THE JOURNAL OF SYMBOLIC LOGIC
Volume 37, Number 2, June 1972

GENERALIZED INTERPOLATION THEOREMS

STEPHEN J. GARLAND[1]

Chang [1], [2] has proved the following generalization of the Craig interpolation theorem [3]: For any first-order formulas φ and ψ with free first- and second-order variables among $\nu_1, \cdots, \nu_n$, R and $\nu_1, \cdots, \nu_n$, S respectively, and for any sequence $Q_1, \cdots, Q_n$ of quantifiers such that Q_i is universal whenever ν_i is a second-order variable, if

$$\models Q_1\nu_1 \cdots Q_n\nu_n(\exists R\varphi \rightarrow \forall S\psi),$$

then there is a first-order formula θ with free variables among $\nu_1, \cdots, \nu_n$ such that

$$\models Q_1\nu_1 \cdots Q_n\nu_n[(\exists R\varphi \rightarrow \theta) \wedge (\theta \rightarrow \forall S\psi)].$$

(Note that the Craig interpolation theorem is the special case of Chang's theorem in which $Q_1, \cdots, Q_n$ are all universal quantifiers.) Chang also raised the question [2, Remark (k)] as to whether the Lopez-Escobar interpolation theorem [6] for the infinitary language $L_{\omega_1\omega}$ possesses a similar generalization. In this paper, we show that the answer to Chang's question is affirmative and, moreover, that several interpolation theorems for applied second-order languages for number theory also possess such generalizations.

Maehara and Takeuti [7] have established independently proof-theoretic interpolation theorems for first-order logic and $L_{\omega_1\omega}$ which have as corollaries both Chang's theorem and its analog for $L_{\omega_1\omega}$. Our proofs are quite different from theirs and rely on model-theoretic techniques stemming from the analogy between the theory of definability in $L_{\omega_1\omega}$ and the theory of Borel and analytic sets of real numbers, rather than the technique of cut-elimination. In addition, we are able to show that slight variants of the seemingly more general results of Takeuti and Maehara can in fact be derived from Chang's theorem and its $L_{\omega_1\omega}$ analog without the use of proof theory.

§1. Preliminaries: languages. $L_{\omega\omega}$ is the usual pure first-order relational language with identity, while $L_{\omega_1\omega}$ is the infinitary language obtained from $L_{\omega\omega}$ by allowing conjunctions and disjunctions of countably infinite sets of formulas. $L^2_{\omega\omega}$ and $L^2_{\omega_1\omega}$ are the second-order languages obtained from $L_{\omega\omega}$ and $L_{\omega_1\omega}$ by the addition of relation variables and quantifiers. *Sentences* in any of these languages have no free first- or second-order variables, but may have countably many individual or finitary relation parameters.

The class of *existential* $L^2_{\omega\omega}$ (or $L^2_{\omega_1\omega}$) formulas is the smallest class containing all $L_{\omega\omega}$ (all $L_{\omega_1\omega}$) formulas and closed under (infinitary) conjunction, (infinitary)

Received May 19, 1971.

[1] Research was supported by a Faculty Fellowship from Dartmouth College while the author was on leave at UCLA during 1970–1971. Preparation of this paper was partially supported by NSF grant GP-22937.

disjunction, quantification over individual variables, and existential quantification over relation variables. Similarly, the class of *universal* $L^2_{\omega\omega}$ (or $L^2_{\omega_1\omega}$) formulas is the smallest class containing all $L_{\omega\omega}$ (all $L_{\omega_1\omega}$) formulas and closed under (infinitary) conjunction, (infinitary) disjunction, quantification over individual variables, and universal quantification over relation variables. Note that the negation of an existential formula is logically equivalent to a universal formula.

The following two lemmas prove to be convenient in the study of classes of structures definable in $L^2_{\omega_1\omega}$, since they enable one to concentrate on definable classes of countably infinite structures.

LEMMA 1. *For any $L^2_{\omega_1\omega}$ sentence φ there is an $L_{\omega_1\omega}$ sentence φ_1 with no infinite models and an $L^2_{\omega_1\omega}$ sentence φ_2 with no finite model such that φ is logically equivalent to $\varphi_1 \vee \varphi_2$.*

PROOF. For each n we construct an $L_{\omega_1\omega}$ sentence φ^n which has as models those models of φ with cardinality at most n by replacing the m-ary relation quantifiers in φ by finite disjunctions or conjunctions over the set of all m-ary relations over n ($= \{i : i < n\}$). In more detail, let $v_0, \cdots, v_{n-1}$ be individual variables not occurring in φ and let φ^n be the $L_{\omega_1\omega}$ sentence

$$\exists v_0 \cdots \exists v_{n-1}\Big[\forall v_n \bigvee_{i<n} v_n \approx v_i \wedge \tilde{\varphi}^n\Big],$$

where $\tilde{\varphi}^n$ is obtained from φ by replacing each m-ary relation quantified subformula $\exists R\psi$ by a disjunction $\bigvee_{R \subseteq {}^m n}\psi^R$, where ${}^m n$ is the set of all functions from m into n and ψ^R is obtained from ψ by replacing each atomic subformula $Rv_0 \cdots v_{m-1}$ by

$$\bigvee_{f \in R} \bigwedge_{i<m} v_i \approx v_{f(i)}.$$

Then let φ_1 be $\bigvee_{n<\omega}\varphi^n$ and let φ_2 be

$$\varphi \wedge \bigwedge_{n<\omega} \neg\exists v_0 \cdots \exists v_{n-1} \forall v_n \bigvee_{i<n} v_n \approx v_i.$$

(A simpler proof exists if φ contains only finitely many parameters since then there are only countably many finite models of φ.)

LEMMA 2 (DOWNWARD LÖWENHEIM-SKOLEM THEOREM FOR EXISTENTIAL $L^2_{\omega_1\omega}$ SENTENCES). *An existential $L^2_{\omega_1\omega}$ sentence has a model if and only if it has a countable model.*

PROOF. A straightforward argument by induction shows that for any existential $L^2_{\omega_1\omega}$ sentence φ, the $L_{\omega_1\omega}$ sentence ψ which is obtained from φ by replacing all bound relation variables in φ by distinct new relation parameters and "erasing" all relation quantifiers possesses the following property: any model of ψ is also a model of φ and any model of φ has an expansion which is a model of ψ. Hence the lemma follows from the downward Löwenheim-Skolem theorem for $L_{\omega_1\omega}$.

The precise manner in which Lemmas 1 and 2 enable one to reduce questions about $L^2_{\omega_1\omega}$-definable classes of structures to questions about $L^2_{\omega_1\omega}$-definable classes of countably infinite structures will be apparent in the proofs of Theorems 6 and 7. As preparation for these theorems, we review some known properties of the last-mentioned classes of structures in the remainder of this section and then establish some further properties in §2 from which the interpolation theorems for $L^2_{\omega_1\omega}$ will follow in §3.

For any sentence φ, any countably infinite model of φ is isomorphic to a model of φ with universe the set ω of natural numbers (in fact, there may be many such models with universe ω, all of which are isomorphic). Hence if one considers φ as a sentence in an applied language in which the universe of any structure is ω, then φ defines a subset of a space T which is a countable cartesian product of copies of the *type* 0 space ω and of the *type* 1 spaces $P^n\omega$, where for any x, Px is the power set $\{y: y \subseteq x\}$ of x; T itself is said to be a type 0 space if all its factor spaces are type 0 and a type 1 space otherwise. For example, if φ involves a binary relation parameter R and two individual parameters a_1 and a_2, then φ defines the subset

$$\{\langle R, a_1, a_2\rangle : \langle \omega, R, a_1, a_2\rangle \models \varphi\}$$

of the space $(P^2\omega) \times \omega \times \omega$. Furthermore, any such subset of a space T is *invariant* under the permutations of T induced by permutations of ω since any permutation of ω is an isomorphism between models of φ.

All such spaces T possess natural topologies, namely the product topologies induced by the discrete topologies on $\{0, 1\}$ and ω. With this topology, all type 1 spaces are homeomorphic, as are all type 0 spaces. Furthermore, any atomic $L_{\omega_1\omega}$ sentence defines an invariant subset of T which is both open and closed, while an arbitrary $L_{\omega_1\omega}$ sentence defines an invariant Borel set, i.e. an invariant set in the smallest class of sets containing the open sets which is closed under complementation (relative to T) and countable union. By a remark of Ryll-Nardzewski (cf. Scott [9] or Lopez-Escobar [6, Theorem 5.3]), it follows from the interpolation theorem for $L_{\omega_1\omega}$ that any invariant Borel subset of T is definable by an $L_{\omega_1\omega}$ sentence. To see this, suppose that A is an invariant Borel subset of T. Then A is definable by an $L_{\omega_1\omega}$ sentence $\varphi(R)$ which involves an additional binary relation parameter R to be interpreted as the usual ordering of ω. Let $\psi(R)$ be the $L_{\omega_1\omega}$ sentence which asserts that R is an ω-type ordering of the universe. Since A is invariant, the following sentence is valid:

$$[\varphi(R) \wedge \psi(R)] \rightarrow [\psi(S) \rightarrow \varphi(S)].$$

By the interpolation theorem for $L_{\omega_1\omega}$, there is an $L_{\omega_1\omega}$ interpolant θ which involves neither R nor S; this sentence θ defines A.

The connection between $L_{\omega_1\omega}$-definable sets and Borel sets extends to a connection between existential $L^2_{\omega_1\omega}$-definable sets and analytic sets. A subset of T is *analytic* if and only if it is the projection of a Borel subset of $T \times T'$ for some space T'. Since the class of invariant analytic sets is closed under countable unions and intersections as well as projections, any existential $L^2_{\omega_1\omega}$-sentence defines an invariant analytic set. Conversely, if A is an invariant analytic subset of T, then A is definable by an existential $L^2_{\omega_1\omega}$-sentence $\varphi(R)$ which involves an additional binary relation parameter R as above, so that A is also definable by the existential $L^2_{\omega_1\omega}$-sentence $\exists R(\varphi(R) \wedge \psi(R))$ with no additional parameters (R is now a relation variable) where ψ also is as above.

§2. **Interpolation theorems for analytic sets.** In this section we establish several interpolation theorems for analytic sets together with analogs of these theorems for invariant analytic sets. As noted by various authors, some of these theorems follow from known interpolation theorems for $L_{\omega_1\omega}$; for example, the Luzin separation theorem (see below) and its corollary for invariant sets is a consequence of the Lopez-Escobar interpolation theorem while the Novikov generalized separation theorem and its corollary follow from a result of Makkai [8] which we establish later as Theorem 7. Our purpose in proving these results first, without reference to $L_{\omega_1\omega}$, is to show that the reverse situation is also possible, i.e. that interpolation theorems for $L_{\omega_1\omega}$ can be derived from results in descriptive set theory about analytic sets together with the fact proved in §1 that every invariant Borel set is definable by a sentence of $L_{\omega_1\omega}$. If another proof of this fact could be given which does not use the Lopez-Escobar interpolation theorem, then one could obtain all the interpolation theorems for $L_{\omega_1\omega}$ mentioned in this paper (including Lopez-Escobar's) as corollaries of theorems in descriptive set theory; at worst, one need prove only one interpolation theorem for $L_{\omega_1\omega}$ in order to derive the others using the tools of descriptive set theory.

We begin by deriving corollaries for invariant sets from two classical theorems of descriptive set theory.

THE LUZIN SEPARATION THEOREM [5, p. 485]. *For any disjoint analytic sets A and B, there is a Borel set C such that $A \subseteq C \subseteq \sim B$.*

COROLLARY 3. *For any disjoint analytic sets A and B, if either A or B is invariant then there is an invariant Borel set C such that $A \subseteq C \subseteq \sim B$.*

PROOF. By symmetry, it suffices to consider the case in which B is invariant. By the Luzin Separation Theorem there is a Borel set C_0 such that $A \subseteq C_0 \subseteq \sim B$. The invariant closure

$$C_0^* = \{x\colon \text{for some permutation } \pi \text{ of } \omega,\ \pi x \in C_0\}$$

of C_0 is an analytic set and also a subset of $\sim B$ since B is invariant. Hence we may iterate the above procedure, first with C_0^* in place of A, in order to obtain a sequence $C_0, C_1, \cdots$ of Borel sets and their invariant closures $C_0^*, C_1^*, \cdots$ such that

$$A \subseteq C_0 \subseteq C_0^* \subseteq C_1 \subseteq C_1^* \subseteq \cdots \subseteq \sim B.$$

Let $C = \bigcup\{C_n\colon n \geq 0\}$. Then C is an invariant Borel set and $A \subseteq C \subseteq \sim B$.

THE NOVIKOV GENERALIZED SEPARATION THEOREM [5, p. 510]. *For any collection $\{A_n\colon n \in \omega\}$ of analytic sets with empty intersection, there is a collection $\{B_n\colon n \in \omega\}$ of Borel sets with empty intersection such that $A_n \subseteq B_n$ for all n.*

COROLLARY 4. *For any collection $\{A_n\colon n \in \omega\}$ of invariant analytic sets with empty intersection, there is a collection $\{B_n\colon n \in \omega\}$ of invariant Borel sets with empty intersection such that $A_n \subseteq B_n$ for all n.*

PROOF. By the Novikov Generalized Separation Theorem there is a collection $\{C_n\colon n \in \omega\}$ of Borel sets with empty intersection such that $A_n \subseteq C_n$ for all n. By Corollary 3, there is, for each n, an invariant Borel set B_n such that $A_n \subseteq B_n \subseteq C_n$. Hence $\{B_n\colon n \in \omega\}$ has empty intersection.

The following theorem is the descriptive set-theoretic analog of Chang's theorem for $L_{\omega\omega}$ from which we shall derive the $L_{\omega_1\omega}$ analog of Chang's theorem in §3. In order to keep the notation simple, we write "$\vec{x}$" for "$\langle x_1, \cdots, x_n\rangle$" and use the quantifiers "$\exists$" and "$\forall$" from the formal language $L_{\omega\omega}$ informally with their usual meanings.

THEOREM 5. *Let $T = T_1 \times \cdots \times T_n$ be a topological space and let $Q_1, \cdots, Q_n$ be a sequence of quantifiers such that Q_i is universal whenever T_i is a type 1 space. For any analytic subsets A and B of T, if*

$$Q_1x_1 \cdots Q_nx_n(\vec{x} \in A \Rightarrow \vec{x} \notin B)$$

then there is a Borel subset C of T such that

$$Q_1x_1 \cdots Q_nx_n(\vec{x} \in A \Rightarrow \vec{x} \in C), \tag{1}$$

$$\forall x_1 \cdots \forall x_n(\vec{x} \in C \Rightarrow \vec{x} \notin B). \tag{2}$$

Furthermore, if either A or B is invariant, then there is an invariant such C.

PROOF. We proceed by induction on the number of existential quantifiers among $Q_1, \cdots, Q_n$. If all the Q_i are universal, then the conclusion follows from the Luzin Separation Theorem and Corollary 3. For the induction step, suppose that

$$\forall x_1 \cdots \forall x_{m-1} \exists x_m Q_{m+1}x_{m+1} \cdots Q_nx_n(\vec{x} \in A \Rightarrow \vec{x} \notin B) \tag{a}$$

and let P be

$$\{\langle x_1, \cdots, x_m\rangle : Q_{m+1}x_{m+1} \cdots Q_nx_n(\vec{x} \in A \Rightarrow \vec{x} \notin B)\}.$$

Then P is a co-analytic set (i.e., the complement of an analytic set) since Q_i is universal whenever T_i is a type 1 space. Furthermore,

$$\forall x_1 \cdots \forall x_{m-1} \exists x_m \langle x_1, \cdots, x_m\rangle \in P. \tag{b}$$

Hence by the Novikov-Kondô uniformization theorem [10, p. 188] there is a co-analytic set P^* such that

$$P^* \subseteq P \tag{c}$$

and

$$\forall x_1 \cdots \forall x_{m-1} \exists ! x_m \langle x_1, \cdots, x_m\rangle \in P^*. \tag{d}$$

Since x_m is existentially quantified in (a), it is an element of ${}^k\omega$ for some k by the hypothesis on Q_m. Furthermore, by (d),

$$\langle x_1, \cdots, x_m\rangle \in P^* \Leftrightarrow \forall u \in {}^k\omega(\langle x_1, \cdots, x_{m-1}, u\rangle \in P^* \Rightarrow u = x_m),$$

so that P^* is also an analytic set. By (c),

$$\forall x_1 \cdots \forall x_m[\langle x_1, \cdots, x_m\rangle \in P^* \Rightarrow Q_{m+1}x_{m+1} \cdots Q_nx_n(\vec{x} \in A \Rightarrow \vec{x} \notin B)],$$

and hence

$$\forall x_1 \cdots \forall x_m Q_{m+1}x_{m+1} \cdots Q_nx_n(\langle x_1, \cdots, x_m\rangle \in P^* \;\&\; \vec{x} \in A \Rightarrow \vec{x} \notin B). \tag{e}$$

Since $\{\vec{x}: \langle x_1, \cdots, x_m\rangle \in P^* \;\&\; \vec{x} \in A\}$ is analytic and the quantifier prefix in (e) has one fewer existential quantifier than the prefix in (a), the induction hypothesis may be applied to obtain a Borel set C (which is invariant if B is, and also if A is as can be seen by transposing the condition on P^* to the other side of the conditional in (e)) such that

$$\forall x_1 \cdots \forall x_m Q_{m+1}x_{m+1} \cdots Q_nx_n(\langle x_1, \cdots, x_m\rangle \in P^* \;\&\; \vec{x} \in A \Rightarrow \vec{x} \in C), \tag{f}$$

$$\forall x_1 \cdots \forall x_n(\vec{x} \in C \Rightarrow \vec{x} \notin B). \tag{g}$$

By (d) and (f),

$$\forall x_1 \cdots \forall x_{m-1} \exists x_m Q_{m+1}x_{m+1} \cdots Q_nx_n(\vec{x} \in A \Rightarrow \vec{x} \in C). \tag{h}$$

The desired conclusion is given by (g) and (h).

An alternative conclusion to Theorem 5 is the existence of a possibly noninvariant Borel set C satisfying (1) and

$$(2') \qquad Q_1x_1 \cdots Q_nx_n(\vec{x} \in C \Rightarrow \vec{x} \notin B)$$

which is a cylinder over the universally quantified x_i's. To see this, notice that instead of introducing P^* in the proof one could have used a Borel Skolem function to eliminate the existentially quantified variable x_m. For example, if

$$\forall x_1 \exists x_2 \forall x_3(\langle x_1, x_2, x_3\rangle \in A \Rightarrow \langle x_1, x_2, x_3\rangle \notin B)$$

and x_2 ranges over ω then there is a Borel function F (i.e., the graph of F is a Borel set) such that

$$\forall x_1 \forall x_3(\langle x_1, F(x_1), x_3\rangle \in A \Rightarrow \langle x_1, F(x_1), x_3\rangle \notin B)$$

(in fact, F is precisely that function with graph P^* as defined above). Since

$$\{\langle x_1, x_3\rangle : \langle x_1, F(x_1), x_3\rangle \in A\} \quad \text{and} \quad \{\langle x_1, x_3\rangle : \langle x_1, F(x_1), x_3\rangle \in B\}$$

are analytic sets, there is a Borel subset C of $T_1 \times T_3$ such that

$$\forall x_1 \forall x_3(\langle x_1, F(x_1), x_3\rangle \in A \Rightarrow \langle x_1, x_3\rangle \in C)$$

and

$$\forall x_1 \forall x_3(\langle x_1, x_3\rangle \in C \Rightarrow \langle x_1, F(x_1), x_3\rangle \notin B).$$

Hence

$$\text{(i)} \qquad \begin{aligned} \forall x_1 \exists x_2 \forall x_3[(\langle x_1, x_2, x_3\rangle \in A &\Rightarrow \langle x_1, x_3\rangle \in C) \\ \& (\langle x_1, x_3\rangle \in C &\Rightarrow \langle x_1, x_2, x_3\rangle \notin B)]. \end{aligned}$$

It may not be possible to find an invariant such C, however, even if both A and B are invariant. For example, if

$$A = \sim B = \{\langle x_1, x_2, x_3\rangle : x_2 = x_3 \in \omega\},$$

then no invariant C satisfies (i).

§3. Interpolation theorems for $L_{\omega_1\omega}$. The following $L_{\omega_1\omega}$ analog of Chang's theorem for $L_{\omega\omega}$ follows easily from the results of the previous sections.

THEOREM 6. *For any existential $L^2_{\omega_1\omega}$ formulas φ and ψ with free first- and second-order variables among $v_1, \cdots, v_n$, and for any sequence $Q_1, \cdots, Q_n$ of quantifiers such that Q_i is universal whenever v_i is a second-order variable, if*

$$\models Q_1v_1 \cdots Q_nv_n(\varphi \rightarrow \neg\psi),$$

then there is an $L_{\omega_1\omega}$ formula θ with free variables among $v_1, \cdots, v_n$ such that

$$(1) \qquad \models Q_1v_1 \cdots Q_nv_n(\varphi \rightarrow \theta),$$

$$(2) \qquad \models \forall v_1 \cdots \forall v_n(\theta \rightarrow \neg\psi).$$

PROOF. Since φ and ψ are existential, they define invariant analytic subsets A and B of an appropriate topological space T (we regard the free variables of φ and ψ as parameters). By Theorem 5 there is an invariant Borel set C "separating" A and B. Let θ' be an $L_{\omega_1\omega}$ sentence which defines C as a subset of T and which furthermore has no finite models; then let θ be $\theta' \vee \varphi_1$, where φ_1 is the $L_{\omega_1\omega}$ sentence corresponding to φ by Lemma 1 (here we reconvert the parameters introduced above to variables). Since the sentences in (1) and (2) are equivalent to universal $L^2_{\omega_1\omega}$ sentences, (1) and (2) hold by the choice of θ and Lemmas 1 and 2.

Although Chang stated the conclusion of his theorem as

$$\models Q_1 v_1 \cdots Q_n v_n [(\varphi \to \theta) \wedge (\theta \to \neg\psi)]$$

to accentuate the similarity with Craig's theorem, his proof actually establishes the stronger conclusions expressed by (1) and (2). The apparent lack of symmetry in the quantifier strings which appear in (1) and (2) disappears when one realizes that they can be interchanged by applying the theorem to $\psi \to \neg\varphi$ rather than to $\varphi \to \neg\psi$.

As mentioned before, Maehara and Takeuti have given proof-theoretic demonstrations of interpolation theorems for $L_{\omega\omega}$ and $L_{\omega_1\omega}$ which include Chang's theorem and Theorem 6 as special cases. We show now that their theorems can be derived using Chang's theorem and Craig's theorem for $L_{\omega_1\omega}$, and using Theorem 6 and the following inconsistency theorem due to Makkai for $L_{\omega_1\omega}$. Since the proof is so natural, we deduce Makkai's theorem from Corollary 4; Makkai [8] gives a different proof and observes that Corollary 4 follows from his theorem.

THEOREM 7 (MAKKAI [8]). *For any inconsistent collection $\{\varphi_n : n \in \omega\}$ of existential $L^2_{\omega_1\omega}$ sentences there is an inconsistent set $\{\psi_n : n \in \omega\}$ of $L_{\omega_1\omega}$ sentences such that $\models \varphi_n \to \psi_n$ for any n.*

PROOF. Each sentence φ_n defines an invariant analytic subset A_n of a topological space T. Since $\{\varphi_n : n \in \omega\}$ is inconsistent, $\{A_n : n \in \omega\}$ has empty intersection. Hence by Corollary 4 there is a collection $\{B_n : n \in \omega\}$ of invariant Borel sets with empty intersection such that $A_n \subseteq B_n$ for all n. For each n, let ψ'_n be an $L_{\omega_1\omega}$ sentence with no finite models which defines B_n as a subset of T, and let ψ_n be $\psi'_n \vee \varphi_{n,1}$, where $\varphi_{n,1}$ is the $L_{\omega_1\omega}$ sentence obtained from φ_n by Lemma 1. Then $\{\psi_n : n \in \omega\}$ is inconsistent and $\models \varphi_n \to \psi_n$ for all n by Lemmas 1 and 2 since $\varphi_n \to \psi_n$ is equivalent to a universal sentence.

In order to state the result of Maehara and Takeuti, we define, for any $L^2_{\omega\omega}$ (or $L^2_{\omega_1\omega}$) formula φ, the class of *universal $L^2_{\omega\omega}$ (or $L^2_{\omega_1\omega}$) formulas relative to* φ to be the smallest class containing φ and all $L_{\omega\omega}$ (or $L_{\omega_1\omega}$) formulas which is closed under (infinitary) conjunction, (infinitary) disjunction, quantification over individual variables, and universal quantification over relation variables. Note that for any φ and ψ there is a natural one-one correspondence between the class of universal formulas relative to φ and the class of universal formulas relative to ψ. In fact, to any universal formula $\Phi(\varphi)$ relative to φ corresponds the universal formula $\Phi(\psi)$ relative to ψ obtained by "substituting" ψ for φ in $\Phi(\varphi)$. Note also that if $\Phi(\varphi)$ is universal relative to a universal formula φ, then $\Phi(\varphi)$ is also universal.

LEMMA 8. *For any formulas φ, ψ, and any formula $\Phi(\varphi)$ universal relative to φ, if $\models \varphi \to \psi$ then $\models \Phi(\varphi) \to \Phi(\psi)$.*

PROOF. The proof is a straightforward induction on the construction of $\Phi(\varphi)$, since φ has only positive occurrences in $\Phi(\varphi)$.

THEOREM 9 (MAEHARA AND TAKEUTI [7]). *For any universal $L^2_{\omega\omega}$ (or $L^2_{\omega_1\omega}$) formula φ and any $L^2_{\omega\omega}$ (or $L^2_{\omega_1\omega}$) sentence $\Phi(\varphi)$ universal relative to φ, if $\models \Phi(\varphi)$ then there is an $L_{\omega\omega}$ (an $L_{\omega_1\omega}$) formula θ with free variables among those of φ such that $\models \Phi(\theta)$ and $\models \theta \to \varphi$.*

PROOF. We show by induction on the construction of $\Phi(\varphi)$ that for any $L_{\omega\omega}$ (or $L_{\omega_1\omega}$) formula ψ, if $\models \psi \to \Phi(\varphi)$, then there is an $L_{\omega\omega}$ (an $L_{\omega_1\omega}$) formula θ with free variables among those of φ such that $\models \psi \to \Phi(\theta)$ and $\models \theta \to \varphi$. This clearly is sufficient. In order to treat $L_{\omega\omega}$ and $L_{\omega_1\omega}$ simultaneously, we let w be either 2 or ω in the following seven cases.

Case 1. φ does not occur in $\Phi(\varphi)$. Let θ be any logically false formula. Then $\Phi(\varphi)$ equals $\Phi(\theta)$ and $\models \theta \to \varphi$.

Case 2. $\Phi(\varphi)$ is φ. Then $\models \psi \to \varphi$ and we may let θ be any interpolating formula with free variables among those of φ.

Case 3. $\Phi(\varphi)$ is $\bigwedge_{n<w}\Phi_n(\varphi)$. Then for any $n < w$, $\models \psi \to \Phi_n(\varphi)$. Hence, by the induction hypothesis, there are $L_{\omega\omega}$ (or $L_{\omega_1\omega}$) formulas θ_n such that $\models \psi \to \Phi_n(\theta_n)$ and $\models \theta_n \to \varphi$. Let θ be $\bigvee_{n<w}\theta_n$. Since $\models \theta_n \to \theta$ for any n, $\models \Phi_n(\theta_n) \to \Phi_n(\theta)$ by Lemma 8. Hence $\models \psi \to \Phi(\theta)$ and $\models \theta \to \varphi$.

Case 4. $\Phi(\varphi)$ is $\bigvee_{n<w}\Phi_n(\varphi)$. By Craig's theorem for $L_{\omega\omega}$ or by Theorem 7 for $L_{\omega_1\omega}$ there is a collection $\{\psi_n : n < w\}$ of $L_{\omega\omega}$ (or $L_{\omega_1\omega}$) formulas such that $\models \bigvee_{n<w}\psi_n$ and $\models \psi_n \to (\psi \to \Phi_n(\varphi))$ for any $n < w$. By the induction hypothesis, there are $L_{\omega\omega}$ (or $L_{\omega_1\omega}$) formulas θ_n for each $n < w$ such that $\models \psi_n \to (\psi \to \Phi_n(\theta_n))$ and $\models \theta_n \to \varphi$. Let θ be $\bigvee_{n<w}\theta_n$. As in Case 3, $\models \psi \to \Phi(\theta)$ and $\models \theta \to \varphi$.

Case 5. $\Phi(\varphi)$ is $\forall v\Phi'(\varphi)$. We may assume that the individual variable v does not have a free occurrence in ψ. Then $\models \psi \to \Phi'(\varphi)$ and by the induction hypothesis there is an $L_{\omega\omega}$ (or an $L_{\omega_1\omega}$) formula θ such that $\models \psi \to \Phi'(\theta)$ and $\models \theta \to \varphi$. Hence $\models \psi \to \forall v\Phi'(\theta)$.

Case 6. $\Phi(\varphi)$ is $\exists v\Phi'(\varphi)$. Again we may assume that the individual variable v does not have a free occurrence in ψ. Then $\models \exists v[\psi \to \Phi'(\varphi)]$. By Chang's theorem for $L_{\omega\omega}$ or by Theorem 6 for $L_{\omega_1\omega}$ there is an $L_{\omega\omega}$ (or an $L_{\omega_1\omega}$) formula ψ' such that $\models \exists v(\psi \to \psi')$ and $\models \forall v(\psi' \to \Phi'(\varphi))$. By the induction hypothesis, there is an $L_{\omega\omega}$ (or an $L_{\omega_1\omega}$) formula θ such that $\models \forall v(\psi' \to \Phi'(\theta))$ and $\models \theta \to \varphi$. Then $\models \exists v\psi' \to \exists v\Phi'(\theta)$ and $\models \psi \to \exists v\psi'$ since v has no free occurrence in ψ. Thus $\models \psi \to \exists v\Phi'(\theta)$.

Case 7. $\Phi(\varphi)$ is $\forall R\Phi'(\varphi)$. The proof is the same as in Case 5.

Having derived Theorem 9 from Chang's theorem and Theorem 6, we note that these two theorems are indeed special cases of Theorem 9: if $\Phi(\psi)$ is the sentence $Q_1\nu_1 \cdots Q_n\nu_n(\neg\varphi \to \psi)$ for some universal formulas φ and ψ and $\models \Phi(\psi)$, then there is a first-order formula θ with free variables among $\nu_1, \cdots, \nu_n$ such that $\models Q_1\nu_1 \cdots Q_n\nu_n(\neg\varphi \to \theta)$ and $\models \theta \to \psi$. Since the proof of Theorem 9 requires only the special case of Chang's theorem or Theorem 6 in which the quantifier prefix consists of universal quantifiers followed by a single existential quantifier, one could separate the proofs of these theorems into an examination of the special prefix followed by an argument by induction. (An even more special case of Chang's theorem is due to Kueker [4].)

From the $L^2_{\omega_1\omega}$ version of Theorem 9 follow generalized separation theorems for analytic sets and for invariant analytic sets in the usual manner. It should also be noted that these theorems, as well as the results of §2, have effective versions in which codes for the Borel sets asserted to exist can be computed effectively from codes for the given analytic sets.

REFERENCES

[1] C. C. Chang, *A generalization of the Craig interpolation theorem*, *Notices of the American Mathematical Society*, vol. 15 (1968), p. 934.

[2] ———, *Two interpolation theorems*, ***Symposia Mathematica***, vol. V, Academic Press, New York, 1970, pp. 5–19.

[3] W. Craig, *Three uses of the Herbrand-Gentzen theorem in relating model theory and proof theory*, this Journal, vol. 22 (1957), pp. 269–285.

[4] D. W. Kueker, *Generalized interpolation and definability*, ***Annals of Mathematical Logic***, vol. 1 (1970), pp. 423–468.

[5] K. Kuratowski, ***Topology***, Academic Press, New York, 1966.

[6] E. G. K. Lopez-Escobar, *An interpolation theorem for denumerably long sentences*, ***Fundamenta Mathematicae***, vol. 57 (1965), pp. 253–272.

[7] S. Maehara and G. Takeuti, *Two interpolation theorems for a* Π_1^1 *predicate calculus*, this Journal, vol. 36 (1971), pp. 262–270.

[8] M. Makkai, *An application of the method of Smullyan to logics on admissible sets*, ***Bulletin de l'Académie Polonaise des Sciences. Série des Sciences Mathématiques, Astronomiques et Physiques***, vol. 17 (1969), pp. 341–346.

[9] D. Scott, *Logic with denumerably long formulas and finite strings of quantifiers*, ***The theory of models***, North-Holland, Amsterdam, 1965, pp. 329–341.

[10] J. R. Shoenfield, ***Mathematical logic***, Addison-Wesley, Reading, Mass., 1967.

DARTMOUTH COLLEGE
HANOVER, NEW HAMPSHIRE 03755

UNIVERSITY OF CALIFORNIA
LOS ANGELES, CALIFORNIA 90024