Stability of Networked Control Systems with Dynamic Controllers in the Feedback Loop

Pedro Henrique de R. Quemel e Assis Santana†, Luis Felipe da Cruz Figueredo†, Eduardo da Silva Alves†, João Yoshiyuki Ishihara, Geovany Araújo Borges and Adolfo Bauchspiess

***Abstract*— Network-induced delays and packet dropouts are relevant issues that can degrade networked control systems' performance and may even lead to instability. This paper concerns the establishment of a stability criterion for networked control systems (NCSs) consisting of an LTI process and a dynamic feedback controller. Conditions for stability are provided in terms of Linear Matrix Inequalities (LMIs), whose solution yields network-induced delay upper bounds. Numerical examples and a simulation ratify the theoretical results.**

***Index Terms*— Networked control systems, stability criterion, dynamic controller, LMI, Lyapunov function**

I. INTRODUCTION

NETWORKED Control Systems refer to a class of control systems whose elements (plant, controller, actuators and sensors) are linked together through a multipurpose shared communication network and the information is exchanged in the form of data packets [1]–[5].

Networked control systems have many advantages compared to the traditional local control architecture, including lower instalation costs, reduced system wiring, greater flexibility and higher reliability [5], [6]. However, the insertion of a shared communication network in the control loop introduces different forms of time-delay uncertainty between sensors, actuators and controllers [5]. The overall delay between sending and eventual decoding at the receiver can be highly variable because both the network access delays and the transmission delays depend on network conditions such as congestion and channel quality [4]. It is well known in control systems that time delays can degrade a system's performance and even cause system instability [2]–[4]. Moreover, there is also a possibility that sensor/control signals may be lost in communication. These problems have stimulated a strong research interest in NCS within the control community [5].

The pioneer contribution is by Halevi *et al.* [7], whose work presents a discrete-time model and analyzes the stability for systems with constant and periodic delays. The *one-channel* feedback NCS, where a continuous-time controller is collocated with the actuator module and only the sensor and the controller modules are linked together through a shared network, is considered in [8]. In Nilsson [1] and references therein, the authors systematically investigate the modelling and analysis problems for NCSs under the assumption that the time-delay from sensor to actuator is less than one sampling period.

The works [9], [10] model the overall NCS by continuous-time delayed differential equations (DDEs). An important advantage of this characterization is that the equations are valid even when the delay exceeds the sampling interval [4]. The problem of stability analysis for continuous-time networked control systems is studied in [11]–[20]. All these works concern the investigation of the delay-dependent stability problem by choosing an appropriate type of Lyapunov function candidate and solving a set of LMIs.

The criteria presented in [19], [20] distinguish themselves from the others in the sense that the obtained network-induced delay bound is less conservative. Nevertheless, [19], [20] only consider proportional state feedback controllers in the stability analysis. This is a significant constraint, since it seriously restricts the controller universe that can be considered. For instance, even ordinary Proportional-Integral (PI) controllers cannot be treated by this criteria. The distinguishing feature in this paper is that it presents a delay-dependent stability criterion for NCSs with dynamic controllers in the feedback loop, being able to deal with a much larger set of control systems. Moreover, when it comes to NCSs with only proportional controllers, our criterion yields network-induced delay bounds equal to the ones presented in [19], [20], which are less conservative than the ones from previous works.

This paper is organized as follows. Section II presents the system description and preliminaries, taking into account network-induced delay and packet loss features. In Section III, a new criterion for stability analysis is proposed, which is obtained by solving a set of LMIs. Numerical examples and a simulation are given in Section IV, followed by the conclusions, which are presented in Section V.

II. SYSTEM DESCRIPTION

Consider the closed-loop NCS shown in Figure 1. It consists of an LTI plant G_p and a controller module G_c, which are connected through a shared network. All the network communication is performed by the *Sender* and the *Receiver* elements, which are responsible for transmiting and acquiring data packets through the network, respectively.

The sensor, the controller, and the actuator modules can either be time-driven or event-driven. Time-driven devices

†These authors contributed equally to this work and should all be considered first authors.

All the authors are with the Robotics, Automation and Computer Vision Group (GRAV), a research group of the Department of Electrical Engineering, University of Brasília, Brazil. e-mails: phrqas@ieee.org, lfc.figueredo@yahoo.com.br, ealves@ieee.org, ishihara@ene.unb.br, gaborges@ene.unb.br, adolfobs@unb.br

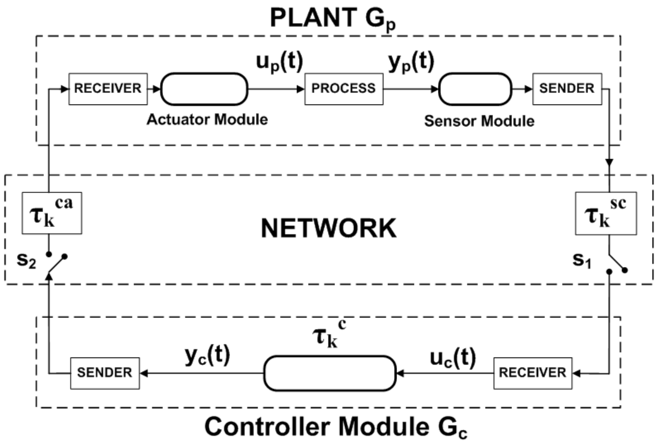

Fig. 1. A NCS with data packet dropout and transmission delays.

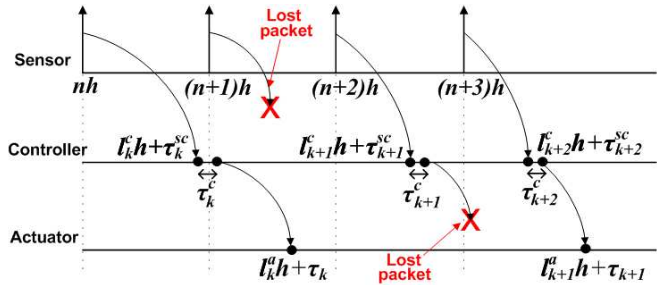

Fig. 2. Time diagram for network-induced delays.

acquire and transmit data periodically, i.e., according to some clock rate, while the event-driven acquisition and transmission are subject to the occurrence of some specific event. Whether a system component should be time-driven or event-driven depends on the control strategy used and on the devices' characteristics. For example, in [19], [21], only the sensors are time-driven, whereas in [22] all devices are time-driven. Moreover, the modelling in [23] considers only event-driven devices.

Throughout this paper, we assume that the sensor module is *clock-driven* and sends its measurements over the network with transmission period h. The *Controller* and *Actuator modules* are *event-driven* and start to process a new packet immediately after its arrival. Single packet transmission is assumed, i.e., all data sent or received over the network is assembled together into one network packet and transmitted at the same time.

Furthermore, the following delays are considered:

- τ_k^{sc}: delay from sensor to controller module for the k-th network packet;
- τ_k^{c}: computation delay for the k-th network packet;
- τ_k^{ca}: delay from controller to actuator module for the k-th network packet;
- τ_k: total delay from sensor to actuator module for the k-th network packet.

The switches S_1 and S_2 in Figure 1 model the possibility of packet loss. In the closed position, packets are able to reach their destinies. Otherwise, they are lost.

A. System's model

At instants nh, where h is the transmission period and $n \in \mathbb{N}^*$, the *Sensor Module* samples data from the plant and sends it over the network to G_c. The controller receives network packets at instants $l_k^c h + \tau_k^{sc}$, where the term $l_k^c h$, $k \in \mathbb{N}^*$, denotes the sample instant of the k-th network packet received by the controller. After computation, the *Controller Module* sends the control signal to the actuator at instant $l_k^c h + \tau_k^{sc} + \tau_k^c$. We denote $l_k^a h + \tau_k$, $k \in \mathbb{N}^*$, the time instant when the actuator receives the k-th control signal. The data flow's time diagram is shown in Figure 2.

Remark 1 If $\{l_1^c, l_2^c, \ldots, l_n^c, \ldots\} = \{1,2,\ldots,n,\ldots\}$, then no packet dropout or disordering occured in the transmission from the sensor to the controller. If the p-th sample was lost, then $\nexists q,\ q \in \mathbb{N}^*$, such that $l_q^c = p$. Packet disordering occurs when one packet reaches its destination later than its successors, i.e., $\exists p,q \in \mathbb{N}^*,\ p > q$, such that $l_q^c > l_p^c$. In this case, the old packet, l_p^c, is dropped and its data discarded. Similarly, if $\{l_1^a, l_2^a, \ldots, l_n^a, \ldots\} = \{l_1^c, l_2^c, \ldots, l_n^c, \ldots\}$, then no packet dropout or disordering occured in the transmission from the controller to the actuator. The packet dropout and disordering for this case have analogous definitions to the previous one.

Similarly to [18], [21], we assume the existence of constants η_j and τ_j, $0 \le \tau_j \le \eta_j$, $j \in \{1,2,3\}$, such that

$$(l_k^c - l_{k-1}^c)h + \tau_k^{sc} \le \ \eta_1, \quad (1)$$
$$(l_k^a - l_{k-1}^a)h + (\tau_k^c + \tau_k^{ca}) \le \ \eta_2, \quad (2)$$
$$(l_k^a - l_{k-1}^a)h + \tau_k \le \ \eta_3, \quad (3)$$

and

$$\tau_1 \le \tau_k^{sc}, \quad \tau_2 \le \tau_k^c + \tau_k^{ca}, \quad \tau_3 \le \tau_k, \forall k \in \mathbb{N}^*.$$

The elements η_j, $j \in \{1,2,3\}$, denote upper bounds to different parts of the total network-induced delay, involving both transmission delays and packet dropouts. The term η_1 limits the total delay from the sensor to the controler module. Similarly, η_2 is the upper bound delay from the controller to the actuator module. Finally, η_3 limits the total network-induced delay. The terms τ_j denote lower bounds and have analogous definitions.

The plant's LTI process has a state space model of the form

$$\dot{x}_p(t) \ = \ A_p x_p(t) + B_p u_p(t), \quad (4)$$
$$y_p(t) \ = \ C_p x_p(t), \quad (5)$$

where $x_p(t) \in \mathbb{R}^{n_p}$ is the plant's state vector, $u_p(t) \in \mathbb{R}^m$ and $y_p(t) \in \mathbb{R}^r$ are the plant's input and output vectors, respectively, and $A_p \in \mathbb{R}^{n_p \times n_p}$, $B_p \in \mathbb{R}^{n_p \times m}$, and $C_p \in \mathbb{R}^{r \times n_p}$ are constant matrices.

The linear dynamic *Controller Module* G_c can be described as

$$\dot{x}_c(t) \ = \ A_c x_c(t) + B_c u_c(t), \quad (6)$$
$$y_c(t) \ = \ C_c x_c(t - \tau_k^c) + D_c u_c(t - \tau_k^c), \quad (7)$$

where $x_c(t) \in \mathbb{R}^{n_c}$ is the controller's state vector, $u_c(t) \in \mathbb{R}^r$ and $y_c(t) \in \mathbb{R}^m$ are the controller's input and output vectors, respectively, and $A_c \in \mathbb{R}^{n_c \times n_c}$, $B_c \in \mathbb{R}^{n_c \times r}$, $C_c \in \mathbb{R}^{m \times n_c}$ and $D_c \in \mathbb{R}^{m \times r}$ are known, real, constant matrices.

Considering the delay from sensor to controller and packet dropouts, the controller's input $u_c(t)$ from (6) can be described as

$$u_c(t) = y_p(l_k^c h) = C_p x_p(l_k^c h), \qquad \forall k \in \mathbb{N}^* \tag{8}$$

where $t \in [l_k^c h + \tau_k^{sc},\ l_{k+1}^c h + \tau_{k+1}^{sc})$.

The plant's input $u_p(t)$ from (4) can be described as

$$\begin{aligned} u_p(t) =& y_c(l_k^a h + \tau_k^{sc} + \tau_k^c) \\ =& C_c x_c(l_k^a h + \tau_k^{sc}) + D_c C_p x_p(l_k^a h), \quad \forall k \in \mathbb{N}^* \end{aligned} \tag{9}$$

where $t \in [l_k^a h + \tau_k, l_{k+1}^a h + \tau_{k+1})$, and $\tau_k = \tau_k^{sc} + \tau_k^c + \tau_k^{ca}$.

Defining $x(t) = [x_p^T(t), x_c^T(t)]^T \in \mathbb{R}^{n_p + n_c}$ and using (4)-(7), (8), (9), the NCS can be described as:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bx(t - d_1(t)) + Cx(t - d_2(t)) \\ &\quad + Dx(t - d_3(t)), & (10) \\ x(t) &= \phi(t), t \in [t_1 - \eta_3,\ t_1], & (11) \\ \tau_j &\le d_j(t) \le \eta_j, j \in \{1,2,3\}, & (12) \\ \dot{d}_1(t) &= 1, t \in [l_k^c h + \tau_k^{sc}, l_{k+1}^c h + \tau_{k+1}^{sc}), & (13) \\ \dot{d}_2(t) &= \dot{d}_3(t) = 1, t \in [l_k^a h + \tau_k, l_{k+1}^a h + \tau_{k+1}), & (14) \end{aligned}$$

where t_1 denotes the instant that the actuator receives the first control signal,

$$\begin{aligned} A &= \begin{bmatrix} A_p & 0 \\ 0 & A_c \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ B_c C_p & 0 \end{bmatrix}, \\ C &= \begin{bmatrix} 0 & B_p C_c \\ 0 & 0 \end{bmatrix}, \ D = \begin{bmatrix} B_p D_c C_p & 0 \\ 0 & 0 \end{bmatrix} \end{aligned} \tag{15}$$

and

$$\begin{aligned} d_1(t) =& t - l_k^c h, & t \in [l_k^c h + \tau_k^{sc}, l_{k+1}^c h + \tau_{k+1}^{sc}), \quad (16) \\ d_2(t) =& t - l_k^a h - \tau_k^{sc}, & t \in [l_k^a h + \tau_k, l_{k+1}^a h + \tau_{k+1}), \quad (17) \\ d_3(t) =& t - l_k^a h, & t \in [l_k^a h + \tau_k, l_{k+1}^a h + \tau_{k+1}). \quad (18) \end{aligned}$$

The function $d_1(t)$ denotes the time-varying delay from sensor to controller module; $d_2(t)$ is the time-varying delay from controller to actuator module; and $d_3(t)$ is the time-varying delay from sensor to actuator module ($d_3(t) = d_1(t) + d_2(t)$). The functions $d_1(t)$, $d_2(t)$, and $d_3(t)$ are discontinuous at the points $l_k^c h + \tau_k^{sc}$, $l_k^a h + \tau_k$, and $l_k^a h + \tau_k$, $\forall k \in \mathbb{N}^*$, respectively.

Moreover, no knowledge about the network behavior is assumed. Thefore, differently from [12], [14], no restrictions on the characteristics of the time-varying delays' derivative are made.

III. STABILITY ANALYSIS

For the system described by (4)-(7), the stability analysis technique based in Lyapunov functions candidates was chosen. These functions are a generalization of energy functions and must fulfill two conditions to guarantee stability: they must be positive definite and have continuous, negative definite partial first derivatives [24].

This section presents a new stability criterion for NCSs with dynamic controllers in the feedback loop. The resulting theorem, written in the form of a set of LMIs, is based on the Lyapunov function candidate

$$V(t) = \sum_{i=1}^{6} V_i(t), \tag{19}$$

where

$$\begin{aligned} V_1(t) =& x^T(t) P x(t), \qquad V_2(t) = \sum_{i=1}^{3} \int_{t-\tau_i}^{t} \left[x^T(s) M_i x(s)\right] ds, \\ V_3(t) =& \sum_{i=1}^{3} \int_{t-\eta_i}^{t} \left[x^T(s) N_i x(s)\right] ds, \quad V_4(t) = \sum_{i=1}^{3} \int_{t-\alpha_i d_i(t)}^{t} \left[x^T(s) Q_i x(s)\right] ds, \\ V_5(t) =& \sum_{i=1}^{3} \int_{-\eta_i}^{0} \int_{t+\beta}^{t} \left[\dot{x}^T(s) S_i \dot{x}(s)\right] ds d\beta, V_6(t) = \sum_{i=1}^{3} \int_{-\eta_i}^{-\tau_i} \int_{t+\beta}^{t} \left[\dot{x}^T(s) Z_i \dot{x}(s)\right] ds d\beta, \end{aligned} \tag{20}$$

and matrices $P = P^T > 0$, $Q_j = Q_j^T \ge 0$, $M_j = M_j^T \ge 0$, $N_j = N_j^T \ge 0$, $Z_j = Z_j^T > 0$, $S_j = S_j^T > 0$, $j \in \{1,2,3\}$.

Remark 2 Similarly to [19], [20], the derivative character of the time-varying delay functions (16)-(18) can be taken into account through the elements α_i, $i \in \{1,2,3\}$, considered in (20). If these constants are not used, some elements in the derivative of (19) cancel out, making the stability criterion more conservative. One should notice that introducing the α_i constants does not affect the monotonic decrease of $V_4(t)$ in (20) over time.

Throughout this subsection, the following results will be useful to derive sufficient conditions for the NCS's stability.

Lemma 1 *For any constants τ and η and matrix M of appropriate dimensions, the following equality holds:*

$$\frac{d}{dt}\left[\int_{-\eta}^{-\tau} \int_{t+\beta}^{t} \left[\dot{x}^T(s) M \dot{x}(s)\right] ds d\beta\right] = (\eta - \tau)\dot{x}^T(t) M \dot{x}(t) - \int_{t-\eta}^{t-\tau} \dot{x}^T(s) M \dot{x}(s) ds$$

Remark 3 Lemma 1 is a simple extension of Liebniz integral rule.

Lemma 2 ([15], [19], [25]) *For given scalars r_1, r_2 and matrix $M \in \mathbb{R}^{m \times m}$ such that $(r_2 - r_1) > 0$ and $M = M^T > 0$, if choosing a vectorial function $x : [r_1, r_2] \longrightarrow \mathbb{R}^m$ yields:*

$$\int_{r_1}^{r_2} x^T(\beta) M x(\beta) d\beta \ge \frac{1}{(r_2 - r_1)} \left(\int_{r_1}^{r_2} x(\beta) d\beta\right)^T M \left(\int_{r_1}^{r_2} x(\beta) d\beta\right) \tag{21}$$

Using (19) as the Lyapunov function candidate for the stability analysis of the NCS described by (4)-(7), a delay-dependent asymptotical stability criterion for NCSs with dynamic controllers is derived as follows.

Theorem 1 *For given scalars $0 \le \tau_i < \eta_i$ and $0 < \alpha_i < 1$, $i \in \{1,2,3\}$, the NCS described by (4)-(7) is asymptotically*

$$
\Omega=\begin{bmatrix}
\Omega_{1,1} & \Omega_{1,2} & \Omega_{1,3} & \Omega_{1,4} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\alpha_1\eta_1}S_1 & \frac{1}{\alpha_2\eta_2}S_2 & \frac{1}{\alpha_3\eta_3}S_3 \\
* & \Omega_{2,2} & \Omega_{2,3} & \Omega_{2,4} & \frac{1}{\eta_1-\tau_1}Z_1 & 0 & 0 & \Omega_{2,8} & 0 & 0 & \frac{1}{(1-\alpha_1)\eta_1}S_1 & 0 & 0 \\
* & * & \Omega_{3,3} & \Omega_{3,4} & 0 & \frac{1}{\eta_2-\tau_2}Z_2 & 0 & 0 & \Omega_{3,9} & 0 & 0 & \frac{1}{(1-\alpha_2)\eta_2}S_2 & 0 \\
* & * & * & \Omega_{4,4} & 0 & 0 & \frac{1}{\eta_3-\tau_3}Z_3 & 0 & 0 & \Omega_{4,10} & 0 & 0 & \frac{1}{(1-\alpha_3)\eta_3}S_3 \\
* & * & * & * & \Omega_{5,5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & \Omega_{6,6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & \Omega_{7,7} & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & \Omega_{8,8} & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & \Omega_{9,9} & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & \Omega_{10,10} & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & \Omega_{11,11} & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & \Omega_{12,12} & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & \Omega_{13,13}
\end{bmatrix} < 0 \qquad (22)
$$

stable if there exist matrices $P=P^T>0$*,* $Q_j=Q_j^T\geq 0$*,* $M_j=M_j^T\geq 0$*,* $N_j=N_j^T\geq 0$*,* $Z_j=Z_j^T>0$*,* $S_j=S_j^T>0$*,* $j\in\{1,2,3\}$*, such that (22) holds, where*

$$
U_1=\left[\sum_{i=1}^{3}\eta_i S_i\right], \qquad U_2=\left[\sum_{i=1}^{3}(\eta_i-\tau_i)Z_i\right],
$$

$$
U_3=\sum_{i=1}^{3}\left(M_i+N_i+Q_i-\frac{1}{\alpha_i\eta_i}S_i\right),
$$

$$
V_i=\left(\frac{1}{(1-\alpha_i)\eta_i}+\frac{1}{\eta_i-\tau_i}\right)S_i, \qquad i\in\{1,2,3\},
$$

$$
W_i=(1-\alpha_i)Q_i+\left(\frac{1}{\alpha_i\eta_i}+\frac{1}{(1-\alpha_i)\eta_i}\right)S_i, \qquad i\in\{1,2,3\},
$$

$$
\begin{aligned}
\Omega_{1,1} &= A^TP+PA+U_3+A^T(U_1+U_2)A,\\
\Omega_{1,2} &= PB+A^T(U_1+U_2)B,\\
\Omega_{1,3} &= PC+A^T(U_1+U_2)C,\\
\Omega_{1,4} &= PD+A^T(U_1+U_2)D,\\
\Omega_{2,2} &= B^T(U_1+U_2)B-V_1-\frac{2}{\eta_1-\tau_1}Z_1,\\
\Omega_{2,3} &= B^T(U_1+U_2)C,\\
\Omega_{2,4} &= B^T(U_1+U_2)D,\\
\Omega_{2,8} &= \frac{1}{\eta_1-\tau_1}(S_1+Z_1),\\
\Omega_{3,3} &= C^T(U_1+U_2)C-V_2-\frac{2}{\eta_2-\tau_2}Z_2,\\
\Omega_{3,4} &= C^T(U_1+U_2)D,\\
\Omega_{3,9} &= \frac{1}{\eta_2-\tau_2}(S_2+Z_2),\\
\Omega_{4,4} &= D^T(U_1+U_2)D-V_3-\frac{2}{\eta_3-\tau_3}Z_3,\\
\Omega_{4,10} &= \frac{1}{\eta_3-\tau_3}(S_3+Z_3),\\
\Omega_{5,5} &= -M_1-\frac{1}{\eta_1-\tau_1}Z_1,\\
\Omega_{6,6} &= -M_2-\frac{1}{\eta_2-\tau_2}Z_2,\\
\Omega_{7,7} &= -M_3-\frac{1}{\eta_3-\tau_3}Z_3,\\
\Omega_{8,8} &= -N_1-\frac{1}{\eta_1-\tau_1}(S_1+Z_1),\\
\Omega_{9,9} &= -N_2-\frac{1}{\eta_2-\tau_2}(S_2+Z_2),\\
\Omega_{10,10} &= -N_3-\frac{1}{\eta_3-\tau_3}(S_3+Z_3),\\
\Omega_{11,11} &= -W_1, \quad \Omega_{12,12} = -W_2, \quad \Omega_{13,13} = -W_3.
\end{aligned}
$$

Proof: Taking the time derivative of the Lyapunov function candidate (19) yields

$$
\begin{aligned}
\dot{V}_1(t) &= \dot{x}^T(t)Px(t)+x^T(t)P\dot{x}(t),\\
\dot{V}_2(t) &= \sum_{i=1}^{3}x^T(t)M_ix(t)-x^T(t-\tau_i)M_ix(t-\tau_i),\\
\dot{V}_3(t) &= \sum_{i=1}^{3}x^T(t)N_ix(t)-x^T(t-\eta_i)N_ix(t-\eta_i),\\
\dot{V}_4(t) &= \sum_{i=1}^{3}x^T(t)Q_ix(t)-(1-\alpha_i)x^T(t-\alpha_id_i(t))Q_ix(t-\alpha_id_i(t)).
\end{aligned}
$$

From Lemma 1, $\dot{V}_5(t)$ and $\dot{V}_6(t)$ can be written as

$$
\begin{aligned}
\dot{V}_5(t) =& \sum_{i=1}^{3}\dot{x}^T(t)\eta_iS_i\dot{x}(t)-\int_{t-\alpha_id_i(t)}^{t}\dot{x}^T(s)S_i\dot{x}(s)ds\\
&-\int_{t-d_i(t)}^{t-\alpha_id_i(t)}\dot{x}^T(s)S_i\dot{x}(s)ds-\int_{t-\eta_i}^{t-d_i(t)}\dot{x}^T(s)S_i\dot{x}(s)ds, \qquad (23)\\
\dot{V}_6(t) =& \sum_{i=1}^{3}\dot{x}^T(t)(\eta_i-\tau_i)Z_i\dot{x}(t)\\
&-\int_{t-d_i(t)}^{t-\tau_i}\dot{x}^T(s)Z_i\dot{x}(s)ds-\int_{t-\eta_i}^{t-d_i(t)}\dot{x}^T(s)Z_i\dot{x}(s)ds. \qquad (24)
\end{aligned}
$$

Applying (21) to (23) and (24) yields

$$
\begin{aligned}
\dot{V}_5(t) \leq& \sum_{i=1}^{3}\dot{x}^T(t)\eta_iS_i\dot{x}(t)-\frac{1}{\alpha_i\eta_i}p^TS_ip\\
&-\frac{1}{(1-\alpha_i)\eta_i}q^TS_iq-\frac{1}{\eta_i-\tau_i}r^TS_ir,
\end{aligned}
$$

$$
\begin{aligned}
\dot{V}_6(t) \leq& \sum_{i=1}^{3}\dot{x}^T(t)(\eta_i-\tau_i)Z_i\dot{x}(t)\\
&-\frac{1}{\eta_i-\tau_i}m^TZ_im-\frac{1}{\eta_i-\tau_i}n^TZ_in,
\end{aligned}
$$

where

$$\begin{aligned} p &= x(t) - x(t-\alpha_i d_i(t)), \\ q &= x(t-\alpha_i d_i(t)) - x(t-d_i(t)), \\ r &= x(t-d_i(t)) - x(t-\eta_i), \\ m &= x(t-\tau_i) - x(t-d_i(t)), \\ n &= x(t-d_i(t)) - x(t-\eta_i). \end{aligned}$$

Replacing $\dot{x}(t)$ by (10) and denoting

$$\begin{aligned} &\delta^T = \\ &[\, x^T(t)\ x^T(t-d_1(t))\ x^T(t-d_2(t))\ x^T(t-d_3(t)) \\ &x^T(t-\tau_1)\ x^T(t-\tau_2)\ x^T(t-\tau_3)\ x^T(t-\eta_1) \\ &x^T(t-\eta_2)\ x^T(t-\eta_3)\ x^T(t-\alpha_1 d_1(t)) \\ &x^T(t-\alpha_2 d_2(t))\ x^T(t-\alpha_3 d_3(t))\,], \end{aligned}$$

stability will be guaranteed if the condition

$$\dot{V}(t) = \sum_{i=1}^{6} \dot{V}_i(t) \le \delta^T \Omega \delta < 0$$

is fulfilled. This completes the proof. ∎

Remark 4 The works in [15]–[19], similarly to Theorem 1, propose delay-depedent stability criteria by using Lyapunov function techniques. Nevertheless, only NCSs with proportional state feedback controllers were considered. Therefore, Theorem 1 is more general and may be applied to a larger set of controllers, including proportional controllers in the feedback loop. Indeed, the combined use of the NCS model proposed in [15], [18], [19] with Theorem 1 yields the same stability criterion presented in [19], which is less conservative than the ones in [15]–[18]. In order to see that [19] is a particular case of Theorem 1, suppose $A_c = 0$ and $B_c = 0$ in the closed loop system parameters (15). Substituting the result in (10) yields

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} A_p & 0 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} x(t-d_1(t)) \\ &+ \begin{bmatrix} 0 & B_pC_c \\ 0 & 0 \end{bmatrix} x(t-d_2(t)) + \begin{bmatrix} B_pD_cC_p & 0 \\ 0 & 0 \end{bmatrix} x(t-d_3(t)) \\ &= A_p x_p(t) + B_pC_c x_c(t-d_2(t)) + B_pD_cC_p x_p(t-d_3(t)). \quad (25) \end{aligned}$$

Using $A_c = 0$ and $B_c = 0$ in (6) yields $\dot{x}_c(t) = 0$. Hence, if we suppose null initial conditions to the controller, (25) reduces to

$$\dot{x}(t) = A_p x_p(t) + B_pD_cC_p x_p(t-d_3(t)).$$

Making $K = D_c$, $C_p = I$, $d(t) = d_3(t)$, $\eta = \eta_3$, $\tau = \tau_3$, $\alpha = \alpha_3$ and using appropriately Schur complements in (22), Theorem 1 reduces to the following corollary.

Corollary 1 ([19]) *For given scalars τ, η $(0 \le \tau < \eta)$ and α $(0 < \alpha < 1)$, the NCS with state feedback gain matrix K and time-varying delay from sensor to actuator $d(t)$ ($\tau \le d(t) \le \eta$) described by*

$$\dot{x}(t) = Ax(t) + BKx(t-d(t))$$

TABLE I

MAXIMUM TOTAL DELAY BOUNDARY FOR DIFFERENT VALUES OF τ_3

τ_3	0s	0.02s	0.04s	0.05s	0.0685s
η_3	0.05985s	0.06163s	0.06434s	0.06579s	0.06858s

is asymptotically stable if there exist matrices $P = P^T > 0$, $Q_i = Q_i^T \ge 0$ $(i = 1,2,3)$, $Z_j = Z_j^T > 0$ $(j = 1,2)$, such that the following LMI holds:

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} & 0 & 0 & \frac{1}{\alpha\eta}Z_1 \\ * & \Phi_{22} & \frac{1}{\eta-\tau}Z_2 & \frac{1}{\eta-\tau}(Z_1+Z_2) & \frac{1}{(1-\alpha)\eta}Z_1 \\ * & * & \Phi_{33} & 0 & 0 \\ * & * & * & \Phi_{44} & 0 \\ * & * & * & * & \Phi_{55} \end{bmatrix} < 0$$

where

$$\begin{aligned} \Phi_{11} &= PA + (PA)^T + \sum_{i=1}^{3} Q_i - \frac{1}{\alpha\eta} Z_1 + A^T U A, \\ \Phi_{12} &= P(BK) + A^T U (BK), \\ \Phi_{22} &= -\left[\frac{1}{(1-\alpha)\eta} + \frac{1}{\eta-\tau}\right] Z_1 - \frac{2}{\eta-\tau} Z_2 + (BK)^T U (BK), \\ \Phi_{33} &= -Q_1 - \frac{1}{\eta-\tau} Z_2, \\ \Phi_{44} &= -Q_2 - \frac{1}{\eta-\tau}(Z_1+Z_2), \\ \Phi_{55} &= -(1-\alpha)Q_3 - \frac{1}{\alpha\eta} Z_1 - \frac{1}{(1-\alpha)\eta} Z_1, \\ U &= \eta Z_1 + (\eta-\tau) Z_2. \end{aligned}$$

Proof: Immediate. ∎

IV. NUMERICAL EXAMPLES

This section presents two examples that ratify the validity of the proposed criterion. The first one investigates the possibility of applying Theorem 1 to a NCS with a proportional state feedback controller. The last one illustrates the effectiveness of our method by applying it to a NCS with a PI dynamic controller in the feedback loop.

Example 1 Consider the same NCS example presented in [3] described by:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix},\ B = 0,\ C = 0,\ D = \begin{bmatrix} 0 & 0 \\ -0.375 & -1.15 \end{bmatrix}.$$

From Theorem 1, with $\alpha_1 = \alpha_2 = 0$, $\alpha_3 = 0.75$, $\tau_1 = \tau_2 = \tau_3 = 0$ s and $\eta_1 = \eta_2 = \frac{\eta_3}{2}$, we obtained that the NCS system is stable for a total delay within the interval of 0 to 1.008 s. The bound value for the total delay is less conservative than the values obtained using the criteria presented in the following works: 0.00045 s [3], 0.0538 s [13], 0.8695 s [17], 0.87 s [16], 0.8871 s [18]. The result is the same as the obtained in [19], which goes in agreement with Corollary 1.

Example 2 For this example, we consider a real NCS composed by a DC motor driven by a PI controller: $G_c(s) = K_p + \frac{1}{s}K_I$. The matrices

$$A_p = \begin{bmatrix} 0 & 1 \\ 0 & -36.17 \end{bmatrix}, \; B_p = \begin{bmatrix} 0 \\ 36.3 \end{bmatrix}, \quad C_p = \begin{bmatrix} 1 & 0 \end{bmatrix},$$
$$A_c = 0, \; B_c = K_I = -47.45, \; C_c = 1, \; D_c = K_p = -11.86,$$

are the same as described in (4)-(7). The plant's model matrices (A_p, B_p and C_p) were obtained experimentally from a Maxon F2140 DC motor. The results from Table I assumed $\tau_1 = \tau_2 = \frac{\tau_3}{2}$, $\eta_1 = \eta_2 = \frac{\eta_3}{2}$, $\alpha_1 = \alpha_2 = 0.5\alpha_3 = 0.375$, and show the relationship between the upper and the lower network delay bounds, establishing sufficient conditions for the asymptotical stability of the NCS described in this example. The simulation results depicted in Figure 3 show that the system starts to behave unstably for $\tau_3 = \eta_3 = 0.078s$, which is close to the value in Table I.

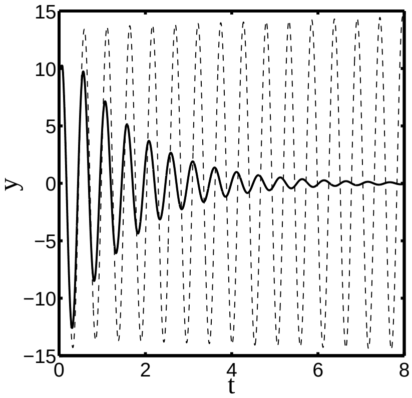

Fig. 3. Simulation results for the NCS described in Example 2 using $\tau_3 = \eta_3 = 0.07s$ (solid) and $\tau_3 = \eta_3 = 0.078s$ (dashed).

V. CONCLUSIONS

This work's main result concerns the establishment of a new stability criterion for NCSs with dynamic controllers in the feedback loop. Network-induced delay bounds are obtained by solving a set of LMIs. The validity of our results is shown through a numerical simulation. Although this paper deals mainly with dynamic controllers, our criterion, when applied to proportional controllers, yields network-induced delay bounds as good as the ones presented in [19], [20], which are shown to be less conservative than the ones from previous works, as indicated in Section IV.

VI. ACKNOWLEDGMENTS

The authors are supported by research grants funded by the National Council of Technological and Scientific Development - CNPq, Brazil.

REFERENCES

[1] J. Nilsson, "Real-time control systems with delays," Ph.D. dissertation, Lund Institute of Technology, Lund, 1998.

[2] M. Branicky, S. Phillips, and W. Zhang, "Stability of networked control systems: explicit analysis of delay," in *Proceedings of the American Control Conference*, vol. 4, 2000, pp. 2352–2357.

[3] W. Zhang, M. S. Branicky, and S. M. Phillips, "Stability of networked control systems," *IEEE Control Systems Magazine*, vol. 21, pp. 84–99, Feb. 2001.

[4] J. Hespanha, P. Naghshtabrizi, and Y. Xu, "A survey of recent results in networked control systems," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 138–162, Jan. 2007.

[5] T. C. Yang, "Networked control system: a brief survey," *Control Theory and Applications*, vol. 153, pp. 403–412, July 2006.

[6] F.-L. Lian, "Analysis, design, modeling and control of networked control systems," Ph.D. dissertation, University of Michigan, 2001.

[7] Y. Halevi and A. Ray, "Integrated communication and control systems: Part 1 analysis," *J. Dyn. Sys., Meas., Control*, vol. 110, pp. 367–373, Dec. 1988.

[8] P. Seiler and R. Sengupta, "Analysis of communication losses in vehicle control problems," *Proceedings of the American Control Conference*, vol. 2, pp. 1491–1496 vol.2, 2001.

[9] M. Yu, L. Wang, T. Chu, and F. Hao, "An lmi approach to networked control systems with data packet dropout and transmission delays," in *43rd IEEE Conference on Decision and Control*, vol. 4, Dec. 2004, pp. 3545–3550.

[10] J.-P. Richard, "Time-delay systems: an overview of some recent advances and open problems," *Automatica*, vol. 39, no. 10, pp. 1667 – 1694, 2003.

[11] E. Fridman and U. Shaked, "An improved stabilization method for linear time-delay systems," *IEEE Transactions on Automatic Control*, pp. 1931–1937, Nov. 2002.

[12] M. Wu, Y. He, J.-H. She, and G.-P. Liu, "Delay-dependent criteria for robust stability of time-varying delay systems," *Automatica*, vol. 40, no. 8, pp. 1435 – 1439, 2004.

[13] H. S. Park, Y. H. Kim, D. S. Kim, and W. H. Kwon, "A scheduling method for network-based control systems," *IEEE Transactions on Control Systems Technology*, vol. 10, no. 3, pp. 318–330, May 2002.

[14] Y. He, Q.-G. Wang, L. Xie, and C. Lin, "Further improvement of free-weighting matrices technique for systems with time-varying delay," *IEEE Transactions on Automatic Control*, vol. 52, no. 2, pp. 293–299, Feb. 2007.

[15] Y. He, Q.-G. Wang, C. Lin, and M. Wu, "Technical communique: Delay-range-dependent stability for systems with time-varying delay," *Automatica*, vol. 43, no. 2, pp. 371–376, 2007.

[16] P. Naghshtabrizi, J. P. Hespanha, and A. R. Teel, "Stability of delay impulsive systems with application to networked control systems," *Transactions of the Institute of Measurement and Control*, 2009.

[17] D. Yue, Q.-L. Han, and C. Peng, "State feedback controller design of networked control systems," *IEEE Transactions on Circuits and Systems II: Express Briefs*, vol. 51, no. 11, pp. 640–644, Nov. 2004.

[18] D. Yue, Q.-L. Han, and J. Lam, "Network-based robust H_∞ control of systems with uncertainty," *Automatica*, vol. 41, no. 6, pp. 999–1007, 2005.

[19] X.-L. Zhu and G.-H. Yang, "New results on stability analysis of networked control systems," *American Control Conference*, pp. 3792–3797, Nov. 2008.

[20] L. Figueredo, P. Santana, E. Alves, J. Ishihara, G. Borges, and A. Bauchspiess, "Robust stability of networked control systems," *IEEE International Conference on Control & Automation, 2009*, December 2009.

[21] X. Jiang, Q.-L. Han, S. Liu, and A. Xue, "A new H_∞ stabilization criterion for networked control systems," *IEEE Transactions on Automatic Control*, vol. 53, no. 4, pp. 1025–1032, May 2008.

[22] U. Ozguner, H. Goktas, H. Chan, J. Winkelman, M. Liubakka, and R. Krtolica, "Automotive suspension control through a computer communication network," in *First IEEE Conference on Control Applications*, Sep 1992, pp. 895–900 vol.2.

[23] Y. Xu and J. Hespanha, "Estimation under uncontrolled and controlled communications in networked control systems," *44th IEEE Conference on Decision and Control, 2005 and 2005 European Control Conference, CDC-ECC '05.*, pp. 842–847, Dec. 2005.

[24] S. Sastry, *Nonlinear Systems: Analysis, Stability and Control*. New York, NY, USA: Springer-Verlag New York, Inc., 1999.

[25] V. K. K. Gu and J. Chen, *Stability of time-delay systems*. Birkhauser, 2003.