MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.003: Signals and Systems—Fall 2002

Tutorial for the week of September 30th - October 4th

Alex's Office Hours
Monday 5pm - 7pm
Thursday 4pm - 6pm

Important Due Dates:

- Lab 1 is due Wednesday, October 2.
- Problem Set 4 is due Friday, October 4.
- Quiz 1 will be held on Thursday, October 17, from 7:30 pm to 9:30 pm in Walker Memorial. Memorial.
- Quiz 1 review sessions are tentatively scheduled for Wednesday, October 9, from 7-9pm, and Thursday, October 10, from 7-9pm. Watch your e-mail for schedule changes and room announcements.
- If you have a conflict with another exam, please contact Prof. Boning.

Materials covered in the past weeks

1. Fourier Series of DT Periodic Signals (most recent)
2. Fourier Series of CT Periodic Signals (most recent)
3. Eigenfunctions of LTI systems (most recent)
4. Frequency response (most recent)
5. DT Convolution
6. CT Convolution

Fourier Series of DT Periodic Signals

Question: What's all this about Fourier Series? What do they *mean*?

Answer: Fourier series are just a way of representing a periodic signal in terms of more elementary periodic signals.

We do this all the time in other areas.

1. Whole numbers - Represent a general number by their decimal coefficients:

$$32618 = (3 \times 10^4) + (2 \times 10^3) + (6 \times 10^2) + (1 \times 10^1) + (8 \times 10^0)$$

2. Vectors - Represent any vector by its orthogonal components:

$$\vec{v} = \sum_i a_i \vec{x}_i$$

   where each $|\vec{x}_i| = 1$, and $\vec{x}_i \cdot \vec{x}_j = 0$ unless $i = j$.

3. Currency - Represent an amount by more elementary denominations:

$$\begin{aligned} \$32.41 &= 1 \times (\text{20 dollar bill}) + 1 \times (\text{10 dollar bill}) + 2 \times (\text{1 dollar bill}) \\ &\quad +1 \times (\text{quarter}) + 1 \times (\text{dime}) + 1 \times (\text{nickel}) + 1 \times (\text{penny}) \end{aligned}$$

4. Discrete Periodic Signals - Represent it as the sum of more basic periodic signals:

$$x[n] = \sum_{k=0}^{N-1} a_k x_k[n]$$

   where $x[n] = x[n+N]$, and $x_k[n] = \mathrm{e}^{-jk\left(\frac{2\pi}{N}\right)n}$.

We choose to use exponentials as our "basis" signals, $x_k[n]$, because it turns out that they have nice properties that we will make use of later. The main difficulty with Discrete Fourier series is that there are many sets of $\{x_k[n]\}$ that you can choose to use depending on how you interpret your period N.

Examples

1. Represent $x[n] = 2$ in terms of its Fourier series coefficients.

2. Represent $x[n] = 1 + \sin\left(\frac{2\pi}{5}n\right) + (-1)^n$ in terms of its Fourier series coefficients.

Work Space

Fourier Series of CT Periodic Signals

What about continuous time? Things are largely the same, except that there is no finite "basis". So if a signal is with period T,

$$x(t) = x(t+T) \qquad \text{for all } t.$$

Then it can be represented as

$$x(t) \;=\; \sum_{k=-\infty}^{\infty} a_k \mathrm{e}^{jk\left(\frac{2\pi}{T}\right)t}.$$

where we can interchangeably use $\omega_0 = \frac{2\pi}{T}$. How can we determine the coefficients, a_k? Use the orthogonality of the complex exponentials:

$$\int_T \mathrm{e}^{j(k-n)\omega_0 t}\mathrm{d}t = \begin{cases} T, & k = n \\ 0, & k \neq n \end{cases}.$$

Multiplying $x(t)$ by $\mathrm{e}^{-jn\omega_0 t}$ gives:

$$\begin{aligned}\int_T x(t)\mathrm{e}^{-jn\omega_0 t}\mathrm{d}t &= \int_T \sum_{-\infty}^{\infty} a_k \mathrm{e}^{j(k-n)\omega_0 t}\mathrm{d}t = \sum_{-\infty}^{\infty} \int_T a_k \mathrm{e}^{j(k-n)\omega_0 t}\mathrm{d}t \\ &= a_n T\end{aligned}$$

So for CT, we get:

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} a_k \mathrm{e}^{jk\omega_0 t} && \text{Synthesis equation} \\ a_k &= \frac{1}{T}\int_T x(t)\mathrm{e}^{-jk\omega_0 t}\mathrm{d}t && \text{Analysis equation.}\end{aligned}$$

and for DT, we can follow a similar process to get:

$$\begin{aligned} x[n] &= \sum_{k=<N>} a_k \mathrm{e}^{k\left(\frac{2\pi}{N}\right)n} && \text{Synthesis equation} \\ a_k &= \frac{1}{N}\sum_{n=<N>} x[n]\mathrm{e}^{-n\left(\frac{2\pi}{N}\right)k} && \text{Analysis equation.}\end{aligned}$$

Eigenfunctions of LTI Systems

Complex exponentials are eigenfunctions of LTI systems for both DT and CT.

$$x(t) = \mathrm{e}^{j\omega_0 t} \longrightarrow \boxed{\begin{array}{c} h(t) \\ H(j\omega) \end{array}} \longrightarrow y(t) = H(j\omega_0)\mathrm{e}^{j\omega_0 t}$$

$$x[n] = \alpha^n \longrightarrow \boxed{\begin{array}{c} h[n] \\ H(\mathrm{e}^{j\Omega}) \end{array}} \longrightarrow y[n] = H(\alpha)\alpha^n$$

This is especially nice if the inputs happen to be periodic signals, because then we can represent them as a linear combination of exponentials. The output can then be represented in terms of the sum of the responses to each of these exponentials.

For example, if $x(t)$ is presented to a system with a frequency response $H(j\omega)$, and $x(t)$ is periodic, then

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \mathrm{e}^{jk\omega_0 t}$$

and the output, $y(t)$ will be given by

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) \mathrm{e}^{jk\omega_0 t}$$

That is, $y(t)$ will have Fourier coefficients $a_k H(jk\omega_0)$.

This tells us *meaning* of the frequency response, $H(j\omega)$:

> The frequency response, $H(j\omega)$, essentially "magnifies" the frequency component of $x(t)$ at ω_0 by the amount $H(j\omega_0)$.

Examples

1. Suppose we have a CT system whose frequency response is as follows:

$$H(j\omega) = \begin{cases} 1 & \text{if} \quad |\omega| = \omega_0 \\ \frac{1}{2} & \text{if} \quad |\omega| = 2\omega_0 \\ 0 & \text{otherwise} \end{cases}$$

Find the output $y(t)$ if the input signal is

$$x(t) = \cos(\omega_0 t) + \cos(2\omega_0 t) + \cos(3\omega_0 t)$$

Work Page

Convolution Review

DT convolution

1. Evaluate the following DT convolution sum:

$$y[n] = (2^n u[n]) * (3^{-n} u[n])$$

CT convolution

1. The impulse response of a system is $h(t) = \mathrm{e}^{-at} u(t)$, $a > 0$. Calculate the step response $s(t)$ of the system.

2. Evaluate the following CT convolution integral:

$$y(t) = [u(10 + t) - u(3 + t)] * \mathrm{e}^{-t}.$$

Work Space

Extra Problems

1. $x[n]$ is a periodic discrete signal with the following properties:

   (a) It is real.

   (b) It is even.

   (c) Its fundamental period is $N = 5$.

   (d) $\sum_{n=-2}^{n=2} x[n] = 0$.

   (e) Its Fourier coefficient $a_6 = 1$.

   (f) $\frac{1}{5}\sum_{n=0}^{n=4} |x[n]|^2 = 2$.

   Find an expression that describe $x[n]$ completely.

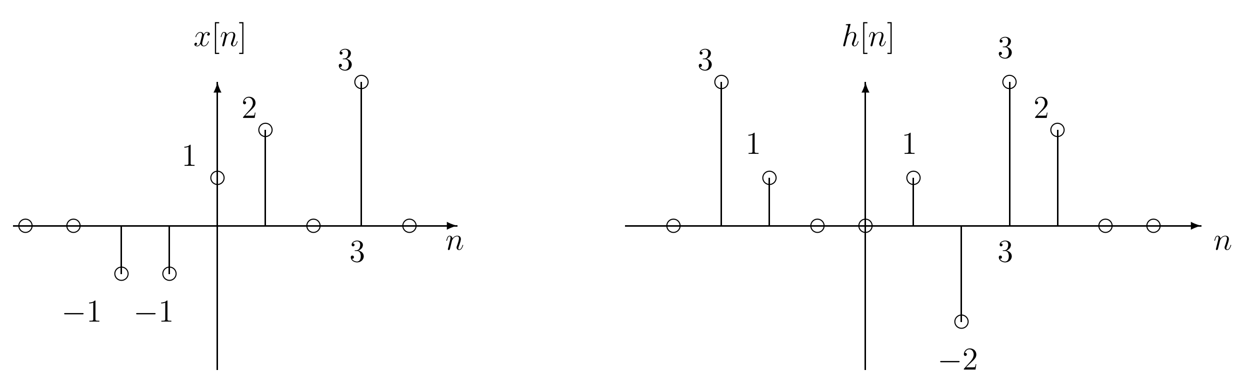
2. Suppose $x[n]$ is a discrete time periodic signal with period 4, such that

$$x[n] = \begin{cases} 2, & n = 0 \\ 1, & n = 1 \\ 0, & n = 2 \\ 1, & n = 3 \end{cases}.$$

   (a) Let $x_a[n] = x[n]$, $x_b[n] = (x[n])^2$, and $x_c[n] = x[-n]$. Find the discrete time Fourier series of each of these signals.

   (b) In addition, if each of these signals is the input to an LTI system, with frequency response $H(e^{j\Omega}) = 3e^{-2j\Omega}$, find the Fourier series of the output signal in each case.

3. Compute the convolution $y[n] = x[n] * h[n]$:



Work Space