MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.003: Signals and Systems—Fall 2002

Tutorial for the week of October 28st - November 1st

Alex's Office Hours
Monday 3-5pm
Tuesday, 6-8pm

Important Due Dates:

- Problem Set 6 due on Wednesday.
- Lab 2 due Friday, November 8th (No late submissions accepted).

Today

1. Time and Frequency Characterizations of signals and systems (Chapter 6)
2. Sampling (Chapter 7)

Visualizing System Characterizations

Up until now, we have learned how to characterize LTI systems in terms of:

- $h(t)$, which is the *impulse response*. This is computed as the output of the system when the input is $\delta(t)$).

$$\text{e.}g. \quad h(t) = e^{-at}u(t)$$

- $H(j\omega)$, which is the *frequency response*. This is computed as the Fourier transform of the impulse response.

$$\text{e.}g. \quad H(j\omega) = \frac{1}{1 + aj\omega}$$

- LCCDE. This is the Linear Constant Coefficient Differential Equation that relates $y(t)$ to $x(t)$.

$$\text{e.}g. \quad \frac{dy}{dt} + ay(t) = x(t)$$

Frequency Characterization - Bode Plots

For continuous time, in particular, Bode plots are a useful way of visualizing a system's frequency response. Bode plots stem from two simple observations

1. $H(j\omega) = |H(j\omega)|e^{\angle H(j\omega)}$
2. $\log(AB) = \log(A) + \log(B)$

Therefore, if we focus on plotting $\log |H(j\omega)|$ separately from $\angle H(j\omega)$, then we can visualize complicated systems as the sum of simpler systems which we know how to plot.

$$\begin{aligned}
H(j\omega) &= H_1(j\omega)H_2(j\omega) \\
&= |H_1(j\omega)||H_2(j\omega)|e^{\angle H_1(j\omega)+\angle H_2(j\omega)} \\
\Rightarrow \log |H(j\omega)| &= \log |H_1(j\omega)| + \log |H_2(j\omega)| \\
\Rightarrow \angle H(j\omega) &= \angle H_1(j\omega) + \angle H_2(j\omega)
\end{aligned}$$

In particular, we note that $H_1(j\omega) = 1 + aj\omega$ looks like 1 for $a\omega$ very small. It looks like $aj\omega$ for $a\omega$ very large. Therefore,

ω	$H_1(j\omega)$	$\log \lvert H_1(j\omega)\rvert$	$\angle H_1(j\omega)$
$\omega << a^{-1}$	1	0	0
$\omega >> a^{-1}$	$a\omega j$	$\log(\omega) - \log(a^{-}1)$	$\frac{\pi}{2}$

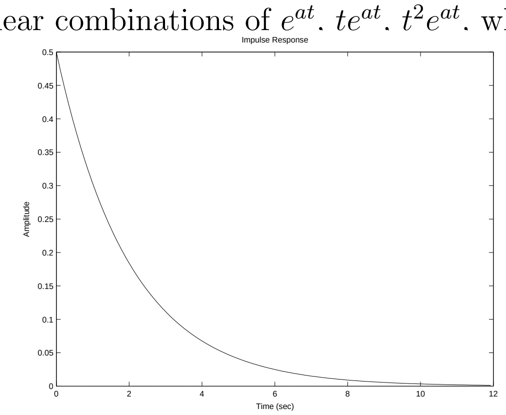
Therefore, if we plot $\log |H(j\omega)|$ versus $\log(\omega)$, then this simple first order system looks asymptotically like a piecewise linear function with a breakpoint at $\omega = a^{-1}$. Knowing this, we can plot the magnitude of any system that is a cascade of first order systems.

$$
\begin{aligned}
H(j\omega &= \frac{(1+a_1 j\omega)\cdots(1+a_N j\omega)}{(1+b_1 j\omega)\cdots(1+b_M j\omega)} \\
\Rightarrow \log|H(j\omega)| &= \sum_{i=1}^{N} \log|(1+a_i j\omega)| - \sum_{k=1}^{M} \log|(1+b_k j\omega)| \\
\Rightarrow \angle H(j\omega) &= \sum_{i=1}^{N} \angle(1+a_i j\omega) - \sum_{k=1}^{M} \angle(1+b_k j\omega)
\end{aligned}
$$

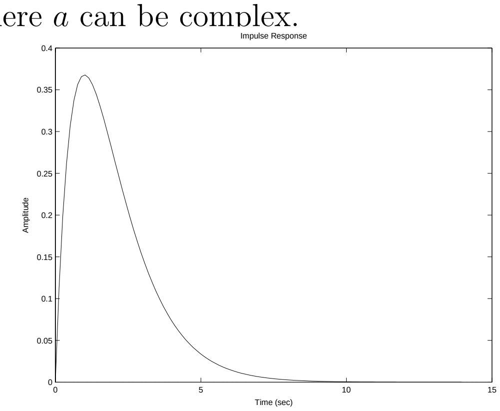
Time Characterization - Impulse and Step Response

Another useful way of visualizing system characteristics is to view the time domain responses by plotting $h(t)$ the response of the system to an impulse, and/or $s(t)$, the response of the system to a unit step.

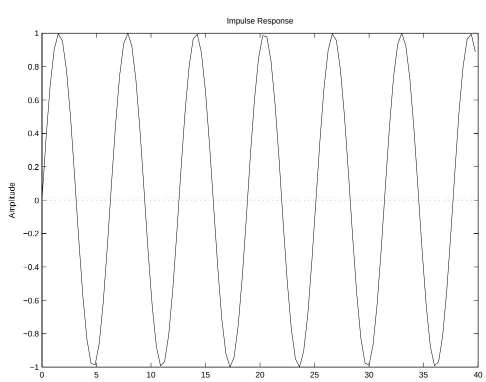
For most of the system examples that we will look at in this course, $h(t)$ will consist of linear combinations of e^{at}, te^{at}, t^2e^{at}, where a can be complex.



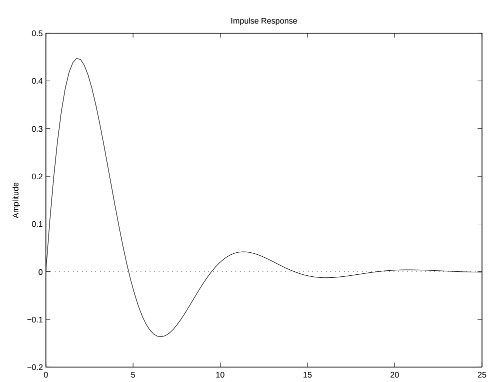
$H(j\omega) = \frac{1}{1+(1/2)j\omega}$
$h(t) = e^{-\frac{1}{2}t}u(t)$



$H(j\omega) = \frac{j\omega}{1+(1/2)j\omega}$
$h(t) = te^{-\frac{1}{2}t}u(t)$



$H(j\omega) = \frac{1}{1+(j\omega)^2}$
$h(t) = \frac{1}{2j}(e^{jt} - e^{-jt})sin(t)u(t)$



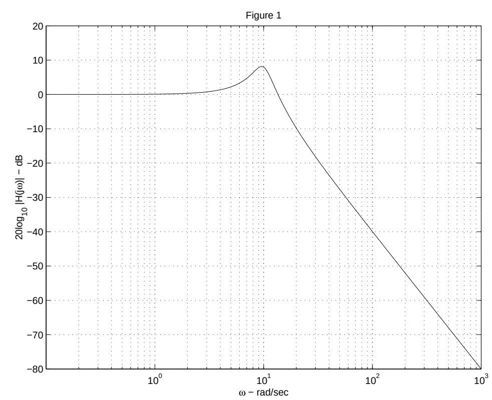
$H(j\omega) = \frac{1}{1+j\omega+2(j\omega)^2}$
$h(t) = e^{-1/4t}(e^{-\sqrt{7}/4jt} + e^{\sqrt{7}/4jt})u(t)$

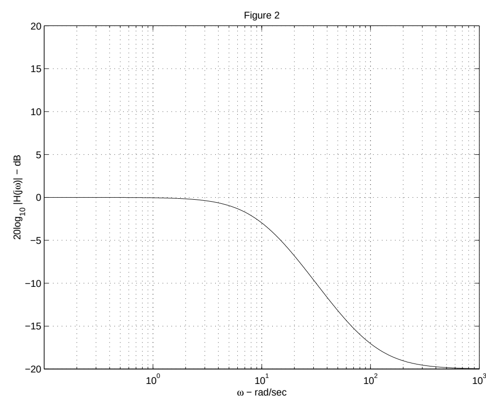
[Example]

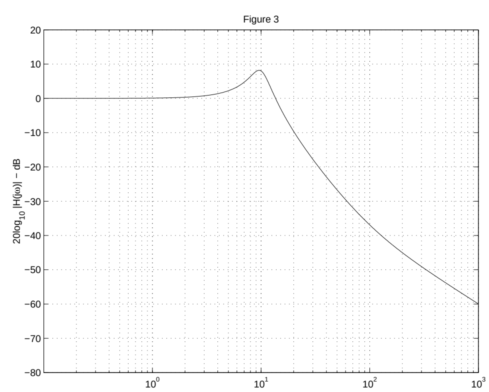
We are given the following frequency responses:

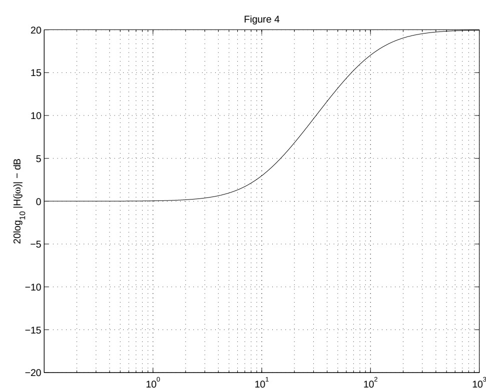
$$H_1(j\omega) = \frac{j\omega+100}{10(j\omega+10)} \qquad H_2(j\omega) = \frac{j\omega-100}{10(j\omega+10)}$$
$$H_3(j\omega) = \frac{10(j\omega+10)}{j\omega+100} \qquad H_4(j\omega) = \frac{100}{(j\omega)^2+4j\omega+100}$$
$$H_5(j\omega) = \frac{j\omega+100}{(j\omega)^2+4j\omega+100} \quad H_6(j\omega) = \frac{j\omega+100}{10j\omega(j\omega+10)}$$

(a) Match each of the frequency response functions above to its corresponding Bode magnitude plot.









(b) Draw the Bode phase plots for $H_1(j\omega)$ and $H_2(j\omega)$.

(c) Does there exist $H_7(j\omega)$ such that $|H_7(j\omega)| = |H_1(j\omega)|$ and $|H_7(j\omega)| = |H_2(j\omega)|$, but $\angle H_7(j\omega)$ is not identical to $\angle H_1(j\omega)$ nor $\angle H_2(j\omega)$? If there exists, find an expression for $H_7(j\omega)$. If not, explain why.

(d) Draw the Bode plots for the frequency responses for which there are not corresponding magnitude plots in (a).

Work Space

Sampling

Sampling is where 6.003 really starts to get interesting. As Victor mentioned on Friday, the majority of signal processing that gets done these days is performed digitally, with computers or microprocessors that can only deal with a discrete amount of data. The idea of *sampling* is to take a continuous time signal, and only look at that signal at certain discrete points in time. The values at those points are known as *samples*. Chapter 7 focuses on some important questions related to sampling.

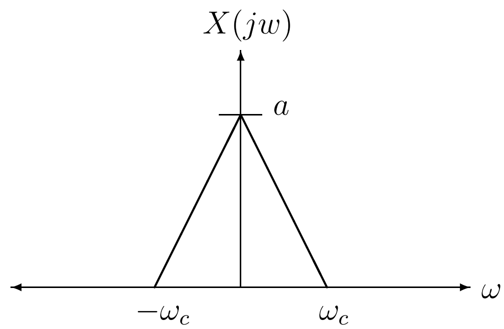
1. When do samples represent an unambiguous representation of a signal?
2. What can we do to the samples of a signal, and what does that do to the signal itself?

The important thing to remember for this chapter is:

PICTURES, PICTURES, PICTURES

It's always easier to understand what's happening by drawing pictures of the Frequency domain representation... Don't get lost in the equations!

Start with a signal, $x(t)$ which has a frequency response, $X(j\omega)$ as below: If we sample



$x(t)$ by taking values only at multiples of T, then we are essentially multiplying $x(t)$ with a periodic impulse train. So our sampled signal is

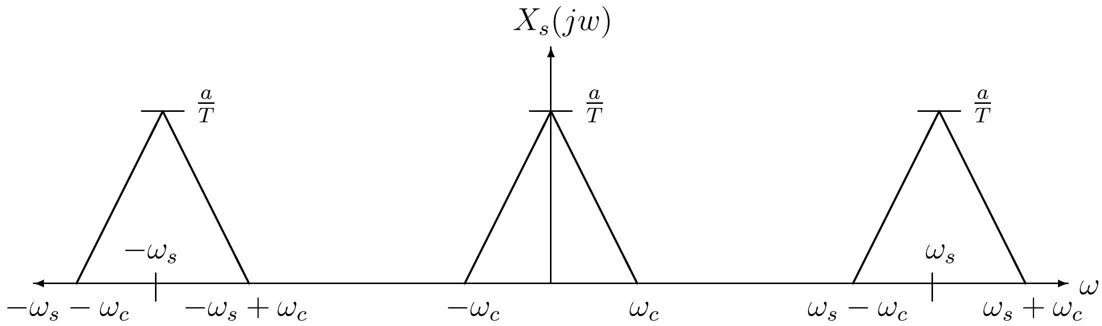
$$x_s(t) = x(t) \cdot \Sigma_{k=-\infty}^{\infty} \delta(t - kT)$$

Since the Fourier transform of an impulse train is another impulse train

$$\Sigma\delta(t - kT) \longleftrightarrow \frac{2\pi}{T}\Sigma\delta(\omega - m\frac{2\pi}{T})$$

Then $X_s(j\omega)$ is the convolution of $X(j\omega)$ with an impulse train. So $X_s(j\omega)$ is just a bunch of $X(j\omega)$'s that are repeated at multiples of $\omega_s = \frac{2\pi}{T}$ and that are scaled by $\frac{1}{T}$. Notice that there is no overlapping of the repeated versions as long as the cut-off frequency, ω_c, is less than half of the sampling frequency, ω_s. If there is some interference or distortion due to $\omega_s \leq 2\omega_c$. This is called *aliasing*.

By the way, aliasing is not always a bad thing!!!



[**Example**]

How can we convert $x(t) = \cos(2\pi t)$ to $y(t) = \cos(\pi t)$?

[**Example**]

Consider $x(t) = \frac{sin(50\pi t)}{\pi t}$.

(a) What is the lowest frequency that $x(t)$ can be sampled at to avoid aliasing?

(b) What about $(x(t))^2$?

[**Example**]

Suppose you have one of those clocks that has a sweeping second hand.

(a) What is the period of this clock?

(b) How fast do we have to sample the clock in order to get the second-hand to seem like it is moving backwards?

Work Space