Indescribability and the Continuum. by Kenneth Kunen
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sets C_ξ is a model of ZF plus the generalized continuum hypothesis and the axiom of choice, thus proving that these propositions are consistent with ZF (Gödel VI 112).

The construction of Cohen's models is more complicated. The sets C_ξ are defined as above, but the values of ξ are just the ordinals in a given countable transitive model A of ZF. Instead of a single sequence a, there is a sequence $a(p)$ for each p in a certain topological space, and the Baire category theorem is used to select a "generic" point p for which the sets C_ξ corresponding to $a = a(p)$ form a model $N(p)$ of ZF. (This use of topology to define the notion of a generic point is due to Ryll-Nardzewski and Takeuti.) For example if b and c are sets in A and $N(p)$ is to contain a new element p of b^c then the topological space is b^c; for each mapping π of a finite subset of c into b, the set of extensions of π in b^c is a basic open set. The sequences $a(p)$ are chosen so that p will belong to $N(p)$. In this way models are constructed in which the axiom of choice and the continuum hypothesis are false (Cohen), or in which they are true but no well-ordering of the real numbers is explicitly definable (Lévy). Easton's theorem on consistent alternatives to the generalized continuum hypothesis is stated but not proved.

The author's careful and detailed proofs give the book a deceptively complex appearance; it is actually easy to read and well motivated at each stage.

There is no attempt to survey alternative approaches to forcing, and the student who wishes to learn Boolean-valued models or finitary versions of the proofs must consult other sources.

Errata are few, and usually occur in proofs, where they will easily be repaired by the reader. On page 2, an axiom asserting the existence of the empty set is needed; page 4, the definition of $\{x\}$ should be changed so that it is the universal class if x is not a set; page 8, it should be noted that '$f \in V^A$' is not to be taken literally; page 101, line 9, '[12]' should be '[13]'.

PERRY SMITH

KENNETH KUNEN. *Indescribability and the continuum.* ***Axiomatic set theory,*** Proceedings of symposia in pure mathematics, vol. 13 part 1, American Mathematical Society, Providence, Rhode Island, 1971, pp. 199–203.

Call a cardinal κ $\bigvee^1_n$ *characterizable* (or $\bigwedge^1_n$ *characterizable*) iff there is an $\bigvee^1_n$ sentence (or an $\bigwedge^1_n$ sentence) of second-order logic which is true in a structure iff the universe of that structure has cardinality κ. It is easy to see that the power of the continuum is $\bigvee^1_2$ characterizable (one way is to assert the existence of relations which make the universe into a complete ordered field). Kunen shows that the continuum does not possess any "natural" $\bigwedge^1_2$ characterizations by showing that it is consistent relative to Zermelo-Fraenkel set theory that $2^{\aleph_0} < \aleph_{\omega_1}$ and that any $\bigwedge^1_2$ sentence true of $2^{\aleph_0}$ is also true of some smaller cardinal. This result answers a question of the reviewer, who showed that for certain α (e.g., for α a Δ^1_2 ordinal), $\aleph_\alpha$ is both $\bigvee^1_2$ and $\bigwedge^1_2$ characterizable (***Axiomatic set theory,*** Proceedings of symposia in pure mathematics, vol. 13 part 2, pp. 127–146).

Kunen's method of proof employs a Boolean-valued extension of the universe of set theory to blow the continuum up to a cardinal which reflects $\bigwedge^1_2$ sentences. Kunen also establishes the relative consistency of certain stronger reflection properties for the continuum.

STEPHEN J. GARLAND

R. BJÖRN JENSEN. *The fine structure of the constructible hierarchy.* ***Annals of mathematical logic,*** vol. 4 no. 3 (1972), pp. 229–308.

In work beginning in the late 1960s, the author has formulated, and proved to hold in L, combinatorial principles whose power lies in the inductive constructions which they allow. These principles strengthen in at least two ways the type of inductive construction of length κ which is normally carried out under the assumption $2^{<\kappa} = \kappa$. In the present paper the author presents the fundamentals of this deep, original, and important work.

Let $\Diamond_\kappa$ mean that for every stationary $T \subseteq \kappa$ there is a sequence $\langle S_\alpha : \alpha \in T\rangle$ such that $S_\alpha \subseteq \alpha$ and for every $X \subseteq \kappa$, $\{\alpha : X \cap \alpha = S_\alpha\}$ is stationary in κ. The author shows that if $V = L$ then $\Diamond_\kappa$ holds for every regular $\kappa > \omega$. The main applications of $\Diamond_\kappa$ are when κ is a successor cardinal; if $\kappa = \lambda^+$, $2^{<\lambda} = \lambda$, and λ is regular, and $\Diamond_\kappa$ then there is a ($< \lambda$-closed) κ-Souslin tree. The proof of $\Diamond_\kappa$ in L uses only a small fragment of the author's theory; the following result requires the main machinery. *Theorem.* If $V = L$ and κ is a regular uncountable