimum Principle of SINR: A Useful Discretization Tool for Wireless Communication

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Abstract

Theoretical study of optimization problems in wireless communication often deals with zero-dimensional tasks. For example, the power control problem requires computing a power assignment guaranteeing that each transmitting station s_i is successfully received at a *single receiver point* r_i. This paper aims at addressing communication applications that require handling 2-dimensional tasks (e.g., guaranteeing successful transmission in entire *regions* rather than in specific points). A natural approach to such tasks is to *discretize* the 2-dimensional optimization domain, e.g., by sampling points within the domain. This approach, however, might incur high time and memory requirements, and moreover, it cannot guarantee exact solutions.

Towards this goal, we establish the *minimum principle* for the SINR function with free-space path loss (i.e., when the signal decays in proportion to the square of the distance between the transmitter and receiver). We then utilize it as a discretization technique for solving two-dimensional problems in the SINR model. This approach is shown to be useful for handling optimization problems over two dimensions (e.g., power control, energy minimization); in providing tight bounds on the number of null-cells in the reception map; and in approximating geometrical and topological properties of the wireless reception map (e.g., maximum inscribed sphere). Essentially, the minimum principle allows us to reduce the dimension of the optimization domain *without* losing anything in the accuracy or quality of the solution. More specifically, when the 2-dimensional optimization domain is bounded and free from any interfering station, the minimum principle implies that it is sufficient to optimize over the *boundary* of the domain, as the "hardest" points to be satisfied reside on boundary and not in the interior. We believe that the minimum principle, as well as the interplay between continuous and discrete analysis presented in this paper, may pave the way to future study of algorithmic SINR in higher dimensions.

Keywords

SINR; rational polynomials; minimum principle;

I. Introduction

Background and motivation: This paper concerns the fundamental goal of developing useful discretization tools for optimization problems in wireless networks. Specifically, we focus on a basic analytic tool known as the minimum principle. We show that the SINR function satisfies the *minimum principle* assuming the free-space model [5] and demonstrate its usefulness.

We study wireless communication in free space; this is simpler than the irregular environment of radio channels in a general setting, which involves reflection and shadowing. We consider the *Signal to Interference-plus-Noise Ratio (SINR)* model, where given a set of stations $S = \{s_1, \ldots, s_n\}$ in $\mathbb{R}^d$ concurrently transmitting with power assignment ψ, and environmental noise N, a receiver at point $p \in \mathbb{R}^d$ successfully receives a message from station s_i if and only if $\mathrm{SINR}(s_i, p) \geq \beta$, where $\mathrm{SINR}(s_i, p) = \frac{\psi_i \cdot \mathrm{dist}(s_i,p)^{-\alpha}}{\sum_{j \neq i} \psi_j \cdot \mathrm{dist}(s_j,p)^{-\alpha} + N}$ for constants α (the *path-loss exponent*) and $\beta > 0$ (the *reception threshold*), and where $\mathrm{dist}()$ denotes Euclidean distance. Throughout, we assume $\alpha = 2$, which is the path-loss exponent in free-space (cf. [5]). The SINR model, as all other physical models for wireless networks, is continuous in space and characterized by a bivariate polynomial of degree $\Theta(n)$.

In practical network optimization tasks, it is usually *insufficient* to achieve a desired property at a *single* target point (i.e., of zero dimension); rather, it is required that a certain (two-dimensional) *region* satisfies a desired property (e,g., successful reception of transmission by a given station). The observation motivating our work is that optimization over two dimensional space is rather complicated when dealing with high degree polynomials as arise by the SINR function. Previous theoretical work in this area avoided this difficulty by focusing on zero dimensional optimization domains. For example, in the power control problem, one is given n communication links $\langle s_i, r_i \rangle$ and a target SINR threshold β and the goal is to compute a *feasible* power assignment ψ with respect to β, that is, a power assignment that achieves $\mathrm{SINR}(s_i, r_i) \geq \beta$ for every $i \in \{1, \ldots, n\}$ where all stations transmit according to ψ. Hence, every station s_i has to be received at a single point r_i. For a comprehensive review on the power control problem, see [3]. In this paper, we aimed at studying 2-dimensional problems in the SINR model, namely, problems pertaining to entire regions rather than single points. The first natural approach to such generalization is to *discretize* the 2-dimensional optimization domain, e.g., by sampling many points in the given 2-dimensional region. This brute-force approach has two main shortcomings. From a quantitative point of view, the resulting time complexity depends upon the *area* of the optimization domain and hence the size of the new program might be very large. From a qualitative point of view, a-priori this approach is doomed to be an *approximation* scheme and can never result in an exact solution (even in cases where exact solution can be obtained in polynomial time for the 0-dimensional case). The uncertainty for unsampled points can be decreased upon increasing the sampling resolution, but it can never be completely avoided. In this paper, we establish the minimum principle of the SINR function in free-space and demonstrate its power as a useful discretization technique. Generally, a function satisfies the minimum principle if its minimum in any closed domain is attained at the domain's boundary. The minimum principle (dually known as the maximum principle) has been widely studied and it is one of the useful tools employed in studying partial differential equations [9], most notably for elliptic, parabolic, and hyperbolic PDE's. We show that the minimum principle of the SINR function has several algorithmic applications. It is proved to be useful in optimization problems over two dimensions (e.g., power control); in providing tight bounds on the number of null-cells; and in approximating geometrical and topological properties of the wireless reception map (e.g., maximum inscribed sphere) faster than before. The power of the minimum principle is that it reduces the dimension of the optimization domain *without* losing anything in the accuracy or quality of the solution. More specifically, as long as the 2-dimensional optimization domain is bounded and free from any interfering station, the minimum principle implies that it is sufficient to optimize over the *boundary* of the domain, as the "hardest" points to be satisfied reside on the boundary of the domain and not on its interior. Clearly, optimization in one dimension is significantly more tractable than optimization in two dimensions, which makes this property useful. The benefit of this approach is thus two fold. First, the time complexity is no longer scaled with the area of the optimization region but rather with its *perimeter*. Second, in certain cases, this approach can yield an exact solution. To get a sense of this effect, consider a *reception testing* problem where one is given a wireless network, a target station s_i and a closed polygon $\mathcal{P}$, defined by rational vertices and free from interfering stations, i.e., $\mathcal{P} \cap (S \setminus \{s_i\}) = \emptyset$. The task is to decide if the entire area of $\mathcal{P}$ is receptive to the transmission of the station s_i (i.e., $\mathrm{SINR}(s_i, p) \geq \beta$, for every $p \in \mathcal{P}$). Without the minimum principle, the best one can do is to sample sufficiently many points within $\mathcal{P}$ and to evaluate the SINR value at each such point. Since there is no guarantee that the unsampled points are receptive, this scheme cannot decide in finite time if $\mathcal{P}$ is receptive. The minimum principle allows us to do so. By exploiting properties of rational *univariate* polynomials, one can decide in polynomial time if every *edge* σ of $\mathcal{P}$ is receptive or not. In particular, in contrast to the 2-dimensional input polygon $\mathcal{P}$, the polygon edge σ is a line-segment (of dimension 1), and thus testing reception on it is more tractable. The minimum principle then implies that $\mathcal{P}$ is receptive iff every edge of it is receptive.

We hope that these new discretization tools will encourage the future study of two-dimensional optimization problems in the SINR model. In particular, we believe that these tools should aid us in handling the generalization of the joint scheduling and power control problem from zero dimension to 2 dimensions. The complexity of this problem (in zero dimension) in the physical model, taking into account the geometry of the problem, is not fully understood. Nevertheless, many algorithms and heuristics have been suggested for it, e.g., [2], [4], [11], [12], [15], [16], [6], [10].

From the topological point of view, the minimum principle also allows us to give a better topological characterization of the wireless communication map. To model the reception regions, we use the convenient representation of an *SINR diagram*, introduced in [1], which partitions the plane into n reception zones, one per station, and the complementary region where no station can be heard. The topology and geometry of SINR diagrams was studied in [1] in the relatively simple setting of *uniform power*, where all stations transmit with the same power level. SINR diagrams under the general *non-uniform* setting (i.e., with arbitrary power assignments) were studied in [7]. The topological features of general SINR diagrams turned out to be more complicated than in the uniform case. Several important properties of SINR diagrams were established in [7]. One of the key results demonstrates that the reception regions in $\mathbb{R}^{d+1}$ (i.e., one dimension higher than that in which the stations are embedded) are hyperbolically convex. Hence, although the d-dimensional map might be highly fractured, drawing the map in one dimension higher "heals" the zones, which become (hyperbolically) connected. So far, the challenge of establishing useful properties that hold in the dimension where the network is embedded remains open. It was conjectured in [7] that certain undesirable configurations are in fact excluded in (general) d-dimensional SINR diagrams. In particular, there is no hole in a reception region that is free of interfering stations (i.e., every reception cell must contain at least one interfering station). This property, termed *"no-free-hole" (NFH)* in [7], is defined as follows. A collection $\mathcal{C}$ of closed domains in $\mathbb{R}^d$ obeys the *NFH property* w.r.t. a station set S if for every station-free domain $C \in \mathcal{C}$, if all its *boundary* points hear s_i, then *all* its (internal) points hear s_i as well. In [7], the NFH property was established only for 1-dimensional networks in free-space (i.e., $\alpha = 2$) and was conjectured to hold for *any* dimension.

By showing that the SINR function satisfies the minimum principle, the NFH conjecture is resolved for *every* dimension $d \geq 1$ and for SINR threshold $\beta > 0$. Consequently, every null-cell ("hole") in a reception region must contain an interfering station.

Contributions: The main technical contribution of this paper involves establishing the minimum principle for the SINR function in free-space (i.e., for path-loss exponent $\alpha = 2$). We then show its applicability as a discertization tool which enables us to study standard two-dimensional problems in the SINR model. Extending the result to other α values remains a challenging open problem.

Resolving the NFH conjecture: From the topological point of view, we improve our understanding of SINR map compared to [7], by resolving the NFH conjecture raised therein. The minimum principle implies that although the reception regions are not convex in general, they enjoy a certain type of convexity (or smoothness) in *station-free* regions. Our reasoning involves a characterization of the "hard" network configurations for which establishing the minimum principle requires a more subtle analysis. An essential step in our analysis is providing a closed and elegant form for the average energy of a station on the boundary of a d-dimensional ball, which might be of independent interest.

We then present several applications of the minimum principle, briefly reviewed next.

Exact and approximate schemes of reception testing: The first application that illustrates the usefulness of our result, is given by the setting of *reception testing*, see Sec. IV-A. The input for this problem is a wireless network, a closed domain $\mathcal{C}$ and a target station s_0. The closed domain is said to be *receptive* for the station s_0 iff the domain is fully contained in the reception region of s_0. It is then required to decide if $\mathcal{C}$ is fully receptive. We provide two alternative reception schemes. First, for the case where the input shape $\mathcal{C}$ is a polygon whose vertices are positioned on rational coordinates in the plane, we provide an exact reception scheme that returns in polynomial time "yes" iff $\mathcal{C}$ is receptive for s_0. Then, for the general case of any closed domain $\mathcal{C}$, present an approximate testing scheme testing procedure that by evaluating the SINR function for the points on the *boundary* of $\mathcal{C}$ can make deductions regarding the reception quality of the *entire* domain $\mathcal{C}$.

The polygonal power control problem: In Sec. IV-C we define the following problem, which is a 2-dimensional generalization of the well-known power control problem. Given n stations $s_1, \ldots, s_n$ along with n polygons $\mathcal{P}_1, \ldots, \mathcal{P}_n$, a desired SINR threshold β, find transmission powers ψ such that the SINR value of any given reception point $p \in \mathcal{P}_i$ with respect to station s_i is at least β when all stations transmit simultaneously according to ψ. We show that this problem corresponds to a convex program and present a separation oracle that can be used as a black box by the Ellipsoid algorithm for solving this problem. The same scheme applies also for the *sum-power minimization problem* or the *min-max power problems*, in which it is also required to minimize the total (resp., maximum) transmission power [3].

Universal bound of the number of null-cells: In Sec. IV-D we consider the theoretical challenge of providing a tight bound for the number of null-cells in the reception map for SINR threshold $\beta > 1$. Note that in the presence of ambient noise there is only one unbounded null-cell (see Lemma 4.8), and hence the number of bounded null-cells equals the number of null-cells minus one. In [7] it is shown, using Milnor-Thom Theorem, that there are $O(n^{2d})$ null-cells for every dimension $d \geq 1$. In this paper, we tighten this into *linear* bound on the number of null-cells for *every* dimension (which is tight up to constants). Our proof strategy combines a topological and continuous characterization of the system on the one hand, along with a discrete analysis of the graph representation induced by the collection of null-cells.

Maximum inscribed sphere inside a reception region: Consider the following problem. Given an n-station network and a target station s_i. Compute the maximum sphere around s_i that is fully contained in the reception region of s_i. We then show that using the minimum principle and in particular the approximate reception scheme describe above, one can compute an approximation for this problem in improved time compared to what could have been done using the standard tools without the minimum principle.

Approximation of the number of null-cells: Finally, in Appendix IV-E, we provide an approximation scheme for the number of null-cells in a given reception region. The motivation of such an approximation (in light of the universal tight upper bound) arises in cases where the number of null-cells in the network is much smaller than the universal upper bound of $O(n)$. Without the minimum principle, there was no lower bound on the area of a null-cell in the map and hence sufficiently small null-cells could not be detected. The minimum principle also implies a lower bound on the area of the null-cells (i.e., the null-cells cannot be arbitrarily small) and in addition, it implies that every null-cell contains an interfering station. These observations lead to an efficient approximate null-cell detection scheme: every null-cell in the SINR map is detected and every detected null-cell exists in an SINR map of slightly smaller SINR threshold. Some of the presented applications (e.g., polygonal power control, universal linear bound on the number of null cells) are technically nontrivial and call for new tools. Others (e.g., computing the maximum inscribed sphere) are mostly built upon existing tools but may be of significant practical interest.

II. Preliminaries

Geometric notions: We consider the d-dimensional Euclidean space $\mathbb{R}^d$ (for $d \in \mathbb{Z}_{\geq 1}$). The *distance* between points p and point q is denoted by $\mathrm{dist}(p, q) = \|q - p\|$. A *ball* of radius r centered at point $p \in \mathbb{R}^d$ is the set of all points at distance at most r from p, denoted by $B^d(p,r) = \{q \in \mathbb{R}^d \mid \mathrm{dist}(p,q) \leq r\}$. The basic notions of open, closed, bounded, compact and connected sets of points are defined in the standard manner. The *closure* of P, denoted $cl(P)$, is the smallest closed set containing P. The *boundary* of a point set P denoted by $\Phi(P)$, is the intersection of the closure of P and the closure of its complement, i.e., $\Phi(P) = cl(P) \cap cl(\bar{P})$. A *maximal connected subset* $P_1 \subseteq P$ is a connected point set such that $P_1 \cup \{p\}$ is no longer connected for every $p \in P \setminus P_1$. A *domain* D in the Euclidean space is an open connected set. We use the term *zone* to describe a point set with some "niceness" properties. Unless stated otherwise, a zone refers to the union of an open connected set and some subset of its boundary. Let $\widetilde{H} : \mathbb{R}^d \to \mathbb{R}$ be a polynomial and let $p \in \mathbb{R}^d$. Then $\widetilde{H}$ is the *characteristic polynomial* of a zone Z if $p \in Z \iff \widetilde{H}(p) \leq 0$. For a non-empty bounded zone $Z \neq \emptyset$ and an internal point p of Z, denote the maximal and minimal radii of Z w.r.t. p by $\delta(p,Z) = \sup\{r > 0 \mid Z \supseteq B(p,r)\}$, $\Delta(p,Z) = \inf\{r > 0 \mid Z \subseteq B(p,r)\}$.

Wireless networks: We consider an $n+1$ station wireless network $\mathcal{A} = \langle d, S, \psi, N, \beta, \alpha \rangle$, where $d \in \mathbb{Z}_{\geq 1}$ is the dimension, $S = \{s_0, s_2, \ldots, s_n\}$ is a set of transmitting *radio stations* embedded in the d-dimensional space, ψ is an assignment of a positive real *transmitting power* ψ_i to each station s_i, $N \geq 0$ is the *background noise*, $\beta \geq 0$ is a constant *reception threshold* (to be explained soon), and $\alpha \geq 1$ is the *path-loss parameter*. The network is assumed to contain at least two stations, i.e., $n \geq 1$. The *energy* of station s_i at point $p \neq s_i$ is defined as $\mathrm{E}_{\mathcal{A}}(s_i,p) = \psi_i \cdot \mathrm{dist}(s_i,p)^{-\alpha}$. The *signal to interference & noise ratio (SINR)* of s_i at point p is defined as

$$\mathrm{SINR}_{\mathcal{A}}(s_i,p) \;=\; \frac{\mathrm{E}_{\mathcal{A}}(s_i,p)}{\sum_{s_j \in (S \setminus \{s_i\})} \mathrm{E}_{\mathcal{A}}(s_i,p) + N} \;=\; \frac{\psi_i \cdot \mathrm{dist}(s_i,p)^{-\alpha}}{\sum_{j \neq i} \psi_j \cdot \mathrm{dist}(s_j,p)^{-\alpha} + N} \; . \tag{1}$$

Observe that $\mathrm{SINR}_{\mathcal{A}}(s_i,p)$ is positive by definition. In certain contexts, it may be more convenient to consider the reciprocal of the SINR function,

$$\mathrm{SINR}_{\mathcal{A}}^{-1}(s_i,p) \;=\; \frac{1}{\psi_i}\left(\sum_{j \neq i} \psi_j \left(\frac{\mathrm{dist}(s_i,p)}{\mathrm{dist}(s_j,p)}\right)^{\alpha} + N \cdot \mathrm{dist}(s_i,p)^{\alpha}\right). \tag{2}$$

When the network $\mathcal{A}$ is clear from the context, we may omit it and write simply $\mathrm{E}(s_i,p)$ and $\mathrm{SINR}(s_i,p)$. The fundamental rule of the SINR model is that the transmission of station s_i is received correctly at point $p \notin S$ if and only if its signal to noise ratio at p is not smaller than the reception threshold of the network, i.e., $\mathrm{SINR}(s_i,p) \geq \beta$. In this case, we say that s_i is *heard* at p. We refer to the set of points that hear station s_i as the *reception zone* of s_i, defined as $\mathcal{H}_i(\mathcal{A}) \;=\; \{p \in \mathbb{R}^d - S \mid \mathrm{SINR}_{\mathcal{A}}(s_i,p) \geq \beta\} \cup \{s_i\}$. This definition is necessary since $\mathrm{SINR}(s_i,\cdot)$ is undefined at points in S and in particular at s_i itself. Note that $\mathcal{H}_i(\mathcal{A})$ is not necessarily connected. A *maximal connected component* within a zone is referred to as a *cell*. In the same manner of we refer to the set of points that hear no station $s_i \in S$ (due to the background noise and interference) and the null zone $\mathcal{H}_\emptyset(\mathcal{A}) = \{p \in \mathbb{R}^d - S \mid \mathrm{SINR}(s_i,p) < \beta, \ \forall s_i \in S\}$. The null zone is not necessarily connected. In general, $\mathcal{H}_\emptyset(\mathcal{A})$ is composed of $\tau_\emptyset(\mathcal{A})$ connected cells, $\mathcal{H}_{\emptyset,j}(\mathcal{A})$. An SINR diagram $\mathcal{H}(\mathcal{A}) = \{\mathcal{H}_i(\mathcal{A}), \ \ 0 \leq i \leq n\} \cup \{\mathcal{H}_\emptyset(\mathcal{A})\}$ is a "reception map" characterizing the reception zones of the stations. For $\beta > 1$, this map partitions the plane into $n+2$ zones; a zone $\mathcal{H}_i(\mathcal{A})$ for each station s_i, $0 \leq i \leq n$, and a null zone $\mathcal{H}_\emptyset(\mathcal{A})$ where no transmissions are received successfully. The following important technical lemma from [1] will be useful in our later arguments.

Lemma 2.1: [1] Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a mapping consisting of rotation, translation, and scaling by a factor of $\sigma > 0$. Consider some network $\mathcal{A} = \langle d, S, \psi, N, \beta, \alpha \rangle$ and let $f(\mathcal{A}) = \langle d, f(S), \psi, N/\sigma^2, \beta, \alpha \rangle$, where $f(S) = \{f(s_i) \mid s_i \in S\}$. Then f preserves the signal to noise ratio, namely,
$\mathrm{SINR}_{\mathcal{A}}(s_i,p) \;=\; \mathrm{SINR}_{f(\mathcal{A})}(f(s_i),f(p))$ for every station s_i and for all points $p \notin S$.

The Minimum Principle ("No-Free-Hole"): A function f satisfies the *minimum principle* (a.k.a the *weak* minimum principle) if the minimum of f in every open connected domain $D \subset \mathbb{R}^d$ is attained on the boundary $\Phi(D)$ of the domain, i.e., $f(p) \geq \min\{f(p') \mid p' \in \Phi(D)\}$, for every $p \in D$. If the minimum is attained only on the boundary, i.e., the above inequality holds with a strict inequality, then the function satisfies the *strong* minimum principle. The strong and weak *maximum* principles are defined analogously.

Main Technical Lemmas: The following technical lemma plays a key role in our analysis. Due to lack of space, missing proofs are deferred to the full version.

Lemma 2.2: For all $x \in (0,1)$, $y_1,...,y_n \in (0,1)$, and $a_1,...,a_n,\alpha \in \mathbb{R}_{>0}$,

$$\max\left\{\; \sum_{i=1}^{n} a_i \left(\frac{x}{y_i}\right)^{\alpha} \;,\; \sum_{i=1}^{n} a_i \left(\frac{1-x}{1-y_i}\right)^{\alpha} \;\right\} \;\geq\; \sum_{i=1}^{n} a_i \; . \tag{3}$$

where equality holds iff $x = y_1 = \ldots = y_n$.

The following lemma is essential for is a generalization of Proposition 3.6 of [1] for wireless system with $n+1$ stations. Consider a noise free wireless network $\mathcal{A} = \langle d = 2, S, \bar{\psi}, N = 0, \beta \geq 1, \alpha \rangle$, where $S = \{s_0, s_1 ..., s_n\}$. Denote the origin point by $q = (0,0)$, let $p_R = (1,0)$, $p_L = (-1,0)$ and define $\rho_i = \mathrm{dist}^2(s_i, q)$, for every $i = 0, ..., n$.

Lemma 2.3: Let $\mathcal{A}$ be a noise-free network ($N = 0$) and let $q \notin S$. Then $\max\{\mathrm{SINR}_{\mathcal{A}}^{-1}(s_0, p_L) \ , \ \mathrm{SINR}_{\mathcal{A}}^{-1}(s_0, p_R)\} \ \geq \ \sum_{i=1}^{n} \frac{\psi_i}{\psi_0} \cdot \left(\frac{\rho_0+1}{\rho_i+1}\right)^{\alpha/2}$.

Proof: Let $\alpha' = \alpha/2$. For ease of analysis, we consider the two dimensional case but it the proof naturally extends to any $d \geq 2$. in addition, for simplicity consider the network $\mathcal{A}' = \langle S' = \{s'_0, ..., s'_n\}, \bar{\psi}, N = 0, \beta = 1, 2\alpha' \rangle$ obtained from $\mathcal{A}$ by rotating each of the stations s_0 (resp., $s_1, ..., s_n$) around the origin point q (maintaining its distance from q) until it reaches the positive (resp., (negative) y-axis, i.e., the stations s'_0 (resp., $s_1, ..., s'_n$) are on the positive (resp., negative) y-axis and preserve the distances of $s_0, ..., s_n$, respectively, from q. Note that $\mathrm{SINR}_{\mathcal{A}}(s_0, q) = \mathrm{SINR}_{\mathcal{A}'}(s'_0, q)$. Formally, the station s'_0 (resp., $s'_i, i > 0$) is located at the point $(0, \sqrt{\rho_0})$ (resp., $(0, -\sqrt{\rho_i})$ for $i \in \{1, ..., n\}$), as illustrated in Figure 1. Define the angle θ_i so that $s_i = (\sqrt{\rho_i}\cos\theta_i, \sqrt{\rho_i}\sin\theta_i)$ for $i = 0, ..., n$. By the cosine theorem (applied to the

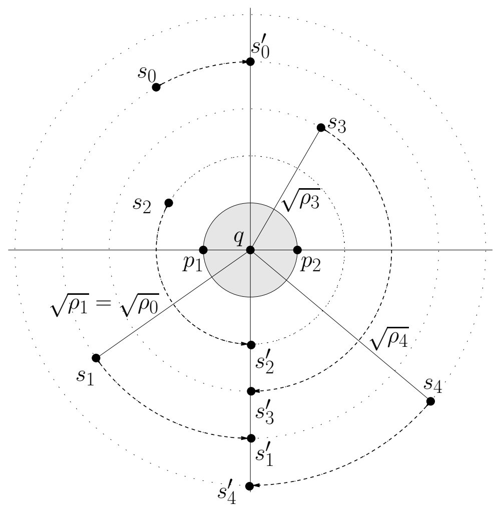

Figure 1. Relocating stations. The interfering stations $s_i \neq s_0$ are mapped to the negative y-axis while the target station s_0 is mapped to the positive y-axis. By that, an interfering station is never co-located with the target station, even when $\rho_i = \rho_0$ (in the figure, $\rho_1 = \rho_0$).

triangle defined by the points $(0,0), (1,0)$ and s_i), $\mathrm{dist}(s_i, p_L)^2 = \rho_i + 2\sqrt{\rho_i}\cos\theta_i + 1$, and analogously $\mathrm{dist}(s_i, p_R)^2 = \rho_i - 2\sqrt{\rho_i}\cos\theta_i + 1$. Thus, for $i \in \{0, \ldots, n\}$, $\mathrm{dist}(s_i, p_L)^2 + \mathrm{dist}(s_i, p_R)^2 = 2(\rho_i + 1)$. Let $x_i = \frac{\mathrm{dist}(s_i, p_L)^2}{2(\rho_i+1)}$ for $i = 0, ..., n$. Then, $x_i \in (0,1)$, and $1 - x_i = \frac{\mathrm{dist}(s_i, p_R)^2}{2(\rho_i+1)}$ for $i = 0, ..., n$. Let $a_i = \frac{\psi_i}{\psi_0}\left(\frac{\rho_0+1}{\rho_i+1}\right)^{\alpha/2}$, for $i \in \{1, \ldots, n\}$. By Eq. (2), $\mathrm{SINR}_{\mathcal{A}}^{-1}(s_0, p_L) = \sum_{i=1}^{n} a_i \left(\frac{x_0}{x_i}\right)^{\alpha/2}$ and $\mathrm{SINR}_{\mathcal{A}}^{-1}(s_0, p_R) = \sum_{i=1}^{n} a_i \left(\frac{1-x_0}{1-x_i}\right)^{\alpha/2}$. Recall that the angles in the polar coordinates of the corresponding stations $s'_0, ..., s'_n$ are $\theta'_0 = ... = \theta'_n = \pi/2$, hence $\mathrm{dist}(s'_i, p_j) = \sqrt{\rho_i + 1}$ for $j = 1, 2$ and therefore $\mathrm{SINR}_{\mathcal{A}'}^{-1}(s'_0, p_L) = \mathrm{SINR}_{\mathcal{A}'}^{-1}(s'_0, p_R) = \sum_{i=1}^{n} a_i$. Applying Lemma 2.2 with $x = x_0$, $y_i = x_i$ for $i = 1, ..., n$, we have that

$$\max\left\{\sum_{i=1}^{n} a_i \left(\frac{x_0}{x_i}\right)^{\alpha} \ , \ \sum_{i=1}^{n} a_i \left(\frac{1-x_0}{1-x_i}\right)^{\alpha}\right\} \geq \sum_{i=1}^{n} a_i$$

for all $x_0, ..., x_n \in (0,1)$. This, in turn, implies that

$$\max\left\{\mathrm{SINR}_{\mathcal{A}}^{-1}(s_0, p_L), \mathrm{SINR}_{\mathcal{A}}^{-1}(s_0, p_R)\right\} \geq \mathrm{SINR}_{\mathcal{A}'}^{-1}(s'_0, p_L) = \mathrm{SINR}_{\mathcal{A}'}^{-1}(s'_0, p_R) = \sum_{i=1}^{n} a_i \ ,$$

the lemma follows. ■

III. The Strong Minimum Principle of the SINR Function

Throughout this section, we restrict attention to $n+1$ station networks $\mathcal{A} = \langle d, S, \psi, N, \beta, \alpha = 2 \rangle$. The function $\mathrm{SINR}_{\mathcal{A}}$ satisfies the strong minimum principle if for every compact domain $D \subset \mathbb{R}^d$ that is free from the stations of S, it holds that

$$\mathrm{SINR}_{\mathcal{A}}(s_i, p) > \min\{\mathrm{SINR}_{\mathcal{A}}(s_i, p') \mid p' \in \Phi(D)\}, \text{ for every } p \in D \setminus \Phi(D) \ . \tag{4}$$

Our main result is the following.

Theorem 3.1: For every network $\mathcal{A} = \langle d, S, \psi, N, \beta, \alpha = 2\rangle$ the function $\mathrm{SINR}_{\mathcal{A}}$ satisfies the *strong minimum principle*. We first show that the strong minimum principle holds for the case where the domain D is a station-free d-dimensional *ball* and then extend it for the general case.

Theorem 3.2: Every $n+1$ station network $\mathcal{A} = \langle d, S, \psi, N, \beta, \alpha = 2\rangle$ satisfies the strong minimum principle on a ball. The following technical tool is found to be useful when studying SINR systems.

Tool: continuous average energy: Our analysis makes use of the notion of *continuous average energy* for a d-dimensional ball $B^d(q, r)$ [8]. For ease of illustration, consider the 2-dimensional case. In addition, assume (w.l.o.g. by Lemma 2.1) that the ball center is $q = (0, 0)$ and let $\mathrm{B} = B(q, r)$. The *average energy of* s_i experienced at the boundary $\Phi(\mathrm{B})$ is given by $\varepsilon(s_i, \mathrm{B}) = \frac{1}{2\pi \cdot r} \int_{p \in \Phi(\mathrm{B})} \mathrm{E}_{\mathcal{A}}(s_i, p)$. In the full version, we generalize this to any $d \geq 2$ and show:[1]

Lemma 3.3: $\varepsilon(s_i, \mathrm{B}) = \frac{\psi_i}{|\mathrm{dist}(s_i, q)^2 - r^2|}$.

Denote the *average signal to the average interference ratio* on the ball's boundary $\Phi(\mathrm{B})$ by

$$\mathrm{AVG}(\mathcal{A}, s_i) = \frac{\varepsilon(s_i, \mathrm{B})}{\sum_{j \neq i} \varepsilon(s_j, \mathrm{B}) + N} \ . \tag{5}$$

The setting and some useful claims: Throughout this section, we use the following conventions. Without loss of generality, we focus on station s_0 and show that the function $\mathrm{SINR}_{\mathcal{A}}(s_0, p)$ satisfies the minimum principle. Consider a point $q \in \mathbb{R}^2$ such that q is *not* equidistant from all of the stations[2], a real $r > 0$ such that all stations are outside the ball $B(q, r)$. That is, there exists a station $s_j \in S$ such that $\mathrm{dist}(s_0, q) \neq \mathrm{dist}(s_j, q)$; and $s_i \notin B(q, r)$ for every $s_i \in S$. Hereafter, to avoid cumbersome notation let $\mathrm{B} = B(q, r)$ and let $\rho_i = \mathrm{dist}^2(s_i, q)$, for every $i = 0, ..., n$. By Lemma 2.1, let us assume without loss of generality, that the radius of the ball B is $r = 1$ and its center $q = (0, 0)$ and that s_0 is located at the positive y-axis (i.e., $s_0 = (0, \sqrt{\rho_0})$). In the ensuing discussion we focus our attention on the four intersection points of $\Phi(\mathrm{B})$ with the x and y axes, namely, $p_L = (-1, 0)$, $p_R = (1, 0)$, $p_D = (0, -1)$ and $p_U = (0, 1)$. Let the minimum value of $\mathrm{SINR}_{\mathcal{A}}(s_i, p)$ on the boundary $\Phi(\mathrm{B})$ of the ball B be denoted by $\mathrm{MIN}(\mathcal{A}, s_i)$, the corresponding dual parameter be denoted by $\mathrm{MAX}^{-1}(\mathcal{A}, s_0)$. That is, $\mathrm{MIN}(\mathcal{A}, s_i) = \min\{\mathrm{SINR}_{\mathcal{A}}(s_i, p) \mid p \in \Phi(\mathrm{B})\}$ and $\mathrm{MAX}^{-1}(\mathcal{A}, s_0) = \frac{1}{\mathrm{MIN}(\mathcal{A}, s_0)}$, and

$$\mathrm{MAX}^{-1}(\mathcal{A}, s_0, p_L, p_R) = \max\{\mathrm{SINR}_{\mathcal{A}}^{-1}(s_0, p_L), \mathrm{SINR}_{\mathcal{A}}^{-1}(s_0, p_R)\} \ . \tag{6}$$

The following claims are useful in our reasoning. (Throughout, missing proofs are deferred to the full version.)

Lemma 3.4: (1) $\mathrm{MAX}^{-1}(\mathcal{A}, s_0) \geq \mathrm{AVG}^{-1}(\mathcal{A}, s_0)$ with equality if and only if $\mathrm{SINR}_{\mathcal{A}}(s_0, p) = \mathrm{AVG}(\mathcal{A}, s_0)$ for every $p \in \Phi(\mathrm{B})$. (2) $\mathrm{MAX}^{-1}(\mathcal{A}, s_0) \geq \mathrm{MAX}^{-1}(\mathcal{A}, s_0, p_L, p_R)$.

The following measure plays a key role in our analysis.

$$\mathrm{SUM} = \sum_{i=1}^{n} \frac{\psi_i \cdot (\rho_0 + 1)}{\psi_0 \cdot (\rho_i + 1)} \ + \ \frac{N \cdot (\rho_0 + 1)}{\psi_0}. \tag{7}$$

Our analysis takes special care in the case where all interfering stations are aligned on the line between s_0 and the ball center point q. As will be shown later, for such networks more delicate characterization (or bounds) can be obtained.

Definition: The network $\mathcal{A}$ is called *y-collinear* if all stations s_i are at the same distance from p_L and p_R (i.e., all stations are aligned on the y-axis).

We proceed by providing a lower bound on $\mathrm{MAX}^{-1}(\mathcal{A}, s_0, p_L, p_R)$ and an exact expression to $\mathrm{AVG}^{-1}(\mathcal{A}, s_0)$ as a function of the $s_i - q$ distances $\rho_0, ..., \rho_n$ and the transmitting powers $\psi_0, ..., \psi_n$.

Lemma 3.5: (1) If the network $\mathcal{A}$ is y-collinear then $\mathrm{MAX}^{-1}(\mathcal{A}, s_0, p_L, p_R) = \mathrm{SUM}$. (2) In all other cases, $\mathrm{MAX}^{-1}(\mathcal{A}, s_0, p_L, p_R) > \mathrm{SUM}$.
(3) $\mathrm{AVG}^{-1}(\mathcal{A}, s_0) = \sum_{i=1}^{n} \frac{\psi_i \cdot (\rho_0 - 1)}{\psi_0 \cdot (\rho_i - 1)} + \frac{N \cdot (\rho_0 - 1)}{\psi_0}$.

Sufficient conditions for the minimum principle on SINR: We now propose two sufficient conditions for establishing the strong minimum principle of SINR on the ball.

[1]Extending this lemma from $\alpha = 2$ to general α is the main barrier to extending the minimum principle to any α.

[2]this assumption is eliminated later

The ball center condition: Given an $n+1$ station network $\mathcal{A} = \langle d, S, \psi, N, \beta, \alpha = 2 \rangle$, the $\mathrm{SINR}_\mathcal{A}$ function is said to satisfy the *ball center condition* if for every station $s_i \in S$,

$$\mathrm{SINR}_\mathcal{A}(s_i, q) > \min\{\mathrm{SINR}_\mathcal{A}(s_i, p) \mid p \in \Phi(B(q, r))\} \ , \tag{8}$$

for every station-free ball $B(q, r)$ such that the ball center q is not equi-distant from all stations. We now show that the ball center condition is sufficient for establishing Thm. 3.2.

Lemma 3.6: If the $n+1$ station network $\mathcal{A}$ satisfies the ball center condition, then Thm. 3.2 holds, namely, the SINR function satisfies the strong minimum principle for *every* ball.

We next provide the second sufficient condition.

The dual minimality condition: The $\mathrm{SINR}_\mathcal{A}$ function satisfies the dual minimality condition on the ball B if and only if

$$\mathrm{SINR}^{-1}_\mathcal{A}(s_0, q) \quad < \quad \max\{\mathrm{MAX}^{-1}(\mathcal{A}, s_0, p_L, p_R), \mathrm{AVG}^{-1}(\mathcal{A}, s_0)\}. \tag{9}$$

The function $\mathrm{SINR}_\mathcal{A}$ satisfies the *weak* dual minimality condition on B if Eq. (9) holds with non-strict inequality.

Lemma 3.7: The dual minimality condition is sufficient for satisfying the ball center condition and hence also the strong minimum principle for the ball.

By Lemma 3.6, we hereafter restrict attention to $s_0 \in S$ and the ball B, and show the following.

Lemma 3.8: The $\mathrm{SINR}_\mathcal{A}$ function satisfies the ball center condition, namely, $\mathrm{SINR}_\mathcal{A}(s_0, q) > \mathrm{MIN}(\mathcal{A}, s_0)$.

Proof overview: We first consider the *noise-free* case. One of the key insights, in this context, is that the expressions $\mathrm{SINR}_\mathcal{A}(s_0, q)$, $\mathrm{AVG}^{-1}(\mathcal{A}, s_0)$ and SUM are all functions of the transmission powers $\psi_0, ..., \psi_n$ and the distances $\rho_0, ..., \rho_n$. Moreover, as will be shown, for the y-collinear case, $\mathrm{MAX}^{-1}(\mathcal{A}, s_0, p_L, p_R) = \mathrm{SUM}$. To establish the principle, we partition the station set S into three subsets according to their distance ρ_i from q compared to the distance ρ_0. Let

$S_{close} = \{s_i \in S \mid \rho_i < \rho_0\}$, $S_{eq} = \{s_i \in S \mid \rho_i = \rho_0\}$ and $S_{far} = \{s_i \in S \mid \rho_i > \rho_0\}$.

Let us now give some intuition by considering two extreme cases. For a network $\mathcal{A}_{far}$ all of whose stations are far ($S = S_{far}$), it holds that $\mathrm{SINR}^{-1}_{\mathcal{A}_{far}}(s_0, q) > \mathrm{MAX}^{-1}(\mathcal{A}_{far}, s_0, p_L, p_R)$. On the other hand, for a network $\mathcal{A}_{close}$ all of whose stations are close ($S = S_{close}$), $\mathrm{SINR}^{-1}_{\mathcal{A}_{close}}(s_0, q) < \mathrm{AVG}^{-1}(\mathcal{A}_{close}, s_0)$. The general case is based on this intuition but is more involved. We then turn to consider the general *noisy* case. To prove the *weak* version of the ball center condition, we use an infinite family $\mathcal{NF}$ of a *noise-free* networks that "behave" similarly to the noisy network at the points of the ball B. The reasoning is by contradiction. Assuming that the weak version of the ball center condition does not hold, implies the existence of noise-free network in the family $\mathcal{NF}$ that violates the strong ball center condition (which was already established for the noise-free case). Turning to the *strong* version of the ball center condition, the analysis considers two cases of network configurations. The first case is of y-collinear networks. In this case, the analysis exploits the fact that the 1-dimensional SINR function satisfies the minimum principle (note that the dimension of y-collinear networks is in fact one). Finally, we consider the remaining case where the network is not y-collinear and yet Ineq. (9) holds with equality. The reasoning for this case includes the following steps. We first claim that $\mathrm{SINR}^{-1}_\mathcal{A}(s_0, q) = \mathrm{AVG}^{-1}(\mathcal{A}, s_0)$. This is shown again by using the family $\mathcal{NF}$. We complete the proof by showing that $\mathrm{AVG}^{-1}(\mathcal{A}, s_0) > \mathrm{MAX}^{-1}(\mathcal{A}, s_0, p_L, p_R)$. To do that, an auxiliary noise-free network which is also y-collinear is constructed. By using the tight bounds previously achieved for y-collinear networks, the desired claim is established.

The structure of the proof of Lemma 3.8 is as follows. In Subsec. III-A, we establish the strong minimum principle for the simplest case where there is no ambient noise. We then considers, in Subsec. III-B, the more general case where there is an ambient $N > 0$.

A. *The noise free case* $N = 0$

In this section, we consider the simpler case where there is no ambient noise (i.e., $N = 0$) and show that the SINR function satisfies the dual minimality condition and hence also the strong minimum principle.

Lemma 3.9: Let $\mathcal{A}_0 = \langle d, S, \psi, N = 0, \beta, \alpha = 2 \rangle$ be an $n+1$ station network with no ambient noise. Then, the $\mathrm{SINR}_{\mathcal{A}_0}$ function satisfies the dual minimality condition (and hence also satisfies the strong minimum principle).

Proof: Since $N = 0$, it follows that

$$\mathrm{SINR}^{-1}_{\mathcal{A}_0}(s_0, q) \ = \ \sum_{i=1}^{n} \frac{\psi_i \cdot \rho_0}{\psi_0 \cdot \rho_i} \ . \tag{10}$$

In addition, by Part 3 of Lemma 3.5

$$\mathrm{AVG}^{-1}(\mathcal{A}_0, s_0) \ = \ \sum_{i=1}^{n} \frac{\psi_i \cdot (\rho_0 - 1)}{\psi_0 \cdot (\rho_i - 1)} \ , \tag{11}$$

and by Part (1,2) of Lemma 3.5,

$$\mathrm{MAX}^{-1}(\mathcal{A}_0, s_0, p_L, p_R) \;\geq\; \sum_{i=1}^{n} \frac{\psi_i}{\psi_0} \cdot \left(\frac{\rho_0+1}{\rho_i+1}\right) \;. \tag{12}$$

We next partition the set of stations S into three subsets depending on their distance ρ_i from q compared to the distance ρ_0. Let

$$S_{close} = \{s_i \mid \rho_i < \rho_0\} \quad , \quad S_{eq} = \{s_i \mid \rho_i = \rho_0\} \;\text{ and }\; S_{far} = \{s_i \mid \rho_i > \rho_0\}. \tag{13}$$

Then,

$$\frac{\rho_0}{\rho_i} > \frac{\rho_0 - 1}{\rho_i - 1} \text{ for every } \; s_i \in S_{close}, \tag{14}$$

$$\frac{\rho_0}{\rho_i} < \frac{\rho_0 + 1}{\rho_i + 1} \;\text{ for every } \; s_i \in S_{far},. \tag{15}$$

$$\frac{\rho_0}{\rho_i} = \frac{\rho_0 + 1}{\rho_i + 1} = \frac{\rho_0 - 1}{\rho_i - 1} \;\text{ for every } \; s_i \in S_{eq}. \tag{16}$$

Note that the last equality holds since $\rho_i > 1$ for $i = 0, ..., n$ (no station is in the ball B). By the definition of q, it holds that $S \setminus S_{eq} \neq \emptyset$. That is, not all stations are positioned at the same distance from the center q.

Let us give some intuition. First, let us rewrite the dual minimality condition for the $\mathrm{SINR}_{\mathcal{A}}$ function on the ball B more explicitly, by replacing Eq. (9) with the following: either

$$\mathrm{SINR}_{\mathcal{A}}^{-1}(s_0, q) \;\; < \;\; \mathrm{MAX}^{-1}(\mathcal{A}, s_0, p_L, p_R), \quad \text{or} \tag{17}$$

$$\mathrm{SINR}_{\mathcal{A}}^{-1}(s_0, q) \;\; < \;\; \mathrm{AVG}^{-1}(\mathcal{A}, s_0). \tag{18}$$

Next, let us consider two extreme cases. If all stations are far ($S = S_{far}$), then the lemma readily follows by establishing property (17). This is shown by combining inequalities (10), (12) and (15), which yields

$$\begin{aligned} \mathrm{SINR}_{\mathcal{A}_0}^{-1}(s_0, q) \;\; &= \;\; \sum_{i=1}^{n} \frac{\psi_i \cdot \rho_0}{\psi_0 \cdot \rho_i} < \sum_{i=1}^{n} \frac{\psi_i \cdot (\rho_0 + 1)}{\psi_0 \cdot (\rho_i + 1)} \\ &\leq \;\; \mathrm{MAX}^{-1}(\mathcal{A}_0, s_0, p_L, p_R). \end{aligned}$$

In contrast, if all stations are close ($S = S_{close}$), then $\sum_{i=1}^{n} \frac{\psi_i \cdot \rho_0}{\psi_0 \cdot \rho_i} > \sum_{i=1}^{n} \frac{\psi_i \cdot (\rho_0+1)}{\psi_0 \cdot (\rho_i+1)}$, so Inequality (12) cannot be applied. Instead, in this case we prove the lemma by establishing property (18). Combining together inequalities (10), (14) and (11), we have

$$\mathrm{SINR}_{\mathcal{A}_0}^{-1}(s_0, q) = \sum_{i=1}^{n} \frac{\psi_i \cdot \rho_0}{\psi_0 \cdot \rho_i} < \sum_{i=1}^{n} \frac{\psi_i \cdot (\rho_0 - 1)}{\psi_0 \cdot (\rho_i - 1)} = \mathrm{AVG}^{-1}(\mathcal{A}_0, s_0) \;.$$

We now turn to discuss the more general case in which S may contain both far and close stations, i.e., $S_{close} \neq \emptyset$ and $S_{far} \neq \emptyset$. Clearly, if Eq. (17) happens to hold, then the lemma follows as well. Hence, hereafter assume that

$$\mathrm{SINR}_{\mathcal{A}_0}^{-1}(s_0, q) \;\geq\; \mathrm{MAX}^{-1}(\mathcal{A}_0, s_0, p_L, p_R), \tag{19}$$

(as will be shown later, this happens when the net interference effect of the far stations S_{far} on the center q dominates the effect of the close stations S_{close}), and establish property (18). By Ineq. (10) and (12), Ineq. (19) can be rewritten as

$$\frac{\psi_i \cdot \rho_0}{\psi_0 \cdot \rho_i} \geq \frac{\psi_i \cdot (\rho_0 + 1)}{\psi_0 \cdot (\rho_i + 1)}.$$

Rearranging, we get

$$\sum_{i=1}^{n} \left(\frac{\psi_i}{\psi_0} \cdot \frac{\rho_0 - \rho_i}{\rho_i(\rho_i + 1)}\right) \geq 0 \;.$$

Partitioning the summation into summations on the close and far stations, and noting that $\rho_0 - \rho_i = |\rho_0 - \rho_i|$ for every $s_i \in S_{close}$; $\rho_0 - \rho_i = 0$ for every $s_i \in S_{eq}$; and $\rho_0 - \rho_i = -|\rho_0 - \rho_i|$, for every $s_i \in S_{far}$, we have that

$$\sum_{s_i \in S_{close}} \left|\frac{\psi_i}{\psi_0} \cdot \frac{\rho_0 - \rho_i}{\rho_i(\rho_i + 1)}\right| - \sum_{s_i \in S_{far}} \left|\frac{\psi_i}{\psi_0} \cdot \frac{\rho_0 - \rho_i}{\rho_i(\rho_i + 1)}\right| \geq 0, \tag{20}$$

(Since for every station $s_i \in S_{eq}$, $\rho_0 - \rho_i = 0$, these stations contribute zero to the summation, so only stations in S_{far} and S_{close} need to be accounted for.) Note that since $\rho_0 > 1$, it follows that $\frac{\rho_0+1}{\rho_0-1} > 0$, so Ineq. (20) can be written as

$$\frac{\rho_0+1}{\rho_0-1}\left(\sum_{s_i\in S_{close}}\left|\frac{\psi_i}{\psi_0}\cdot\frac{\rho_0-\rho_i}{\rho_i(\rho_i+1)}\right| - \sum_{s_i\in S_{far}}\left|\frac{\psi_i}{\psi_0}\cdot\frac{\rho_0-\rho_i}{\rho_i(\rho_i+1)}\right|\right) \geq 0. \tag{21}$$

Moreover, for $s_i \in S_{close}$, it holds that $\rho_0 > \rho_i > 1$, implying that $\frac{\rho_i+1}{\rho_i-1} > \frac{\rho_0+1}{\rho_0-1}$, hence

$$\sum_{s_i\in S_{close}}\left|\frac{\psi_i}{\psi_0}\cdot\frac{\rho_0-\rho_i}{\rho_i(\rho_i-1)}\right| > \frac{\rho_0+1}{\rho_0-1}\cdot\sum_{s_i\in S_{close}}\left|\frac{\psi_i}{\psi_0}\cdot\frac{\rho_0-\rho_i}{\rho_i(\rho_i+1)}\right|. \tag{22}$$

Similarly, for $s_i \in S_{far}$, $\rho_i > \rho_0$, and hence $\frac{\rho_i+1}{\rho_i-1} < \frac{\rho_0+1}{\rho_0-1}$, concluding that

$$\sum_{s_i\in S_{far}}\left|\frac{\psi_i}{\psi_0}\cdot\frac{\rho_0-\rho_i}{\rho_i(\rho_i-1)}\right| < \frac{\rho_0+1}{\rho_0-1}\cdot\sum_{s_i\in S_{far}}\left|\frac{\psi_i}{\psi_0}\cdot\frac{\rho_0-\rho_i}{\rho_i(\rho_i+1)}\right|. \tag{23}$$

Combining the above three inequalities (21-23) we get that,

$$\sum_{s_i\in S_{close}}\left|\frac{\psi_i}{\psi_0}\cdot\frac{\rho_0-\rho_i}{\rho_i(\rho_i-1)}\right| - \sum_{s_i\in S_{far}}\left|\frac{\psi_i}{\psi_0}\cdot\frac{\rho_0-\rho_i}{\rho_i(\rho_i-1)}\right| > 0. \tag{24}$$

We now complete the proof by noting that

$$\begin{aligned}\mathrm{SINR}^{-1}_{\mathcal{A}_0}(s_0,q) - \mathrm{AVG}(\mathcal{A}_0,s_0) &= \sum_{i=1}^{n}\frac{\psi_i\cdot\rho_0}{\psi_0\cdot\rho_i} - \sum_{i=1}^{n}\frac{\psi_i\cdot(\rho_0-1)}{\psi_0\cdot(\rho_i-1)} = \sum_{i=1}^{n}\left(\frac{\psi_i}{\psi_0}\cdot\frac{\rho_i-\rho_0}{\rho_i(\rho_i-1)}\right)\\ &= \sum_{s_i\in S_{far}}\left|\frac{\psi_i}{\psi_0}\cdot\frac{\rho_0-\rho_i}{\rho_i(\rho_i-1)}\right| - \sum_{s_i\in S_{close}}\left|\frac{\psi_i}{\psi_0}\cdot\frac{\rho_0-\rho_i}{\rho_i(\rho_i-1)}\right| < 0\ ,\end{aligned}$$

where the first equality holds by Ineq. (10) and Ineq. (11); the third equality holds, since $\rho_i \leq \rho_0$, for every $s_i \in S_{close}$ and $\rho_i > \rho_0$, for every $s_i \in S_{far}$; and the last inequality holds by Ineq. (24). Lemma 3.9 follows by Lemma 3.7. ■

B. The noisy case $N > 0$

We now turn to consider the more general case where the network $\mathcal{A}$ has a positive noise $N > 0$. The reasoning for this case is more involved and consists of the following stages. In Subsec. III-B1, we first prove that the $\mathrm{SINR}_{\mathcal{A}}$ function satisfies the *weak* minimum principle. The following two subsections establish the *strong* minimum principle for the $\mathrm{SINR}_{\mathcal{A}}$ function, where Subsec. III-B2 considers the extreme case of y-collinear networks where all the stations are aligned on the y-axis. Finally, Subsec. III-B3 deals with the complementary case where not all stations of $\mathcal{A}$ are aligned on the line between s_0 and q (i.e., the y-axis).

1) The Weak Minimum Principle for the Noisy Case $N > 0$: We begin by establishing the weak version of the minimum principle for the noisy setting. Specifically, we show that the $\mathrm{SINR}_{\mathcal{A}}$ function satisfies the dual minimality condition in the weak sense, i.e., it satisfies Eq. (9) with non strict inequality.

Our proof technique makes use of an infinite sequence of networks $\mathcal{A}_\ell$, for $\ell \in \mathbb{R}_{>1}$ that mimics the noisy network $\mathcal{A}$ at the points of the ball B as ℓ gets sufficiently large.

Mimicking $\mathcal{A}$ by an infinite sequence of noise-free networks: Let $\mathcal{NF} = \{\mathcal{A}(\ell) \mid \ell > 1\}$ be an infinite family of noise-free networks $\mathcal{A}(\ell)$. Each of these networks consists of $n+2$ stations, namely, the $n+1$ stations S of $\mathcal{A}$ plus an additional station, s_ℓ, whose position and transmission power are parameterized by ℓ. Intuitively, $\mathcal{A}(\ell)$ is obtained from $\mathcal{A}$ by replacing the noise N with an additional station s_ℓ that plays a similar role to that of the noise at the points of the ball B as ℓ tend to infinity. The "noise-simulating" station s_ℓ is located at the point $(0,\ell)$ and its transmission power is $\psi_\ell = N\cdot\ell^2$, hence the received signal strength of s_ℓ at point q equals that of the noise N. Formally, $\mathcal{A}(\ell) = \langle d, S_\ell, \bar{\psi}_\ell, N = 0, \beta, \alpha\rangle$, where $S_\ell = S \cup \{s_\ell\}$, $s_\ell = (0,\ell)$, $\bar{\psi}_\ell = (\psi_0,...,\psi_n,\psi_\ell)$ and $\mathrm{E}_{\mathcal{A}(\ell)}(s_\ell,q) = \psi_\ell\cdot\mathrm{dist}(s_\ell,q)^{-2} = N$.

Lemma 3.10: The following properties hold for every $\mathcal{A}_\ell \in \mathcal{NF}$.

(P1) $\mathrm{SINR}^{-1}_{\mathcal{A}(\ell)}(s_0,q) = \mathrm{SINR}^{-1}_{\mathcal{A}}(s_0,q)$;
(P2) $\lim_{\ell\to\infty}\mathrm{AVG}^{-1}(\mathcal{A}(\ell),s_0) = \mathrm{AVG}^{-1}(\mathcal{A},s_0)$;
(P3) $\lim_{\ell\to\infty}\mathrm{MAX}^{-1}(\mathcal{A}(\ell),s_0,p_L,p_R) = \mathrm{MAX}^{-1}(\mathcal{A},s_0,p_L,p_R)$; and
(P4) $\mathrm{MAX}^{-1}(\mathcal{A}(\ell),s_0,p_L,p_R)\ <\ \mathrm{MAX}^{-1}(\mathcal{A},s_0,p_L,p_R)$.

Proof: (P1) holds since $\mathrm{E}_{\mathcal{A}(\ell)}(s_\ell, q) = N$. Recall that for every $p \in \Phi(\mathrm{B})$, $\mathrm{E}_{\mathcal{A}}(s_\ell, p) = \frac{N\cdot\ell^2}{\mathrm{dist}(s_\ell,p)^2}$ and by the triangle inequality, $\ell - 1 \le \mathrm{dist}(s_\ell, p) \le \ell + 1$. Hence, $N \cdot \left(\frac{\ell}{\ell+1}\right)^2 \le \mathrm{E}_{\mathcal{A}}(s_\ell, p) \le N \cdot \left(\frac{\ell}{\ell-1}\right)^2$, and thus $\lim_{\ell\to\infty} \mathrm{E}_{\mathcal{A}(\ell)}(s_\ell, p) = N$, for every $p \in \Phi(\mathrm{B})$. Properties (P2) and (P3) follow. It remains to consider (P4). Since $\mathrm{dist}(s_\ell, p_i)^2 = \ell^2 + 1$, for $i \in \{L, R\}$, the received signal strength of s_ℓ at point p_i is $\mathrm{E}_{\mathcal{A}(\ell)}(s_\ell, p_i) = N \cdot \frac{\ell^2}{\ell^2+1} < N$. Hence $\mathrm{SINR}_{\mathcal{A}(\ell)}(s_0, p_i) > \mathrm{SINR}_{\mathcal{A}}(s_0, p_i)$ or $\mathrm{SINR}^{-1}_{\mathcal{A}(\ell)}(s_0, p_i) < \mathrm{SINR}^{-1}_{\mathcal{A}}(s_0, p_i)$ for $i \in \{L, D\}$. Property (P4) follows. ■

We are now ready to show that when $N \ge 0$, the minimum principle holds in the *weak* sense (non strict inequality).

Lemma 3.11: Suppose that $N \ge 0$. Then the $\mathrm{SINR}_{\mathcal{A}}$ satisfies the weak dual minimality condition, i.e., $\mathrm{SINR}^{-1}_{\mathcal{A}}(s_0, q) \le \max\left\{\mathrm{MAX}^{-1}(\mathcal{A}, s_0, p_L, p_R), \mathrm{AVG}^{-1}(\mathcal{A}, s_0)\right\}$.

Proof: Suppose, towards contradiction, that

$$\mathrm{SINR}^{-1}_{\mathcal{A}}(s_0, q) > \mathrm{AVG}^{-1}(\mathcal{A}, s_0) \text{ and } \mathrm{SINR}^{-1}_{\mathcal{A}}(s_0, q) > \mathrm{MAX}^{-1}(\mathcal{A}, s_0, p_L, p_R).$$

This implies, together with properties (P2) and (P3) of Lemma 3.10, that there exists a sufficiently large number ℓ^* such that $\mathcal{A}(\ell^*) \in \mathcal{NF}$ and $\mathrm{SINR}^{-1}_{\mathcal{A}}(s_0, q) > \mathrm{AVG}^{-1}(\mathcal{A}(\ell^*), s_0)$ and $\mathrm{SINR}^{-1}_{\mathcal{A}}(s_0, q) > \mathrm{MAX}^{-1}(\mathcal{A}(\ell^*), s_0, p_L, p_R)$. Combining this with property (P1) of Lemma 3.10, we get that $\mathrm{SINR}^{-1}_{\mathcal{A}(\ell^*)}(s_0, q) > \mathrm{AVG}^{-1}(\mathcal{A}(\ell^*), s_0)$ and $\mathrm{SINR}^{-1}_{\mathcal{A}(\ell^*)}(s_0, q) > \mathrm{MAX}^{-1}(\mathcal{A}(\ell^*), s_0, p_L, p_R)$. Since $\mathcal{A}(\ell^*)$ is a noise-free network, this is in contradiction with Lemma 3.9 (taking $\mathcal{A}_0 = \mathcal{A}(\ell^*)$). The claim follows. ■

2) The Strong Minimum Principle for Noisy Collinear Networks: In this subsection, we establish the strong minimum principle for y-collinear networks where all stations are aligned on the y-axis. The analysis of this case exploits the weak minimum principle for 1-dimension that was established in [7].

Fact 3.12 (Minimum principle in $\mathbb{R}^1$ [7]): In a 1-dimensional network, (i.e., all stations are aligned on a line), the SINR function satisfies the minimum principle.

As explained in [7], Fact 3.12 immediately implies the following corollary (adapted to the current setting where the stations of the 1-dimensional network are embedded on the y-axis instead on the x-axis).

Corollary 3.13 (Weak minimum principle in $\mathbb{R}^1$): Consider a network $\mathcal{A}$ where each $s_i \in S$ is positioned on $(0, y_i)$. Let q_1 and q_2 be two points on the y-axis, such that there is no station of S on the segment $\overline{q_1\, q_2}$. Then for every $p \in \overline{q_1\, q_2}$

$$\mathrm{SINR}_{\mathcal{A}}(s_0, p) \ge \mathrm{MIN}(\mathcal{A}, s_0, q_1, q_2)\ . \tag{25}$$

We begin by showing that the minimum principle holds also in the strong sense for the 1-dimensional case.

Lemma 3.14 (Strong minimum principle in $\mathbb{R}^1$): For a network $\mathcal{A}$ and points as in Cor. 25, Eq. (25) holds with strict inequality.

Proof: First, note that the SINR function restricted to any straight line segment $\overline{p\, q}$ is not constant. This holds as the characteristic polynomial of the SINR function has finite and positive degree, hence its derivative cannot be zero over a segment. Assume, towards contradiction, that there is an internal point $p \in \overline{q_1\, q_2}$ such that $\mathrm{SINR}_{\mathcal{A}}(s_0, p) = \mathrm{MIN}(\mathcal{A}, s_0, q_1, q_2)$. Without loss of generality, assume that $\mathrm{SINR}_{\mathcal{A}}(s_0, q_1) \le \mathrm{SINR}_{\mathcal{A}}(s_0, q_2)$. We consider two cases.

Case (a): $\mathrm{SINR}_{\mathcal{A}}(s_0, q_1) < \mathrm{SINR}_{\mathcal{A}}(s_0, q_2)$. Since the SINR function is not constant in the segment $\overline{q_1\, p}$, there exists some point $w_1 \in \overline{q_1\, p}$ such that $\mathrm{SINR}_{\mathcal{A}}(s_0, w_1) \ne \mathrm{SINR}_{\mathcal{A}}(s_0, q_1)$ and by Cor. 3.13, necessarily, $\mathrm{SINR}_{\mathcal{A}}(s_0, w_1) > \mathrm{SINR}_{\mathcal{A}}(s_0, q_1)$. Since p is an internal point in the segment $\overline{w_1\, q_2}$ but $\mathrm{SINR}_{\mathcal{A}}(s_0, p) < \mathrm{SINR}_{\mathcal{A}}(s_0, w_1), \mathrm{SINR}_{\mathcal{A}}(s_0, q_2)$, we end with contradiction to Cor. 3.13.

Case (b): $\mathrm{SINR}_{\mathcal{A}}(s_0, q_1) = \mathrm{SINR}_{\mathcal{A}}(s_0, q_2)$. Since the SINR function is not constant in the segments $\overline{q_1\, p}$ and $\overline{p\, q_2}$, there exists an internal point w_1 (resp., w_2) in the segment $\overline{q_1\, p}$ (resp., $\overline{p\, q_2}$) such that $\mathrm{SINR}_{\mathcal{A}}(s_0, w_1), \mathrm{SINR}_{\mathcal{A}}(s_0, w_2) \ne \mathrm{SINR}_{\mathcal{A}}(s_0, p)$. Combining this with Cor. 3.13 for the points $w_1, w_2 \in \overline{q_1\, q_2}$, it holds that

$$\mathrm{SINR}_{\mathcal{A}}(s_0, w_1), \mathrm{SINR}_{\mathcal{A}}(s_0, w_2) > \mathrm{SINR}_{\mathcal{A}}(s_0, p)\ . \tag{26}$$

Finally, by applying Cor. 3.13 for the segment $\overline{w_1\, w_2}$, since $p \in \overline{w_1\, w_2}$, it holds that $\mathrm{SINR}_{\mathcal{A}}(s_0, p) > \mathrm{MIN}(\mathcal{A}, s_0, w_1, w_2)$. In contradiction by Eq. (26), the claim follows. ■

Lemma 3.15: Let $\mathcal{A}$ be a y-collinear network. Then, $\mathrm{SINR}_{\mathcal{A}}$ satisfies the strong minimum principle on B.

Proof: Recall that $p_D = (0, -1)$ and $p_U = (0, 1)$ are the two points on the intersection of y-axis and $\Phi(\mathrm{B})$, hence

$$\mathrm{MAX}^{-1}(\mathcal{A}, s_0) \ge \mathrm{MAX}^{-1}(\mathcal{A}, s_0, p_D, p_U). \tag{27}$$

Since the segment $\overline{p_D\, p_U} \subset \mathrm{B}$ and B is free from stations, by the strong minimum principle for $\mathbb{R}^1$ (Lemma 3.14), it holds that $\mathrm{SINR}^{-1}_{\mathcal{A}}(s_0, q) < \mathrm{MAX}^{-1}(\mathcal{A}, s_0, p_D, p_U)$, by combining with Eq. (27), Lemma 3.8 follows. Thus by Lemma 3.6, Lemma 3.15 follows as well. ■

3) The Strong Minimum Principle for Noisy Non-Collinear Networks: In this section, we consider a non-collinear network $\mathcal{A}$ in which not all its stations are aligned on the y-axis. We have shown before (in Lemma 3.11) that such a network satisfies the *weak* dual minimality condition. If it also satisfies the *strong* dual minimality condition, then we are done by Lemma 3.7. Hence, it remains to consider the case where

$$\mathrm{SINR}_{\mathcal{A}}^{-1}(s_0, q) = \max\{\mathrm{MAX}^{-1}(\mathcal{A}, s_0, p_L, p_R), \mathrm{AVG}^{-1}(\mathcal{A}, s_0)\}. \tag{28}$$

Indeed, we believe that this can be realized by some networks. Yet, as will be shown now, the strong minimum principle still holds. We show that the $\mathrm{SINR}_{\mathcal{A}}$ function satisfies the strong minimum principle even in this case.

We begin by showing that if Equality (28) holds then the SINR value at the center point q is equal to the average SINR value on the ball's boundary, i.e., $\mathrm{SINR}_{\mathcal{A}}^{-1}(s_0, q) = \mathrm{AVG}^{-1}(\mathcal{A}, s_0)$. To prove this, Lemma 3.16 exploits properties (P1), (P2) and (P4) of Lemma 3.10 to show in that in this special case, the dominating parameter is the average SINR^{-1} value on the ball's boundary, namely,

$$\mathrm{AVG}^{-1}(\mathcal{A}, s_0) \geq \mathrm{MAX}^{-1}(\mathcal{A}, s_0, p_L, p_R). \tag{29}$$

By Inequality (29) there are now only two cases to consider. The first case is where $\mathrm{AVG}^{-1}(\mathcal{A}, s_0) = \mathrm{MAX}^{-1}(\mathcal{A}, s_0, p_L, p_R)$. Lemma 3.17 shows that this case can be attained only for y-collinear networks. Hence it remains to consider only the complementally case, where $\mathrm{AVG}^{-1}(\mathcal{A}, s_0) > \mathrm{MAX}^{-1}(\mathcal{A}, s_0, p_L, p_R)$. Lemma 3.18, shows that under this setting, the strong minimum principle is guaranteed to exists. We now describe the proof in details.

Lemma 3.16: If Eq. (28) holds, then Eq. (29) holds, i.e., $\mathrm{SINR}_{\mathcal{A}}^{-1}(s_0, q) = \mathrm{AVG}^{-1}(\mathcal{A}, s_0)$.

Proof: To prove the claim, it is sufficient to show that if $\mathrm{SINR}_{\mathcal{A}}^{-1}(s_0, q) > \mathrm{AVG}^{-1}(\mathcal{A}, s_0)$, then $\mathrm{SINR}_{\mathcal{A}}^{-1}(s_0, q) < \mathrm{MAX}^{-1}(\mathcal{A}, s_0, p_L, p_R)$. To show this, assume towards contradiction that (CON1): $\mathrm{SINR}_{\mathcal{A}}^{-1}(s_0, q) > \mathrm{AVG}^{-1}(\mathcal{A}, s_0)$; and (CON2): $\mathrm{SINR}_{\mathcal{A}}^{-1}(s_0, q) \geq \mathrm{MAX}^{-1}(\mathcal{A}, s_0, p_L, p_R)$. Let ℓ^* be a sufficiently large real number such that $\mathcal{A}(\ell^*) \in \mathcal{NF}$ and

$$\mathrm{SINR}_{\mathcal{A}}^{-1}(s_0, q) > \mathrm{AVG}^{-1}(\mathcal{A}(\ell^*), s_0). \tag{30}$$

(By Property (P2) of Lemma 3.10, it holds that $\lim_{\ell \to \infty} \mathrm{AVG}^{-1}(\mathcal{A}(\ell), s_0) = \mathrm{AVG}^{-1}(\mathcal{A}, s_0)$, hence such a real number ℓ^* exists by the contradictory assumption (CON1).)

By the contradictory assumption (CON2), $\mathrm{SINR}_{\mathcal{A}}^{-1}(s_0, q) \geq \mathrm{MAX}^{-1}(\mathcal{A}, s_0, p_L, p_R)$. Combining with Property (P4) of Lemma 3.10, we have

$$\mathrm{SINR}_{\mathcal{A}}^{-1}(s_0, q) > \mathrm{MAX}^{-1}(\mathcal{A}(\ell^*), s_0, p_L, p_R)\ . \tag{31}$$

Finally, by Property (P1) of Lemma 3.10, it holds that $\mathrm{SINR}_{\mathcal{A}(\ell^*)}^{-1}(s_0, q) = \mathrm{SINR}_{\mathcal{A}}^{-1}(s_0, q)$. Combining this with Eq. (30) and (31), we have that $\mathrm{SINR}_{\mathcal{A}(\ell^*)}^{-1}(s_0, q) > \max\{\mathrm{MAX}^{-1}(\mathcal{A}(\ell^*), s_0, p_L, p_R), \mathrm{AVG}^{-1}(\mathcal{A}(\ell^*), s_0)\}$. Since $\mathcal{A}(\ell^*)$ is a noise-free network, we end with contradiction with Lemma 3.9. The claim follows. ■

Lemma 3.17: If $\mathcal{A}$ is a non-collinear network, satisfying Eq. (28), then

$$\mathrm{AVG}^{-1}(\mathcal{A}, s_0) > \mathrm{MAX}^{-1}(\mathcal{A}, s_0, p_L, p_R).$$

Proof: By Eq. (29) that was established in Lemma 3.16, it is sufficient to show that $\mathrm{AVG}^{-1}(\mathcal{A}, s_0) \neq \mathrm{MAX}^{-1}(\mathcal{A}, s_0, p_L, p_R)$.

Assume, towards contradiction that $\mathrm{AVG}^{-1}(\mathcal{A}, s_0) = \mathrm{MAX}^{-1}(\mathcal{A}, s_0, p_L, p_R)$. Combining with Lemma 3.16, we have that

$$\mathrm{SINR}_{\mathcal{A}}^{-1}(s_0, q) = \mathrm{AVG}^{-1}(\mathcal{A}, s_0) = \mathrm{MAX}^{-1}(\mathcal{A}, s_0, p_L, p_R). \tag{32}$$

We next construct two networks $\mathcal{A}'$ and $\mathcal{A}''$ with stations s'_0 and s''_0 respectively, such that

(Q1) $\mathrm{SINR}_{\mathcal{A}}^{-1}(s_0, q) = \mathrm{SINR}_{\mathcal{A}'}^{-1}(s'_0, q) = \mathrm{SINR}_{\mathcal{A}''}^{-1}(s''_0, q)$;
(Q2) $\mathrm{AVG}^{-1}(\mathcal{A}'', s_0) < \mathrm{AVG}^{-1}(\mathcal{A}', s'_0) = \mathrm{AVG}^{-1}(\mathcal{A}, s_0)$;
(Q3) $\mathrm{MAX}^{-1}(\mathcal{A}', s'_0, p_L, p_R), \mathrm{MAX}^{-1}(\mathcal{A}'', s''_0, p_L, p_R) < \mathrm{MAX}^{-1}(\mathcal{A}, s_0, p_L, p_R)$.

Note that the existence of such networks results in a contradiction. Specifically, by combining Eq. (32) with properties (Q1) and (Q2), we have that $\mathrm{AVG}^{-1}(\mathcal{A}'', s''_0) < \mathrm{SINR}_{\mathcal{A}''}^{-1}(s''_0, q)$, and by combining properties (Q1) and (Q3), $\mathrm{MAX}^{-1}(\mathcal{A}'', s''_0, p_L, p_R) < \mathrm{SINR}_{\mathcal{A}''}^{-1}(s''_0, q)$, in contradiction to the weak minimum principle of Lemma 3.11, which implies Lemma 3.17.

It remains to describe the construction of the networks $\mathcal{A}'$ and $\mathcal{A}''$. This first network $\mathcal{A}' = \langle S' = \{s'_0, ..., s'_n\}, \bar{\psi}, N, \beta, \alpha \rangle$ is obtained from the original network $\mathcal{A}$ by rotating separately each of its stations s_0 (resp., $s_1, ..., s_n$) around the origin point q until it reaches the positive (resp., negative) y-axis and preserving the transmitting powers (similarly to the construction of $\mathcal{A}'$ in the proof of Lemma 2.3). That is, the station s_0 (resp., s'_i) is located at $(0, \sqrt{\rho_0})$ (resp., $(0, -\sqrt{\rho_i})$) and its transmitting

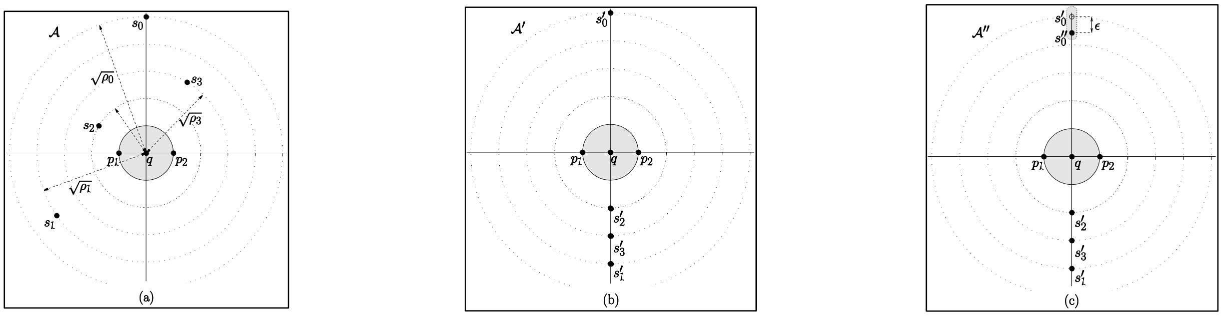

Figure 2. Three wireless systems. (a) network $\mathcal{A}$, in (b) $\mathcal{A}'$ obtained from $\mathcal{A}$ and (c) $\mathcal{A}''$ obtained from $\mathcal{A}'$.

power is ψ_i (for $i \in \{0, ..., n\}$), as illustrated in Figure 2. The stations $s_0', ..., s_n'$ preserve their transmission powers as well as their distances to the ball center q, hence

$$\mathrm{SINR}^{-1}_{\mathcal{A}}(s_0, q) = \mathrm{SINR}^{-1}_{\mathcal{A}'}(s_0', q) = \sum_{i=1}^{n} \frac{\psi_i \cdot \rho_0}{\psi_0 \cdot \rho_i} + \frac{N \cdot \rho_0}{\psi_0} \tag{33}$$

implying property (Q1) for network $\mathcal{A}'$. Since the average energy $\mathrm{AVG}^{-1}(\mathcal{A}, s_0)$ is a function of the distances ρ_i, it also holds that

$$\mathrm{AVG}^{-1}(\mathcal{A}, s_0) = \sum_{i=1}^{n} \frac{\psi_i \cdot (\rho_0 - 1)}{\psi_0 \cdot (\rho_i - 1)} + \frac{N(\rho_0 - 1)}{\psi_0} = \mathrm{AVG}^{-1}(\mathcal{A}', s_0') \ , \tag{34}$$

implying property (Q2) for network $\mathcal{A}'$. Furthermore, since $\mathcal{A}'$ is a y-collinear and $\mathcal{A}$ is not, by Parts 1 and 2 of Lemma 3.5 we have

$$\mathrm{MAX}^{-1}(\mathcal{A}', s_0', p_L, p_R) \quad = \quad \mathrm{SUM} < \mathrm{MAX}^{-1}(\mathcal{A}, s_0, p_L, p_R) \tag{35}$$

Hence $\mathcal{A}'$ satisfies its desired properties. We now turn to construct the second network $\mathcal{A}''$, which can be obtained from $\mathcal{A}'$ in the following manner. Fix a sufficiently small positive real $\epsilon \in (0, \rho_0 - 1)$ and use $\mathcal{A}'$ to construct an $(n+1)-$ station network $\mathcal{A}'' = \langle S'' = \{s_0'', s_1' ..., s_n'\}, \bar{\psi}'' = (\psi_0'', \psi_1, ..., \psi_n), N, \beta, \alpha \rangle$ by substituting station s_0' with station s_0'' preserving the energy at q (i.e., $\mathrm{E}_{\mathcal{A}''}(s_0'', p) = \mathrm{E}_{\mathcal{A}'}(s_0', q)$). The station s_0'' is still located on the positive y-axis but it is a bit closer to q than s_0'. That is, s_i'' is located at $(0, \sqrt{\rho_0''})$, where $\rho_0'' = \rho_0 - \epsilon$, and its transmission power is $\psi_0'' = \frac{\psi_0(\rho_0 - \epsilon)}{\rho_0}$. Hence, $\mathrm{E}_{\mathcal{A}''}(s_0'', q) = \frac{\psi_0''}{\rho_0 - \epsilon} = \frac{\psi_0}{\rho_0} = \mathrm{E}_{\mathcal{A}'}(s_0', q)$. Thus,

$$\mathrm{SINR}_{\mathcal{A}}(s_0, q) = \mathrm{SINR}_{\mathcal{A}'}(s_0', q) = \mathrm{SINR}_{\mathcal{A}''}(s_0'', q), \tag{36}$$

implying property (Q1). Note that $\frac{\rho_0'' - 1}{\psi_0''} < \frac{\rho_0 - 1}{\psi_0}$, hence by part (3) of Lemma 3.5, for every $\epsilon \in (0, \rho - 1)$,

$$\mathrm{AVG}^{-1}(\mathcal{A}'', s_0'') \quad = \quad \frac{\rho_0'' - 1}{\psi_0''} \cdot \left(\sum_{i=1}^{n} \frac{\psi_i}{\rho_i - 1} + N \right) < \frac{\rho_0 - 1}{\psi_0} \cdot \left(\sum_{i=1}^{n} \frac{\psi_i}{\rho_i - 1} + N \right) = \mathrm{AVG}^{-1}(\mathcal{A}', s_0') \ , \tag{37}$$

where the last equality follows by Eq. (34). This implies property (Q2). It remains to show that $\mathcal{A}''$ satisfies property (Q3), namely, that $\mathrm{MAX}^{-1}(\mathcal{A}'', s_0'', p_L, p_R) < \mathrm{MAX}^{-1}(\mathcal{A}, s_0, p_L, p_R)$. Since, $\mathrm{SINR}^{-1}_{\mathcal{A}''}(s_0'', p_i)$ is continuous in ϵ, it holds that $\lim_{\epsilon \to 0} \mathrm{SINR}^{-1}_{\mathcal{A}''}(s_0'', p_i) = \mathrm{SINR}^{-1}_{\mathcal{A}'}(s_0', p_i)$, for $i \in \{L, R\}$. In addition, by Eq. (32) and properties (Q1) and (Q3) for the network $\mathcal{A}'$, $\mathrm{MAX}^{-1}(\mathcal{A}', s_0', p_L, p_R) < \mathrm{SINR}^{-1}_{\mathcal{A}'}(s_0', p_i)$, and by property (Q1), $\mathrm{MAX}^{-1}(\mathcal{A}', s_0', p_L, p_R) < \mathrm{SINR}^{-1}_{\mathcal{A}''}(s_0'', p_i)$. Since all these functions are continuous in ϵ, there exists a sufficiently small $\epsilon > 0$ satisfying $\mathrm{MAX}^{-1}(\mathcal{A}'', s_0'', p_L, p_R) < \mathrm{SINR}^{-1}_{\mathcal{A}''}(s_0'', q)$. Combining this with property (Q1) and Eq. (32) $\mathrm{SINR}^{-1}_{\mathcal{A}''}(s_0'', q) = \mathrm{SINR}^{-1}_{\mathcal{A}}(s_0, q) = \mathrm{MAX}^{-1}(\mathcal{A}, s_0, p_L, p_R)$ establishes property (Q3) for $\mathcal{A}''$. Lemma 3.17 follows. ∎

We are now ready to establish the strong minimum principle for non-collinear networks.

Lemma 3.18: If $\mathcal{A}$ is a non-collinear network satisfying Eq. (28), then $\mathrm{SINR}_{\mathcal{A}}$ satisfies the strong minimum principle.

Proof: By Lemma 3.17, it holds that $\mathrm{AVG}^{-1}(\mathcal{A}, s_0) \neq \mathrm{MAX}^{-1}(\mathcal{A}, s_0, p_L, p_R)$. By Part 1 of Lemma 3.4, we have that $\mathrm{AVG}^{-1}(\mathcal{A}, s_0) < \mathrm{MAX}^{-1}(\mathcal{A}, s_0)$. Hence, by Lemma 3.16, it holds that also $\mathrm{SINR}_{\mathcal{A}}(s_0, q) < \mathrm{MAX}^{-1}(\mathcal{A}, s_0)$, as required for Lemma 3.8. The claim follows by applying Lemma 3.6. ∎

Overall, by combining Lemma 3.11, Lemma 3.15, and Thm. 3.18, Lemma 3.8 is established. Hence, Thm. 3.2 followed by Lemma 3.6.

A note about general dimension $d > 2$: The proof of the strong minimum principle for the ball goes by induction on the dimension d of the network. For the induction base consider $d = 1$. By Lemma 3.14, the property holds in this case. Next, assume that this property holds for any $d - 1 \geq 1$ and consider dimension d. Note that as long as $\mathrm{MAX}^{-1}(\mathcal{A}, s_0, p_L, p_R) > \mathrm{SUM}$ is satisfied, the proof follows the exact same line as for non-collinear network. The only extreme case that it remains to consider is where $\mathrm{MAX}^{-1}(\mathcal{A}, s_0, p_L, p_R) = \mathrm{SUM}$. By Lemma 3.5(1), in $\mathbb{R}^2$, this happens iff the network is y-collinear. Hence the only adaptation required for general dimension $d > 2$ is the extension of the definition of y-collinear networks to higher dimensions. In particular, note that Lemma 3.5(1) holds iff the distances of all stations to the points p_L and p_R (the intersection of the d-dimensional ball with the x-axis) are the same (i.e., $\mathrm{dist}(s_i, p_L) = \mathrm{dist}(s_i, p_R)$ for every $s_i \in S$). In the 2-dimensional case, this happens only when all stations are aligned on a line (i.e., y-collinear). Generally, in higher dimension $d > 2$, this happens when all stations are aligned on a common $d - 1$ hyperplane, namely, one dimension *lower* than the considered dimension d. By the induction assumption for $d - 1$, the strong minimum principle holds for any $d - 1$ dimensional ball, in particular it holds for the $d - 1$ dimensional ball B_{d-1} obtained by intersecting the given d-dimensional ball B_d with the $d - 1$ hyperplane in which the stations are embedded. This in turn implies that the ball center q of the ball B_d attains an SINR value strictly better than the minimum value of the points of $\Phi(B_{d-1}) \subset B_d$, the induction step for d follows. We next prove Thm. 3.1 for every domain D.

Proof of Thm. 3.1 for $d \geq 2$: Let $D \subset \mathbb{R}^d$ be compact domain and assume the $D \cap S = \emptyset$ to ensure that $\mathrm{SINR}_{\mathcal{A}}(s_i, p)$ is continuous on D. Define $\min(D) = \min\{\mathrm{SINR}_{\mathcal{A}}(s_0, p) \mid p \in D \setminus \Phi(D)\}$ and let $\check{p} \in D \setminus \Phi(D)$ be the point attaining the minimum. We show that $\mathrm{SINR}_{\mathcal{A}}(s_0, \check{p}) = \min(D) > \min\{\mathrm{SINR}_{\mathcal{A}}(s_0, p) \mid p \in \Phi(D)\}$. Assume toward contradiction that $\mathrm{SINR}_{\mathcal{A}}(s_0, \check{p}) \leq \min\{\mathrm{SINR}_{\mathcal{A}}(s_0, p) \mid p \in \Phi(D)\}$. Let $B(\check{p}, r) \subseteq D$ be the ball centered at $\check{p}$. Since $\check{p} \in D$, $B(\check{p}, r)$ exists for a sufficiently small radius r. By Thm. 3.2, there exists a point $w \in \Phi(B(p, r))$ such that $\mathrm{SINR}_{\mathcal{A}}(s_0, w) < \mathrm{SINR}_{\mathcal{A}}(s_0, \check{p})$. But $w \in D$, contradicting the minimality of $\check{p}$ in D. ∎

The following corollary demonstrates that SINR diagrams enjoy the practical implications of the minimum principle, namely, there are no free null-cells in a reception region. In addition, it characterizes the minimum points of the SINR function.

Corollary 3.19: (1) [No-Free-Holes in $\mathbb{R}^d$.] Let $D \subseteq \mathbb{R}^d$ be an compact domain of points such that $\Phi(D) \subseteq \mathcal{H}_i(\mathcal{A})$ and D is free from interfering stations (i.e., $D \cap (S \setminus \{s_i\}) = \emptyset$ but s_i might be in D). Then $D \subseteq \mathcal{H}_i(\mathcal{A})$. (2) [Minimum points of the SINR function.] The function $\mathrm{SINR}_{\mathcal{A}}(s_i, p)$ has no minimum points in $\mathbb{R}^d \setminus S$.

IV. Applications

Most of our applications are based on the *characteristic polynomial* of s_i's reception region $\mathcal{H}_i(\mathcal{A})$. This polynomial is hereafter referred to as the SINR *polynomial*. Throughout this section, assume a two-dimensional space ($d = 2$) where every station $s_j \in S$ is located at point (a_j, b_j) in the plane. The SINR polynomial is given by

$$\begin{aligned} \widetilde{H}_{i,\mathcal{A}}(x, y) &= \beta \left[\sum_{j \neq i} \psi_j \cdot \prod_{k \neq j} \left((a_k - x)^2 + (b_k - y)^2 \right) + N \prod_{j} \left((a_j - x)^2 + (b_j - y)^2 \right) \right] \\ &- \psi_i \prod_{j \neq i} \left((a_j - x)^2 + (b_j - y)^2 \right). \end{aligned} \tag{38}$$

Throughout, it is assumed that the stations are located on rational coordinates which results in a rational SINR polynomial. In many of our applications, we are given a line-segment $\sigma = [a, b]$ for rational points a, b and we consider the univariate SINR polynomial $\widetilde{H}_{i,\mathcal{A},\sigma}(x)$ obtained by the restriction of the SINR polynomial $\widetilde{H}_{i,\mathcal{A}}(x, y)$ of Eq. (38) on σ. In this section, we describe a collection of tools that utilize the properties of univariate rational polynomials. The time complexities of all algorithms described, are measured in terms of *arithmetic* operations.

Our algorithms are based on isolating the roots of the rational polynomial $\widetilde{H}_{i,\mathcal{A},\sigma}(x)$.

Root isolation: Isolation of the real roots of a polynomial in a given segment σ is the process of finding open disjoint intervals such that each interval contains exactly one real root and every real root is contained in each interval. Since the polynomial $\widetilde{H}_{i,\mathcal{A},\sigma}(x)$ has at most $2n$ roots, this isolation procedure results in a partition of σ into $O(n)$ subsegments, each containing at most one root. By computing the Sturm sequence [13] of $\widetilde{H}_{i,\mathcal{A},\sigma}(x)$, one can show the following.

Lemma 4.1: A root isolation for the polynomial $\widetilde{H}_{i,\mathcal{A},\sigma}(x)$ can be computed in $O(n^3 \cdot \log n)$ arithmetic operations.

A. Reception Testing and Segment Testing

For a given wireless system $\mathcal{A} = \langle d = 2, S, \psi, N, \beta, \alpha = 2 \rangle$ and a target station s_i, a point p is *receptive* if $\mathrm{SINR}_{\mathcal{A}}(s_i, p) \geq \beta$. The shape $\mathcal{C}$ is *receptive* if every point $p \in \mathcal{C}$ is receptive, i.e., $\mathrm{SINR}_{\mathcal{A}}(s_i, p) \geq \beta$ at every point $p \in \mathcal{C}$, otherwise it is *non-receptive*. In the setting of reception testing, one is given a wireless network $\mathcal{A}$ and a closed domain $\mathcal{C}$ that is free from interfering stations, i.e., $\mathcal{C} \cap (S \setminus \{s_i\}) = \emptyset$. The task is to decide if $\mathcal{C}$ is receptive with respect to the station s_i. The key observation here is that thanks to the minimum principle it is sufficient to test reception on the boundary of the curve in order to deduce about all its internal points.

The Basic Tool - Segment Testing: An important ingredient in our applications is Procedure SegTest, which tests reception on a line segment. It receives as input a line-segment $\sigma = [a, b]$, a network $\mathcal{A}$ and a target station s_i, and outputs "yes" only if $\sigma \subseteq \mathcal{H}_i(\mathcal{A})$. This is done by applying a root isolation procedure on $\widetilde{H}_{i,\mathcal{A},\sigma}(x)$, i.e., the restriction of the SINR polynomial of Eq. (38) on the segment σ. The output of the root isolation procedure is a partition of σ into $O(n)$ subsegments, each containing at most one root. To decide whether σ is receptive, or not, Procedure SegTest evaluates the SINR function at the endpoints of each subsegment. The answer is positive if and only if all the endpoints have SINR value at least β.

Lemma 4.2: Proc. SegTest outputs "yes" if and only if σ is receptive for s_i within $O(n^3 \cdot \log n)$ arithmetic operations.

B. Exact Reception Testing for Polygonal Regions

Let $\mathcal{P}$ be a polygon free from interfering stations with m vertices located at rational positions in $\mathbb{R}^2$. Note that without the minimum principle, it is not possible to decide if the entire polygon is receptive, even when using an arbitrarily large finite set of sampled points in $\mathcal{P}$. Using the minimum principle, we now describe an exact algorithm for this problem.

Procedure PolygonRecepTest invokes Proc. SegTest for every segment $\sigma \in \mathcal{P}$ of $\mathcal{P}$'s boundary and tests its receptiveness for a reception to s_i. It returns "yes", if and only if every edge segment of the polygon is receptive for s_i. The correctness of Proc. PolygonRecepTest follows immediately by the No-Free-Hole property of Cor. 3.19. We have the following.

Theorem 4.3: Given a polygon $\mathcal{P}$ of m vertices, an n-station network $\mathcal{A}$ and target station $s_i \in S$, it can be verified in (arithmetic) time $O(m \cdot n^3 \log n)$ if $\mathcal{P}$ is receptive or not with respect to s_i.

In the full-version, we also presrent an *approximate* testing scheme that tests the reception on any shape in time that is proportial to the perimeter of the shape (rather than to its area). This scheme is later used for computing an approximation for the maximum incribed reception sphere centered at a given station.

We next use the segment test procedure to provide an exact solution for the Polygonal Power Control problem.

C. The Polygonal Power Control Problem

In the *feasibility* variant of the *power control problem*, one is given n communication links $\langle s_i, r_i \rangle$ and a target SINR threshold β and the goal is to compute a *feasible* power assignment ψ with respect to β, that is, a power assignment that achieves $\mathrm{SINR}(s_i, r_i) \geq \beta$ for every $i \in \{1, \ldots, n\}$ where all stations transmit according to ψ. In the *optimization* variant of the problem, the parameter β is not given; rather, the goal is to compute the maximum SINR threshold β^* for which there exists a feasible power assignment ψ^* with respect to β^*. Note that the optimization problem can be approximated up to some desired ratio by using an algorithm for the feasibility problem in order to search for the best β via binary search. Note that in the standard setting considered so far, every transmitting station s_i was required to be successfully received only at a single reception point r_i (i.e., of zero dimension). However, due to stability considerations, communication applications usually require a successful transmission in a *two-dimensional region* rather than in a fixed number of points. In this section, we focus on the feasibility variant and consider a 2-dimensional generalization of this problem.

In the *Power Control for Polygons* (PCPG) problem, one is given a network of n stations $S = \{s_0, \ldots, s_n\}$ in the plane, a target SINR threshold β and a collection of n polygons $\mathcal{P}_1, \ldots, \mathcal{P}_n$ that are free from interfering stations (i.e., $\mathcal{P}_i \cap (S \setminus \{s_i\}) = \emptyset$ for every i). The goal is to find a power assignment vector ψ satisfying that $\mathrm{SINR}(s_i, p) \geq \beta$ for every $p \in \mathcal{P}_i$ and for every $s_i \in S$. This yields the following formulation.

$$\text{Given } \beta, S, N \text{ and polygons } \mathcal{P}_1, \ldots, \mathcal{P}_n, \text{ find powers } \psi_1, \ldots, \psi_n > 0: \tag{39}$$
$$\mathrm{SINR}(s_i, p) \;=\; \frac{\psi_i \cdot \mathrm{dist}(s_i, p)^{-\alpha}}{\sum_{s_j \in (S \setminus \{s_i\})} \psi_j \cdot \mathrm{dist}(s_j, p)^{-\alpha} + N} \geq \beta \text{ for every } s_i \in S \text{ and } p \in \mathcal{P}_i \ .$$

In Sec. IV-C, we show that Program (39) is convex and in addition it can be solved via the Ellipsoid method despite the fact that it contains infinitely many constraints. This holds since we are able to provide a polynomial *separation oracle* based on Procedure SegTest discussed above. We show the following.

Theorem 4.4: Given a set of n stations S, target SINR threshold β and a set of n polygons $\mathcal{P}_1, \ldots, \mathcal{P}_n$ free of interfering stations (i.e., $\mathcal{P}_i \cap (S \setminus \{s_i\}) = \emptyset$ for every i) and whose endpoints vertices located at rational coordinates, there exists a polynomial time exact algorithm for PCPG when: (1) $\beta < \beta^*$ (where β^* is the optimum SINR threshold of the network), or (2) $\beta = \beta^*$ and the optimum power assignment is rational.

Note that without the minimum principle, the best one can do is to sample sufficiently many points inside each polygon $\mathcal{P}_i$ and solve a linear program consisting of the corresponding SINR constraints. Not only does this approach require a large preprocessing time that depends on the area of the polygons, the number of stations n and the fatness of the reception regions (whose bounds are large), but moreover, it can never guarantee the successful transmission in the *entire* polygon region, as there is no guarantee that the unsampled polygon points receive the transmission with the desired SINR threshold of β, but rather with some $\beta - \epsilon$, where $\epsilon \in (0, \beta)$ depends on the density of the sampled points within the polygon.

In contrast, using the no-free-hole property enables us to provide an exact solution for the PCPG problem for the case where $\beta < \beta^*$ or when the optimum power assignment for the given β^* is rational. We first establish convexity.

Observation 4.5: Program (39) is convex.

A combinatorial algorithm for *exactly* solving a convex program is possible only if it admits rational solutions [14]. A nonlinear convex program is rational if, for any setting of its parameters to rational numbers such that it has a finite optimal solution, it admits an optimal solution that is rational and can be written using polynomially many bits in the number of bits needed to write all the parameters. Note that when taking $\beta < \beta^*$, there is a continuous non-empty region of feasible powers and in particular there is rational solution. Hence, if the optimal power assignment for β^* is rational, our algorithm can compute it exactly.

We now show that this convex program can be solved exactly via the Ellipsoid method despite the fact that it contains infinitely many constraints. This holds since we are able to provide a *separation oracle*. A separation oracle is polynomial time algorithm that determines if a given candidate solution is feasible (i.e., it satisfies all linear constraints) or returns a violated constraint if it is not feasible. The separation oracle in our context is based on Proc. SegTest of Sec. IV-A. It is easy to see that Proc. SegTest can be modified to return a non-receptive point on σ if such exists. This non-receptive point is then used to identify a violated constraint to be supplied to the Ellipsoid algorithm. We now formally describe the separation oracle. Let ψ' be a candidate solution (e.g., the center of the current ellipsoid in which the feasible set of solutions reside). Define $\mathcal{A}' = \langle d, S, \psi', N, \beta, \alpha \rangle$. Apply $\mathsf{SegTest}(\mathcal{A}', s_i, \sigma_j)$ for every edge σ_j of the input polygon $\mathcal{P}_i$ and every $i \in \{1, \ldots, n\}$. If every such $\sigma_j \in \mathcal{P}_i$ is receptive for s_i, then ψ' is a feasible power assignment. Else, let $\sigma_j \in \mathcal{P}_\ell$ be a non-receptive segment with respect to s_ℓ and let $p \in \sigma_j$ be a non-receptive point (this point can be returned by Proc. SegTest). Then, the violated constraint by ψ' is $\mathrm{SINR}_{\mathcal{A}'}(s_\ell, p) \geq \beta$. This completes the description of the separation oracle.

Note that the same scheme can be extended to the problem of *sum-power minimization* or the *min-max power problems*, in which it is also required to minimize the total (resp., max) transmit power [3].

D. Universal Bound for the Number of Null-Cells

We now show that the minimum principle can be utilized for providing a tight linear bound on the number of null-cells, improving over the $O(n^{2d})$ bound of [8]. Note that the No-Free-Hole property implies that any null-cell in a reception of station s_i contains some interfering station s_j. This implies that there are $O(n)$ null-cells in the reception zone of s_i and since there are n stations, we have the following immediate corollary.

Corollary 4.6: For every $\beta > 0$ and for every dimension $d \geq 1$, there are $O(n^2)$ null-cells for every n station network $\mathcal{A} = \langle d, S, \psi, N, \beta, \alpha = 2 \rangle$.

In this section, we show that for $\beta > 1$, there are in fact only $O(n)$ null-cells, for every dimension d, and this bound is tight.

Theorem 4.7: For $\beta > 1$ and $N > 0$, the null zone $\mathcal{H}_\emptyset(\mathcal{A}) \subseteq \mathbb{R}^d$ contains at most $\tau_\emptyset(\mathcal{A}) = O(n)$ cells.

We begin by showing that as long as $N > 0$, there exists exactly one infinite null-cell and all other cells are bounded.

Lemma 4.8: There exists exactly one unbounded null-cell $\mathcal{H}_{\emptyset,j}(\mathcal{A})$.

Proof: Since $N > 0$, no station can be received in $\overline{B} = \mathbb{R}^d \setminus B(s_0, r)$ for a sufficiently large r. We therefore have that $\overline{B}$ is fully contained in the null zone $\mathcal{H}_\emptyset(\mathcal{A})$. Note that $\overline{B}$ is connected since for every two null points $p, q \in \overline{B}$, there is a curve connecting these points that is fully contained in $\overline{B}$ as well. ■

It remains to bound the number of bounded cells. We begin by considering the case of $d = 2$ and towards the end of this section extend it to general d.

For every bounded null-cell $H_j = \mathcal{H}_{\emptyset,j}(\mathcal{A})$, denoted by J_j its outer Jordan curve (since a null-cell is closed, this is well defined). Let J_j^- (resp., J_j^+) denote the region outside (resp., inside) J_j. The following observation is essential in our analysis.

Observation 4.9: For every Jordan curve J_j, there exists a station $s_{j'} \in S$ satisfying that $J_j \subset \mathcal{H}_{j'}(\mathcal{A})$.

Proof: By definition of J_j, some transmission is received at every point $p \in J_j$, i.e., for every $p \in J_j$ there must exist a station s_p such that $\mathrm{SINR}_\mathcal{A}(s_p, p) \geq \beta$. By the continuity of J_j and since $\beta > 1$, it must hold that $s_{p_1} = s_{p_2}$ for every $p_1, p_2 \in J_j$, since otherwise there must be a point p_0 on J_j where the identity of the received transmitter switches from some s to some other s', which cannot happen for $\beta > 1$. ■

Note that it may be possible that $s(J_j) \notin J_j^+$. For an illustration of Obs. 4.9, see Fig. 3(a). Two Jordan curves J_1 and J_2 are *independent* if $J_1^+ \not\subseteq J_2^+$ and $J_2^+ \not\subseteq J_1^+$, otherwise they are *dependent*. Let $\mathcal{J} = \{J_1, \ldots, J_\ell\}$ be the collection of

Jordan curves corresponding to the null-cells $\mathcal{H}_{\emptyset}(\mathcal{A}) = \{\mathcal{H}_{\emptyset,1}(\mathcal{A}), \ldots, \mathcal{H}_{\emptyset,\ell}(\mathcal{A})\}$. The following observation plays a key role in our analysis.

Observation 4.10: For every $1 \le j, j_1, j_2 \le \ell$,
(1) If $J^+_{j_1} \cap J^+_{j_2} \neq \emptyset$ then either $J_{j_1} \subset J_{j_2}$ or $J_{j_2} \subset J_{j_1}$. (2) $J_j \cap S = \emptyset$. (3) J^+_j contains an interfering station $s \neq s(J_j)$, i.e., $(S \setminus \{s(J_j)\}) \cap J^+_j \neq \emptyset$.

Proof: Claim (1) holds since two Jordan curves correspond to disconnected null-cells and hence they cannot intersect, although they might touch. Hence, the only possible overlap relations between them are that either they are independent or that one is contained in the other. Claim (2) follows immediately by the fact that J_j is the boundary of a null-cell. Finally, claim (3) follows by Cor. 3.19. ■

The null-cells digraph: Our analysis is based on inducing a directed forest $F = (\mathcal{J}, A)$ on the set of Jordan curves, where a directed edge $a_{i_1,i_2} \in A$ connects J_{i_1} to J_{i_2} iff $J^+_{i_2} \subset J^+_{i_1}$ and there is no other curve J_k satisfying that $J_{i_2} \subset J_k \subset J_{i_1}$. It is easy to verify that F is a directed forest, and each of its components is a tree T_i rooted at some $J(T_i)$, with edges directed downwards from the root, where $J^+_j \subseteq J^+(T_i)$ for every curve $J_j \in T_i$. For an illustration, see Fig. 3. We may

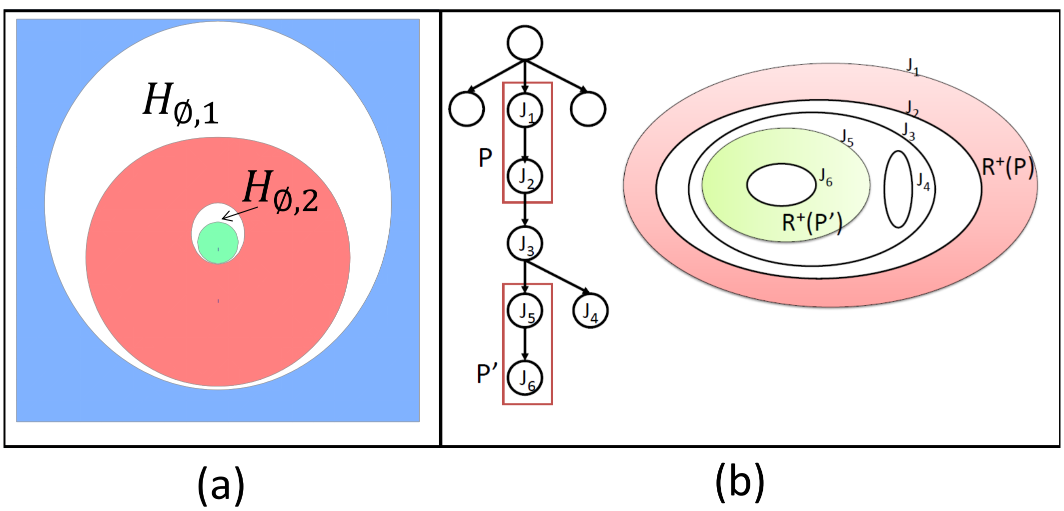

Figure 3. (a) Illustration of a 3-station network with 2 null-cells, $H_{\emptyset,1}$ and $H_{\emptyset,2}$. For every Jordan curve J_j there is station s_j that is continuously received on that curve. The two Jordan curves are dependent, and their null-cells digraph corresponds to a single directed edge (J_1, J_2). (b) Schematic illustration of null-cells tree T_i and the regions associated with the paths of the forest F_i.

refer to $J \in T_i$ as either a vertex in the forest or a curve in the diagram. Let $R_i = J^+(T_i)$ be the region inside the Jordan curve of the root of T_i and let S_i be the set of stations restricted to the region R_i. (Note that S_i does not necessarily equal $\{s(J) \mid J \in V(T_i)\}$.) We then bound the number of vertices (i.e., null-cells) by showing that for every i, $|T_i| \le c \cdot |S_i|$ for some constant $c \ge 1$, where $|T_i|$ is the number of vertices in T_i. Since the sets S_i are disjoint (as they reside in disjoint regions R_i), this would establish the bound.

From now on, we focus on a specific tree T_i and denote by V^{high}_i the vertices with outdegree at least 2 in T_i and by V^{low}_i as the complementary set of vertices with outdegree at most 1. Within this set, let V^{leaf}_i be the set of leaves (outdegree 0) in T_i and let $V_{leaf} = \bigcup_i V^{leaf}_i$. It is straightforward to verify that the forest F satisfies the following properties.

Observation 4.11: (1) If $J_j \subset J_i$ then J_i is an ancestor of J_j in F. (2) Every two leaves $J_k, J_{k'} \in V^{leaf}_i$ are independent.

Proof: Begin with (1). We prove it by the induction on the depth of J_j in its tree (i.e., the distance of J_j from the root). For the base of the induction, J_j is the son of the root (i.e., depth 1). In this case, by the definition of an edge, the root is the only curve that contains J_j and hence the root is J_i. Assume that claim holds for every J_j at depth $\leq k-1$ and consider a vertex J_j at depth k. Let $\mathcal{J}_j = \{J' \in \mathcal{J} \mid J_j \subset J'\}$ be the set of curves containing J_j. Since $J_j \subset J_i$, $\mathcal{J}_j \neq \emptyset$. Let $J' \in \mathcal{J}_j$ be the a curve satisfying that there exists no $J'' \in \mathcal{J}_j$ such that $J_j \subset J'' \subset J'$. Since $\mathcal{J}_j$ is finite, such J' is guaranteed to exist. By the definition of an edge, there must be a directed edge from J' to J_j. Hence, J' is at depth $k-1$ in the tree. We now claim that $J' \subseteq J_i$. Since both curves contain J_j, they are dependent so it remains to refute the possibility that $J_i \subset J'$. This holds as otherwise, we get that $J_j \subset J_i \subset J'$ in contradiction to the definition of J'. Hence, $J' \subseteq J_i$ and by the induction assumption for $k-1$, there is a directed path from J_i to J'. Overall, there is a directed path from J_i to J_j the goes through the arc (J', J_j). The claim follows. Part (2) follows immediately by part (1). ∎

By Obs. 4.11 and the NFH property of Cor. 3.19, we bound the total number of high degree vertices in the trees T_i by n.

Lemma 4.12: $\sum_{T_i \in F} |V_i^{high}| \leq n$.

Proof: We first claim that the total number of leaves $|V_{leaf}|$ in the forest F is at most n. By the minimum principle, see Cor. 3.19, every J_i must contain an interfering station $s_j \neq s(J_i)$ inside J_i^+. By Obs. 4.11(2), every two leaves J_{i_1} and J_{i_2} are independent, hence $J_{i_1}^+ \cap J_{i_2}^+ = \emptyset$, implying the claim. The lemma now follows by noting that $|V_i^{high}| \leq |V_i^{leaf}|$ for every $T_i \in F$. ∎

Bounding the set V_i^{low}: Let F_i be the forest induced by $T_i \setminus \left(V_i^{high} \cup \{J(T_i)\}\right)$. Then, F_i is a collection of up to $|V_i^{high}| + 2$ vertex-disjoint paths. For every path $P = [J_{i_1}, \ldots, J_{i_k}]$ in F_i, we define two sets of stations, $S_{unique}(P)$ and $S_{inter}(P)$, and a multi-set $S_{same}(P)$. Let

$$S_{unique}(P) = \{s(J_j) \mid J_j \in P \text{ and } s(J_j) \neq s(J_k) \text{ for every } J_k \in P \setminus \{J_j\}\} \setminus \{s(J_1)\}$$

be the set of stations that are received on exactly *one* Jordan curve vertex on P (excluding perhaps the station of the first vertex). Let $S_{same}(P) = \{s(J) \mid J \in P \setminus \{J_1\}\} \setminus S_{unique}(P)$ be a multi-set of stations that are received on at least two Jordan curves on P. Since $s(J_{i'})$ is either in S_{unique} or in S_{same}, for every $J_{i'} \in P \setminus \{J_{i_1}\}$, we have that

$$|P| = |S_{unique}(P)| + |S_{same}(P)| + 1 \ . \tag{40}$$

Finally, we associate two regions with P, namely, $R(P) = J_{i_1}^+$, $R^+(P) = R(P) \setminus J_{i_k}^+$. Then let $S_{inter}(P) = S \cap R^+(P)$ be the set of stations restricted to the region $R^+(P)$. For a schematic illustration see Fig. 3.

Observation 4.13: For every $P, P' \in F_i$, it holds that $R^+(P) \cap R^+(P') = \emptyset$ and hence $S_{inter}(P) \cap S_{inter}(P') = \emptyset$.

Proof: Let $P = [J_{x_1}, \ldots, J_{x_z}]$ and $P' = [J_{y_1}, \ldots, J_{y_{z'}}]$. By choice, $J_{x_1} \neq J_{y_1}$. If J_{x_1} and J_{y_1} are independent, then the claim clearly follows. Otherwise, without loss of generality, assume that $J_{y_1}^+ \subset J_{x_1}^+$. See Fig. 3. We now claim that in such a case, for every vertex $J_{x'}$ on P, contains $J_{y_1}^+ \subset J_{x'}^+$. This holds by Obs. 4.11(1) and due to the fact that every vertex $J_{x'}$ on P has outdegree one. Hence, by Obs. 4.11(1), $J_{y_1}^+$ is descendent of J_{x_z}. The claim follows by noting that $R^+(P) \cap J_{x_z}^+ = \emptyset$ but $R^+(P') \subseteq J_{y_1}^+ \subset J_{x_z}^+$. ∎

The following claims are crucial is this context.

Claim 4.14: For every directed path $J_{i_1} - J_{i_2}$ in T_i (i.e., $J_{i_2}^+ \subset J_{i_1}^+$) such that $s(J_{i_1}) \neq s(J_{i_2})$:
(a) $s(J_{i_2}) \in J_{i_1}^+$. (b) $\psi_1 > \psi_2$ where ψ_z is the transmission energy of $s(J_{i_z})$ for $z \in \{1, 2\}$.

We next establish an important corollary of Cl. 4.14.

Corollary 4.15: Let $J_1, J_2, J_3 \in \mathcal{J}$ be such that $J_3^+ \subset J_2^+ \subset J_1^+$, where $s(J_2) \neq s(J_1)$ and $s(J_2) \neq s(J_3)$. Then $s(J_1) \neq s(J_3)$.

The following claim relates the cardinalities of $S_{inter}(P)$ and $S_{same}(P)$.

Claim 4.16: $|S_{same}(P)| \leq 2|S_{inter}(P)|$.

Proof: Let S'_{same} be the unique set (without repetition) of the multi-set $S_{same}(P)$. By Cor. 4.15, for every station $s \in S'_{same}$ there is a unique subpath $P_s \subseteq P$ such that $s(J) = s$ for every vertex $J \in P_s$. These subpaths are disjoint. For an arc $a_{x,y}$ define its region by $R_{x,y} = J_x^+ \setminus J_y^+$. By the definition of the arc, it follows that the regions of any two arcs $a_{x,y}$ and $a_{x',y'}$ are disjoint, for $x \neq x'$, i.e., $R_{x,y} \cap R_{x',y'} = \emptyset$. We now show that there exists an interfering station $s' \neq s$ in the region $R_{x,y}$ for every $s \in S'_{same}$ and for every arc $a_{x,y} \in P_s$. To see this, note that as $s(J_x) = s(J_y) = s$, by Cor. 3.19, it holds that there must be an interfering station $s_{x,y} \neq s$ in the closed region $R_{x,y}$. Let $S' = \{s_{x,y} \mid a_{x,y} \in P_s, s \in S_{same}\}$. Then $S' \subseteq S_{inter}(P)$ and in addition, $|S_{inter}(P)| \geq |S'| = \sum_{s \in S'_{same}} (|V(P_s)| - 1) \geq |S_{same}(P)|/2$, where the last inequality follows as by definition, $|P_s| \geq 2$ for every $s \in S'_{same}$. The claim follows. ∎

We now proceed with the second set $S_{unique}(P)$. Recall that $s(J_1) \notin S_{unique}(P)$ where J_1 is the first vertex of P.

Claim 4.17: (a) $S_{unique}(P)$ is inside $R(P)$, and (b) $S_{unique}(P_1) \cap S_{unique}(P_2) = \emptyset$ for every P_1, P_2.

We are now ready to complete the proof and show the following.

Claim 4.18: $\sum_{T_i \in F} |V_i^{low}| \leq n$.

Proof: For every P, let $S^+(P) = S_{inter}(P) \cup S_{unique}(P)$. Since the station set $S^+(P)$ reside in $R(P)$, it holds that for every $P \in F_i$ and $P' \in F_j$, where $i \neq j$, $S^+(P)$ and $S^+(P')$ are disjoint. In addition, for two paths P and P' that are in the same forest F_i, we have that $S_{inter}(P)$ and $S_{inter}(P')$ are disjoint by Cl. 4.16 and by Cl. 4.17, we also have that $S_{unique}(P)$ and $S_{unique}(P')$ are disjoint. From now on, we consider only paths $P \in F_i$ of length at least 2. Note that there are at most $O(|V_{leaf}|) = O(n)$ paths of length 1, hence this would increase the bound by an additive factor of $O(n)$. Let

$$\widetilde{\mathcal{P}} = \{P \subseteq F_i \mid |P| \geq 2 \text{ for every } T_i \subseteq F\}$$

be the collection of paths considered. Define $V_{low} = \bigcup_{P \in \widetilde{\mathcal{P}}} V(P)$, $\mathcal{S}_1 = \sum_{P \in \widetilde{\mathcal{P}}} |S_{unique}(P)|$ and $\mathcal{S}_2 = \sum_{P \in \widetilde{\mathcal{P}}} |S_{same}(P)|$. By Eq. (40) and since every $P \in \widetilde{\mathcal{P}}$ is of length at least 2, it then holds that $|V_{low}| \leq 2(|\mathcal{S}_1| + |\mathcal{S}_2|)$. We now consider two cases. First, assume that $\mathcal{S}_1 \geq \mathcal{S}_2$. Then, $|V_{low}| \leq 4|\mathcal{S}_1|$. Since $S_{unique}(P_1) \cap S_{unique}(P_2) = \emptyset$, for every $P, P' \in \widetilde{\mathcal{P}}$ it holds that $\mathcal{S}_1 = \left|\bigcup_{P \in \widetilde{\mathcal{P}}} S_{unique}(P)\right| \leq n$, hence the claim follows. Alternatively, if $\mathcal{S}_2 > \mathcal{S}_1$, then, $|V_{low}| \leq 4|\mathcal{S}_2|$. By Cl. 4.16 and the disjointness of the $S_{inter}(P)$ sets, we have that $\mathcal{S}_2 = \sum_{P \in \widetilde{\mathcal{P}}} |S_{same}(P)| \leq 2 \sum_{P \in \widetilde{\mathcal{P}}} |S_{inter}(P)| = 2\left|\bigcup_{P \in \widetilde{\mathcal{P}}} S_{inter}(P)\right| \leq 2n$. The claim follows. ■

Thm. 4.7 for $d = 2$ follows by Lemma 4.12 and Lemma 4.18.

General $d \geq 2$: To bound the number of null-cells for the general case of $d \geq 2$, the following definitions are now extended. For a null-cell, $\mathcal{H}_{\emptyset,j}$, let J_j be its outer boundary (i.e., the notion of outer boundary replaces the terminology of a Jordan curve used in $d = 2$). Note that for every point $p \in \mathcal{H}_{\emptyset,j}$, the shortest path to the unique infinite null-cell intersects with the outer boundary J_j. We now associate a d-dimensional region with every J_j that is free from null-cells. Let $s(J_j)$ be the station received on J_j (i.e., letting $s_{j'} = s(J_j)$ then $J_j \subset \mathcal{H}_{j'}(\mathcal{A}')$). By Obs. 4.9, this is well defined. Let J_j^+ be the open area inside J_j obtained by filling the (possible) null-cells. Formally, define J_j^+ as the collection of all points $\ell \in \overline{pq} \setminus \{p, q\}$ for every $p, q \in J_j$ satisfying that the outer boundary of $J_j \cup \overline{pq}$ is J_j. That is the open set J_j^+ is obtained by filling null-cells while preserving the outer boundary of the shape. Analogously to the 2-dimensional case, we say that two cells $H_1 = \mathcal{H}_{\emptyset,j_1}(\mathcal{A})$ and $H_2 = \mathcal{H}_{\emptyset,j_2}(\mathcal{A})$, with outer boundaries J_1 and J_2 respectively, are *dependent* if $J_1^+ \subset J_2^+$ or vice-verse. We begin by establishing some topological properties on null-cells.

Claim 4.19: (1) Every null-cell $\mathcal{H}_{\emptyset,j}$ is an open connected subset in $\mathbb{R}^d$, and (2) The dimension of $\Phi(\mathcal{H}_{\emptyset,j})$ is $d-1$.

To apply the proof of the 2-dimensional case, it is then sufficient to establish Obs. 4.10(1) for general d. (Claims (2) and (3) are extended naturally to $d \geq 2$.)

Proof of Observation 4.10(1) for every dimension $d \geq 2$: We first claim that J_j^+ is an open set for every $J_j \in \mathcal{J}$. I.e., it is required to show that for every $p \in J_j^+$, there exists a sufficiently small d-dimensional ball $B(p, \epsilon)$ inside J_j^+. If $p \in \mathcal{H}_{\emptyset,j}(\mathcal{A})$, this holds by Cl. 4.19(1). So, it remains to consider the case where $p^+ \in J_j^+ \setminus \mathcal{H}_{\emptyset,j}(\mathcal{A})$. Since the outer boundary J_j is the limit of the non-reception points in $\mathcal{H}_{\emptyset,j}(\mathcal{A})$. Hence, we conclude that J_j^+ is an open set in $\mathbb{R}^d$.

Assume, towards contradiction, that the claim assertion does not hold and consider two d-dimensional null-cells $H_{\emptyset,1} = \mathcal{H}_{\emptyset,j_1}(\mathcal{A})$ and $H_{\emptyset,2} = \mathcal{H}_{\emptyset,j_2}(\mathcal{A})$ with a non-empty intersection of the interior regions, $J_1^+ \cap J_2^+ \neq \emptyset$ and yet $J_2^+ \nsubseteq J_1^+$ and $J_1^+ \nsubseteq J_2^+$. Denote

$$M = (J_1^+ \cup J_1) \cap (J_2^+ \cup J_2).$$

Note that M is obtained by the intersection of two d-dimensional closed sets and since $J_1^+ \cap J_2^+ \neq \emptyset$, it is a d-dimensional closed set as well. In particular, it holds that $\Phi(M) \subseteq J_1 \cup J_2$. Let $p \in M$ be a non-reception point (e.g., by taking p to be sufficiently close to the boundary $\Phi(M)$). Without loss of generality, let p belongs to J_1^+ and since p is a non-reception point, it also holds that $p \in H_{\emptyset,1}$. Consider now another null point $q \in H_{\emptyset,1} \setminus M$. Since $J_1 \nsubseteq M$, the point q can be given by taking a sufficiently close internal point to the boundary $J_1 \setminus \Phi(M)$. On the one hand, the null points p and q are in the same connected subset $H_{\emptyset,1}$, but on the other hand, p is in a d-dimensional subset M whose boundary $\Phi(M)$ has no intersection with $\mathcal{H}_{\emptyset}(\mathcal{A})$ (since every point on $\Phi(M)$ is in $J_1 \cup J_2$ and hence receptive) and in particular has no intersection with $H_{\emptyset,1}$. Contradiction. ■

Once establishing Obs. 4.10, we can safely define a null-cells digraph for the collection of d-dimensional null-cells. The proof reasonings on that graph are invariant to dimension.

E. Approximation of the number of null-cells for any $\beta > 0$

To this end, we use the following notation. Let $\mathcal{A}_{\beta'}$ be a network identical to $\mathcal{A}$ except its SINR threshold is β' instead of β. To avoid cumbersome notation, we focus on the station s_i and define $\tau(\beta', i)$ as the number of null-cells (holes) in the reception zone of s_i in the network $\mathcal{A}_{\beta'}$, i.e., $\mathcal{H}_i(\mathcal{A}_{\beta'})$. In this section, we present a scheme that given an approximation parameter $\epsilon \in (0, 1)$ and a target station s_i, returns an approximation $X(i, \epsilon)$ satisfying that $\tau(\beta, i) \leq X(i, \epsilon) \leq \tau(\beta_\epsilon, i)$

where $\beta_\epsilon = ((1-\epsilon)/(1+\epsilon))^\alpha \cdot \beta$. For $\beta > 1$, it then holds that

$$\tau_\emptyset(\mathcal{A}_\beta) \le \sum_{i=1}^{n} X(i,\epsilon) \le \tau_\emptyset(\mathcal{A}_{\beta_\epsilon}),$$

where $\tau_\emptyset(\mathcal{A})$ is the total number of null-cells in the network $\mathcal{A}$. The scheme of this section is focused on the setting where $\beta > 1$. This is because, the crucial property (used in the previous section) that exactly one station is received on the boundary of every reception cell is guaranteed only when $\beta > 1$. In contrast, when $0 < \beta \le 1$, it might be the case that different regions on the boundary of the null-cell are received by different stations and hence the minimum principle (e.g., the NFH property) cannot be directly applied. We note however that the presented scheme can also be applied for the case of $\beta \le 1$ upon slightly modifying the definition of a null-cell and focusing on a given fixed station. Specifically, for a given station s_i and *any* $\beta > 0$, our scheme can count the number of "holes" in s_i's map where the holes in s_i's map are the connected regions is which s_i cannot be correctly received, i.e., the scheme approximates the number of connected regions in $\mathbb{R}^d \setminus \mathcal{H}_i(\mathcal{A}_\beta)$.

We begin by showing, using the minimum principle, that the number of null-cells (i.e., holes) in $\mathcal{H}_i(\mathcal{A}_\beta)$ is monotone *decreasing* with β. Note that this property does not hold for the number of *reception* cells in $\mathcal{H}_i(\mathcal{A}_\beta)$. In particular, in the extreme cases where β is either infinitely small or extremely large, $\mathcal{H}_i(\mathcal{A}_\beta)$ consists of one connected reception cell.

Lemma 4.20: Let $0 < \beta_1 \le \beta_2$. Then $\tau(\beta_1, i) \ge \tau(\beta_2, i)$.

Proof: By the minimum principle, every null-cell in the map of s_i contains an interfering station $s_j \in S \setminus \{s_i\}$, hence there are at most $n-1$ bounded null-cells plus one infinite null-cell in $\mathcal{H}_i(\mathcal{A}_\beta)$ for every $\beta > 0$. (This is tight for a sufficiently small β). Let $H_1, \ldots, H_\ell$ be the null-cells in $\mathcal{H}_i(\mathcal{A}_{\beta_2})$ such that H_j contains the interfering station $s_{k_j} \neq s_i$. By the definition of null-cells, the interfering stations s_{k_j} are distinct. We now show that for every $1 \le j \le \ell$, there exists a corresponding null-cell H'_j in $\mathcal{H}_i(\mathcal{A}_{\beta_1})$ that contains s_{k_j}. Let $\Phi(H_j)$ be the boundary of the null-cell H_j. By definition, $\Phi(H_j) \subseteq \mathcal{H}_i(\mathcal{A}_{\beta_2})$ and $s_{k_j} \in H_j$. Since $\beta_1 \le \beta_2$, we get that $\Phi(H_j) \subseteq \mathcal{H}_i(\mathcal{A}_{\beta_1})$ as well, and since $s_j \in H_j$ is a null point in s_i's reception region $\mathcal{H}_i(\mathcal{A}_\beta)$ for every $\beta > 0$, we get that there exists a null-cell H'_j in $\mathcal{H}_i(\mathcal{A}_{\beta_1})$ that contains s_{k_j}. The claim follows. ■

Note that without the minimum principle one encounters two main difficulties when approximating the number of null-cells. First, since a priori the null-cell can be located anywhere in the map, one has to sample $\Omega(\widetilde{\Delta}/(\epsilon \cdot \widetilde{\delta}))$ points in the large circle that contains $\mathcal{H}_i(\mathcal{A}_\beta)$ and evaluate the SINR function on each such point. In addition, this brute-force scheme cannot detect all null-cells in the zone $\mathcal{H}_i(\mathcal{A}_\beta)$ as there might be infinitesimally small null-cells that are not captured by the sampling. However, thanks to the minimum principle, and in particular the NFH property, every null-cell must contain an interfering station and hence the interfering stations can be a useful starting point for detecting the null-cells in the map. In addition, as shown next, the minimum principle also implies that the null-cells cannot be too small, hence there exists a fixed sampling precision that guarantees the detection of *every* null-cell in $\mathcal{H}_i(\mathcal{A}_\beta)$. Let $\kappa = \min\{\mathrm{dist}(s_i, s_j) \mid i > 1\}$ denote the minimum distance between any two stations in the network.

Claim 4.21: For every $\beta > 0$, the area of every null-cells is $\Omega(\kappa^2 \cdot \widehat{\beta}/(\psi \cdot n))$ where $\widehat{\beta} = \min\{\beta, 1/\beta\}$.

Let $\Psi_{\max} = \max_i \psi_i$ be the maximal transmission energy and $\widetilde{\delta} = \kappa/4\sqrt{\beta \cdot \Psi_{\max} \cdot n}$. In the full version, we present Alg. ApproxHoles that approximates the number of null-cells in the reception region of station s_i and show the following.

Theorem 4.22: There exists an algorithm that given an approximation parameter $\epsilon \in (0,1)$, a network $\mathcal{A}_\beta$ and a target station s_i, returns a number $X(i,\epsilon)$ such that $\tau(\beta, i) \le X(i,\epsilon) \le \tau(\beta_\epsilon, i)$ by using $O(n^3/(\widetilde{\delta} \cdot \epsilon) + n \cdot |\Phi(\mathcal{H}_i(\mathcal{A}))|/(\epsilon \cdot \widetilde{\delta}) + \widetilde{\Delta} \cdot n)$ arithmetic operations, where $|\Phi(\mathcal{H}_i(\mathcal{A}))|$ is the perimeter of $\Phi(\mathcal{H}_i(\mathcal{A}))$.

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REFERENCES

[1] C. Avin, Y. Emek, E. Kantor, Z. Lotker, D. Peleg, and L. Roditty. SINR diagrams: Convexity and its applications in wireless networks. *J. ACM*, 59(4), 2012.

[2] S. Borbash and A. Ephremides. Wireless link scheduling with power control and SINR constraints. IEEE Tr. Inf. Theory, 2006.

[3] M. Chiang, P. Hande, T. Lan, and C.W. Tan. Power control in wireless cellular networks. *Foundations and Trends in Networking*, 2(4:381–533, 2008.

[4] R. L. Cruz and A. V. Santhanam. Optimal routing, link scheduling and power control in multi-hop wireless networks. In *Proc. INFOCOM*, 2003.

[5] H.T. Friis. A note on a simple transmission formula. In *Proc. IRE*, pages 34(5):254–256, 1946.

[6] M.M. Halldorsson and P. Mitra. Wireless capacity with oblivious power in general metrics. In *Proc. SODA*, 2011.

[7] E. Kantor, Z. Lotker, M. Parter, and D. Peleg. The topology of wireless communication. In *Proc. STOC*, 2011.

[8] E. Kantor, Z. Lotker, M. Parter, and D. Peleg. The topology of wireless communication. In *http://arxiv.org/pdf/1103.4566v2.pdf*, 2011.

[9] W. Kaplan. *Maximum Principles in Differential Equations*. Addison-Wesley, 1984.

[10] T. Kesselheim. A constant-factor approximation for wireless capacity maximization with power control in the SINR model. In *Proc. SODA*, 2011.

[11] T.-H. Lee, J.-C. Lin, and Y. T. Su. Downlink power control algorithms for cellular radio systems. *IEEE Tr. Vehic. Technol*, 44:89–94, 1995.

[12] T. Moscibroda and R. Wattenhofer. The complexity of connectivity in wireless networks. In *Proc. INFOCOM*, 2006.

[13] J.T. Schwartz, M. Sharir, and J.E. Hopcroft. *Planning, geometry, and complexity of robot motion.* Intellect Books, 1987.

[14] V. Vazirani. The notion of a rational convex program, and an algorithm for the arrow-debreu nash bargaining game. *J. ACM*, 59(2), 2012.

[15] K. Wang, C. Chiasserini, R. Rao, and J. Proakis. A joint solution to scheduling and power control for multicasting in wireless ad hoc networks. J. Applied Signal Processing, 2, 2005.

[16] J. Zander. Performance of optimum transmitter power control in cellular radiosystems. *IEEE Tr. Vehic. Technol.*, 41:57–62, 1992.