MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.003: Signals and Systems—Fall 2002

Tutorial for the week of October 7th - October 11th

Office Hours
Monday 3-5pm
By Appointment

Important Due Dates:

- NO PROBLEM SET THIS WEEK!!!
- Quiz 1 will be held on Thursday, October 17, from 7:30 pm to 9:30 pm in Walker Memorial. Memorial.
- Quiz 1 review sessions are tentatively scheduled for Wednesday, October 9, from 7-9pm, and Thursday, October 10, from 7-9pm. Watch your e-mail for schedule changes and room announcements.

Today: Part I (Review)

1. CT and DT Fourier Series
2. Frequency Response
3. Convolution
4. System properties

Today: Part II (Looking forward)

1. Continuous Time Fourier Transform and related concepts (New stuff)

Continuous Time Fourier Series

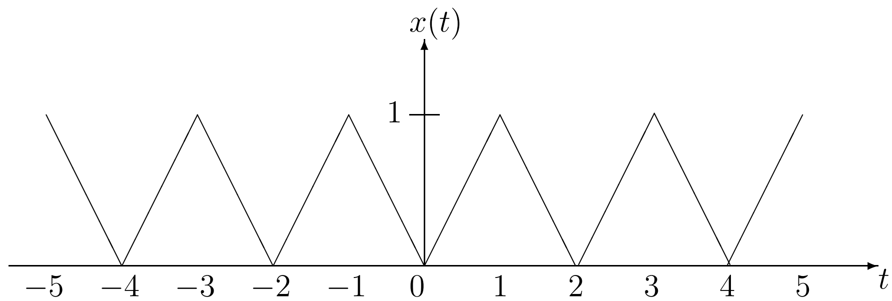
Recall:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t}$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_o t}dt \text{ , where } \omega_o = \frac{2\pi}{T}.$$

[**Example**]

1. The periodic waveform, $x(t)$ shown below, has Fourier coefficients a_k.



   (a) Find a_0.

   (b) Find the Fourier series for $\frac{\mathrm{d}x(t)}{\mathrm{d}t}$.

   (c) Use the result of (b) and properties of the continuous-time Fourier series to help determine the Fourier coefficients of $x(t)$.

Discrete Time Fourier Series

Recall:

$$\begin{aligned} x[n] &= \sum_{k=\langle N \rangle} a_k e^{jk\omega_o n} \\ a_k &= \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jn\omega_o k} \text{ , where } \omega_o = \frac{2\pi}{N}. \end{aligned}$$

[Example]

1. Suppose we are given $x[n]$, $y[n]$, and $z[n]$ as follows

$$\begin{aligned} x[n] &= 1 + \cos(\frac{7\pi}{3} n) \\ y[n] &= \sin(\frac{\pi}{3} n + \frac{\pi}{4}) \\ z[n] &= x[n] y[n] \\ g[n] &= x[n] * y[n] \end{aligned}$$

Find the Fourier series coefficients for $z[n]$ and $g[n]$.

Frequency Response

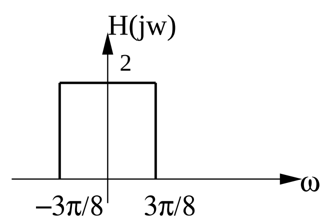
Recall: If we put an input $x(t) = e^{j\omega t}$ into an LTI system we get out:

$$\begin{aligned} y(t) &= \int_{\infty}^{\infty} h(\tau)x(t-\tau)d\tau \\ &= \int_{\infty}^{\infty} h(\tau)e^{j\omega t}e^{-j\omega\tau}d\tau \\ &= e^{j\omega t}\int_{\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau \\ &= e^{j\omega t}H(j\omega) \end{aligned}$$

where $H(j\omega) = \int_{\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau$. We call $H(j\omega)$ is the *frequency response* of the system.

[**Example**]

1. An LTI system has a frequency response depicted below:



   If the input, $x(t) = \cos(\frac{\pi}{2}t) + \cos(\frac{\pi}{4}t)$, what will $y(t)$, the output be?

2. Plot the magnitude of the following frequency response:

$$H(e^{j\omega}) = 1 - e^{-j\omega}$$

Work Space

Convolution and System Properties

Recall: When we put a signal $x(t)$ (or $x[n]$) into a system characterized by an impulse response of $h(t)$ (or $h[n]$), then the output will be $y(t)$ (or $y[n]$), which is given by

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

$$y[n] = \sum_{m=-\infty}^{\infty} h[m]x[n-m]$$

Use Flip and Shift method to determine bounds on t or n (ie, regions where $h[n]$ and $x[n]$ overlap). Then go through the motions of actually calculating the output.

For system properties, there are

- Linearity - Check whether the output of $ax_1(t) + bx_2(t)$ is equal to $ay_1(t) + by_2(t)$, where $y_1(t)$ and $y_2(t)$ are the results of inputting $x_1(t)$ and $x_2(t)$, respectively.
- Time Invariance - Check whether the output of $x_1(t - t_0)$ is equal to $y_1(t - t_0)$, where $y_1(t)$ is the result of inputting $x_1(t)$.
- Stability - Check whether the impulse response, $h(t)$ (or $h[n]$), is absolutely integrable (summable)

  $$\int_{-\infty}^{\infty} |h(t)|dt < \infty$$

  If it **is** absolutely integrable, then the system is stable.
- Causality - Check whether the impulse response, , $h(t)$ (or $h[n]$), is non-zero for negative values of t (or n).

  $$h(t) \neq 0 \quad \text{for} \quad t < 0$$

  If the above condition is true, then the system is **non**-causal.

[**Example**]

1. Suppose a system takes an input $x(t)$ and produces the output

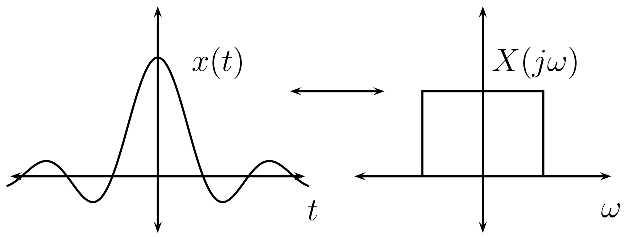
   $$y(t) = x(-t+5)$$

   Is the system time invariant?

Continous Time Fourier Transform

Today, we'll focus on exploring properties of the CTFT via the sinc-box transform pair.

A function of the form $x(t) = \frac{\sin(W\pi t)}{\pi t}$ is known as a *sinc*. This function has zero crossings whenever $\sin(W\pi t)$ has zero crossings, except at $t = 0$. Therefore, it has zero crossings at



The Fourier Transform of any $x(t)$ of the form $\frac{\sin(At)}{Bt}$ will be a box in frequency. That is, it will be of the form

$$X(j\omega) = \begin{cases} C & \text{for } |\omega| \leq D \\ 0 & \text{otherwise} \end{cases}$$

Likewise, a box in time, will have a Fourier Transform which is a sinc in frequency. All you have to really remember is

$$\text{box} \longleftrightarrow \text{sinc}$$

Everything else works out by using CTFT properties

[Example]

Find the Fourier Transform of the function $x(t) = \frac{\sin(At)}{Bt}$.

Solution

First, let's characterize this function.

- Zero crossings - We know that $x(t)$ has zero crossings whenever At is a multiple of π. Therefore, the zero crossings occur at $\frac{k\pi}{A}$ for all non-zero integers k.
- $x(0)$ - We can do this by limits:

$$\begin{aligned} \lim_{t \to 0} x(t) &= \frac{\lim_{t\to 0} \sin(At)}{\lim_{t\to 0} Bt} \\ &= \frac{\lim_{t\to 0} A\cos(At)}{\lim_{t\to 0} B} \\ \Rightarrow x(0) &= \frac{A}{B} \end{aligned}$$

Next, let's figure out what it's Fourier Transform is going to be. We're going to make use of the following important properties:

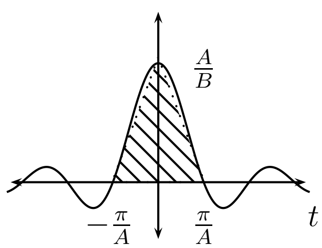
$$\begin{aligned} X(j0) &= \int_{-\infty}^{\infty} x(t)dt \\ \int_{-\infty}^{\infty} X(j\omega)d\omega &= 2\pi x(0) \end{aligned}$$

Since we know that $X(j\omega)$ is of the form

$$X(j\omega) = \begin{cases} C & \text{for } \ |\omega| \leq D \\ 0 & \text{otherwise} \end{cases}$$

we just need to determine C and D.

- C - We know that $C = X(j0)$. This is just the area under $x(t)$. Note that the area under the sinc is just given by the area of the inscribed triangle



$$\int_{-\infty}^{\infty} x(t)dt = \frac{1}{2} \times \left(\frac{A}{B}\right) \times \left(\frac{2\pi}{A}\right)$$

  Therefore, $C = X(j0) = \frac{\pi}{B}$.

- D - This is the width of the box. Since we know that the area under $X(j\omega)$ is equal to $C \times (2D)$, we can use

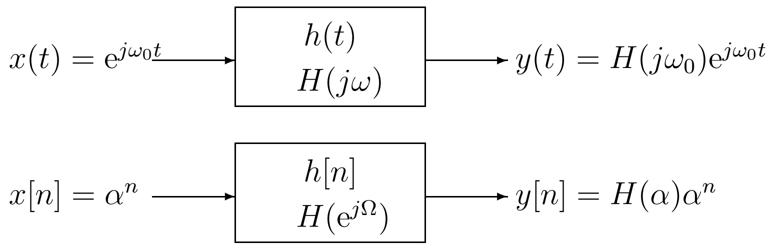
$$\begin{aligned} \int_{-\infty}^{\infty} X(j\omega)d\omega &= 2\pi x(0) \\ C \times 2D &= 2\pi \times \frac{A}{B} \\ \frac{\pi}{B} \times D &= \pi \times \frac{A}{B} \\ \Rightarrow D &= A \end{aligned}$$

Therefore,

$$X(j\omega) = \begin{cases} \frac{\pi}{B} & \text{for } \ |\omega| \leq A \\ 0 & \text{otherwise} \end{cases}$$

Eigenfunctions of LTI Systems

Complex exponentials are eigenfunctions of LTI systems for both DT and CT.



This is especially nice if the inputs happen to be periodic signals, because then we can represent them as a linear combination of exponentials. The output can then be represented in terms of the sum of the responses to each of these exponentials.

For example, if $x(t)$ is presented to a system with a frequency response $H(j\omega)$, and $x(t)$ is periodic, then

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

and the output, $y(t)$ will be given by

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{jk\omega_0 t}$$

That is, $y(t)$ will have Fourier coefficients $a_k H(jk\omega_0)$.

This tells us *meaning* of the frequency response, $H(j\omega)$:

> The frequency response, $H(j\omega)$, essentially "magnifies" the frequency component of $x(t)$ at ω_0 by the amount $H(j\omega_0)$.

Examples

1. Suppose we have a CT system whose frequency response is as follows:

$$H(j\omega) = \begin{cases} 1 & \text{if } |\omega| = \omega_0 \\ \frac{1}{2} & \text{if } |\omega| = 2\omega_0 \\ 0 & \text{otherwise} \end{cases}$$

   Find the output $y(t)$ if the input signal is

$$x(t) = \cos(\omega_0 t) + \cos(2\omega_0 t) + \cos(3\omega_0 t)$$

Work Page

Convolution Review

DT convolution

1. Evaluate the following DT convolution sum:

$$y[n] = (2^n u[n]) * (3^{-n} u[n])$$

CT convolution

1. The impulse response of a system is $h(t) = \mathrm{e}^{-at} u(t)$, $a > 0$. Calculate the step response $s(t)$ of the system.

2. Evaluate the following CT convolution integral:

$$y(t) = [u(10 + t) - u(3 + t)] * \mathrm{e}^{-t}.$$

Work Space

Extra Problems

1. $x[n]$ is a periodic discrete signal with the following properties:

   (a) It is real.

   (b) It is even.

   (c) Its fundamental period is $N = 5$.

   (d) $\sum_{n=-2}^{n=2} x[n] = 0$.

   (e) Its Fourier coefficient $a_6 = 1$.

   (f) $\frac{1}{5}\sum_{n=0}^{n=4} |x[n]|^2 = 2$.

   Find an expression that describe $x[n]$ completely.

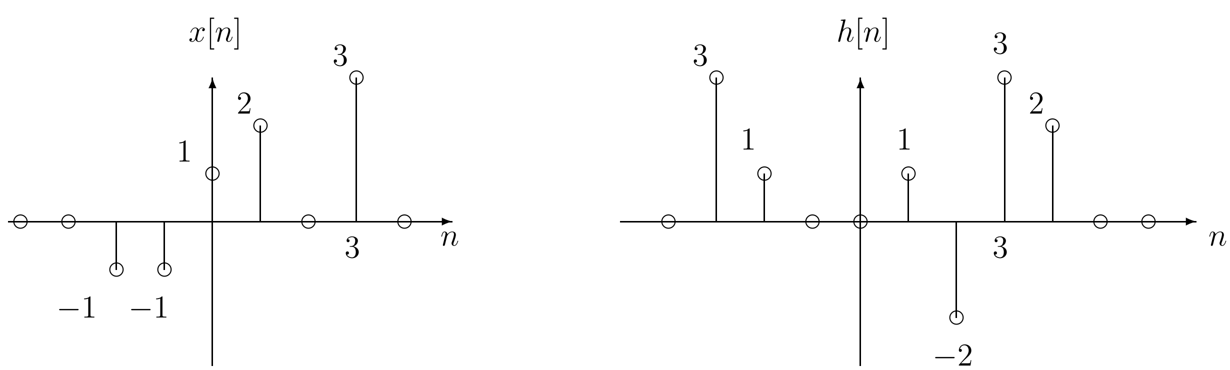
2. Suppose $x[n]$ is a discrete time periodic signal with period 4, such that

$$x[n] = \begin{cases} 2, & n = 0 \\ 1, & n = 1 \\ 0, & n = 2 \\ 1, & n = 3 \end{cases}.$$

   (a) Let $x_a[n] = x[n]$, $x_b[n] = (x[n])^2$, and $x_c[n] = x[-n]$. Find the discrete time Fourier series of each of these signals.

   (b) In addition, if each of these signals is the input to an LTI system, with frequency response $H(e^{j\Omega}) = 3e^{-2j\Omega}$, find the Fourier series of the output signal in each case.

3. Compute the convolution $y[n] = x[n] * h[n]$:



Work Space